Chapter 1

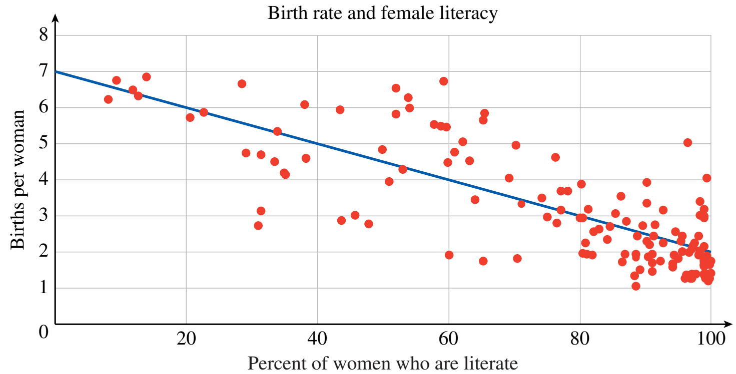
Functions and Their Graphs

You may have heard that mathematics is the language of science. In fact, professionals in nearly every discipline take advantage of mathematical methods to analyze data, identify trends, and predict the effects of change. This process is called **mathematical modeling**.

A **model** is a simplified representation of reality that helps us understand a process or phenomenon. Because it is a simplification, a model can never be completely accurate. Instead, it should focus on those aspects of the real situation that will help us answer specific questions. Here is an example.

The world's population is growing at different rates in different nations. Many factors, including economic and social forces, influence the birth rate. Is there a connection between birth rates and education levels?

The figure shows the birth rate plotted against the female literacy rate in 148 countries. Although the data points do not all lie precisely on a line, we see a generally decreasing trend: the higher the literacy rate, the lower the birth rate.



The **regression line** provides a model for this trend, and a tool for analyzing the data. In this chapter we study the properties of linear models and some techniques for fitting a linear model to data.

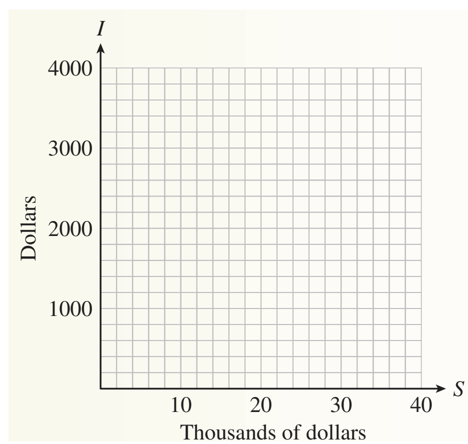
1.1 Linear Models

Investigation 1 Sales on Commission. Delbert is offered a part-time job selling restaurant equipment. He will be paid \$1000 per month plus a 6% commission on his sales. The sales manager tells Delbert he can expect to sell about \$8000 worth of equipment per month. To help him decide whether to accept the job, Delbert does a few calculations.

1. Based on the sales manager's estimate, what monthly income can Delbert expect from this job? What annual salary would that provide?

2. What would Delbert's monthly salary be if he sold only \$5000 of equipment per month? What would his salary be if he sold \$10,000 worth per month? Compute monthly incomes for each sales total shown in the table.

Sales	Income
5000	
8000	
10,000	
12,000	
15,000	
18,000	
20,000	
25,000	
30,000	
35,000	



3. Plot your data points on a graph, using sales, S, on the horizontal axis and income, I, on the vertical axis, as shown in the figure. Connect the data points to show Delbert's monthly income for all possible monthly sales totals.

4. Add two new data points to the table by reading values from your graph.

5. Write an algebraic expression for Delbert's monthly income, I, in terms of his monthly sales, S. Use the description in the problem to help you:

He will be paid: \$1000 . . . plus a 6% commission on his sales.

Income =______________

6. Test your formula from part (5) to see if it gives the same results as those you recorded in the table.
7. Use your formula to find out what monthly sales total Delbert would need in order to have a monthly income of \$2500.
8. Each increase of \$1000 in monthly sales increases Delbert's monthly income by ______________.
9. Summarize the results of your work: In your own words, describe the relationship between Delbert's monthly sales and his monthly income. Include in your discussion a description of your graph.

1.1.1 Tables, Graphs and Equations

The first step in creating a model is to describe relationships between variables. In Investigation 1, p. 2, we analyzed the relationship between Delbert's sales and his income. Starting from a verbal description, we represented the relationship in three different ways.

1. A **table of values** displays specific data points with precise numerical values.
2. A **graph** is a visual display of the data. It is easier to spot trends and describe the overall behavior of the variables from a graph.
3. An **algebraic equation** is a compact summary of the model. It can be used to analyze the model and to make predictions

We begin our study of modeling with some examples of **linear models**. In the examples that follow, observe the interplay among the three modeling tools, and how each contributes to the model.

Example 1.1.1 Annelise is on vacation at a seaside resort. She can rent a bicycle from her hotel for \$3 an hour, plus a \$5 insurance fee. (A fraction of an hour is charged as the same fraction of \$3.)

a Make a table of values showing the cost, C, of renting a bike for various lengths of time, t.

b Plot the points on a graph. Draw a curve through the data points.

c Write an equation for C in terms of t.

Solution.

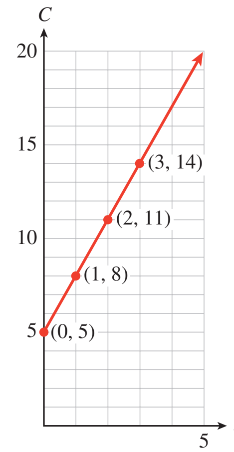
a To find the cost, we multiply the time by \$3, and add the result to the \$5 insurance fee. For example, the cost of a 1-hour bike ride is

$$\text{Cost} = (\$5 \text{ insurance fee}) + (\$3 \text{ per hour}) \times (\mathbf{1} \text{ hour})$$
$$C = 5 + 3(\mathbf{1}) = 8$$

A 1-hour bike ride costs \$8. We record the results in a table, as shown here:

Length of rental (hours)	Cost of rental (dollars)		(t, C)
1	8	$C = 5 + 3(\mathbf{1})$	$(1, 8)$
2	11	$C = 5 + 3(\mathbf{2})$	$(2, 11)$
3	14	$C = 5 + 3(\mathbf{3})$	$(3, 14)$

b Each pair of values represents a point on the graph. The first value gives the horizontal coordinate of the point, and the second value gives the vertical coordinate. The points lie on a straight line, as shown in the figure. The line extends infinitely in only one direction, because negative values of t do not make sense here.



c To write an equation, we let C represent the cost of the rental, and we use t for the number of hours:

$$\text{Cost} = (\$5 \text{ insurance fee}) + (\$3 \text{ per hour}) \times (\text{number of hours})$$
$$C = 5 + 3 \cdot t$$

□

Example 1.1.2 Use the equation $C = 5 + 3 \cdot t$ you found in Example 1.1.1, p. 3 to answer the following questions. Then show how to find the answers by using the graph.

a How much will it cost Annelise to rent a bicycle for 6 hours?

b How long can Annelise bicycle for \$18.50?

Solution.

a We substitute $t = \mathbf{6}$ into the expression for C to find

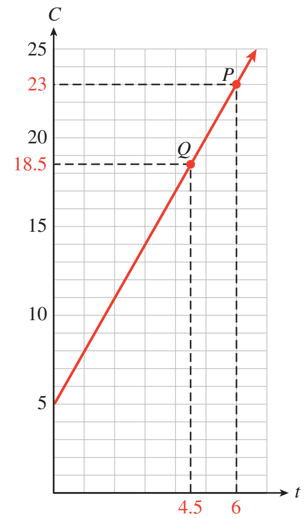
$$C = 5 + 3(\mathbf{6}) = 23$$

A 6-hour bike ride will cost \$23. The point P on the graph in the figure represents the cost of a 6-hour bike ride. The value on the C-axis at the same height as point P is 23, so a 6-hour bike ride costs \$23.

We substitute $C = \mathbf{18.50}$ into the equation and solve for t.

$$\begin{aligned} \mathbf{18.50} &= 5 + 3t \\ 13.50 &= 3t \\ t &= 4.5 \end{aligned}$$

b For \$18.50 Annelise can bicycle for 4½ hours. The point Q on the graph represents an \$18.50 bike ride. The value on the t-axis below point Q is 4.5, so \$18.50 will buy a 4.5 hour bike ride.



□

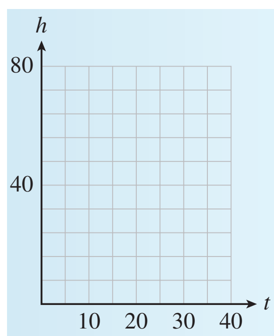
Note 1.1.3 In Example 1.1.2, p. 4, notice the different algebraic techniques we used in parts (a) and (b).

- In part (a), we were given a value of t and we **evaluated the expression** $5 + 3t$ to find C.
- In part (b) we were given a value of C and we **solved the equation** $C = 5 + 3t$ to find t.

Checkpoint 1.1.4

Frank plants a dozen corn seedlings, each 6 inches tall. With plenty of water and sunlight they will grow approximately 2 inches per day. Complete the table of values for the height, h, of the seedlings after t days.

t	0	5	10	15	20
h					



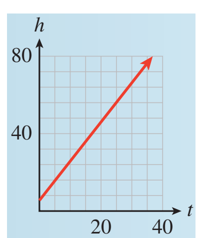
a Write an equation for the height of the seedlings in terms of the number of days since they were planted.

b Graph the equation.

Answer.

a $h = 6 + 2t$

b



Checkpoint 1.1.5 Use your equation from Checkpoint 1.1.4, p. 5 to answer the questions. Illustrate each answer on the graph.

a How tall is the corn after 3 weeks?

b How long will it be before the corn is 6 feet tall?

Hint. For part (b), convert feet to inches.

Answer.

a 48 inches tall

b 33 days

1.1.2 Choosing Scales for the Axes

To create a useful graph, we must choose appropriate scales for the axes.

- The axes must extend far enough to show the values of the variables.
- The tick marks should be equally spaced.
- Usually we should use no more than 10 or 15 tick marks.

Example 1.1.6 In 1990, the median price of a home in the US was $92,000. The median price increased by about $4700 per year over the next decade.

a Make a table of values showing the median price of a house in 1990, 1994, 1998, and 2000.

b Choose suitable scales for the axes and plot the values you found in part (a) on a graph. Use t, the number of years since 1990, on the horizontal axis and the price of the house, P, on the vertical axis. Draw a curve through the points.

c Write an equation that expresses P in terms of t.

d How much did the price of the house increase from 1990 to 1996? Illustrate the increase on your graph.

Solution.

a In 1990 the median price was $92,000. Four years later, in 1994, the price had increased by $\mathbf{4}(4700) = 18,800$ dollars, so

$$P = 92,000 + \mathbf{4}(4700) = 110,800$$

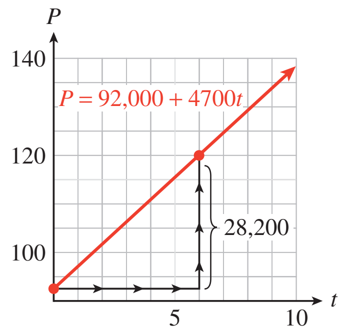
In 1998 the price had increased by $\mathbf{8}(4700) = 37,600$ dollars, so

$$P = 92,000 + \mathbf{8}(4700) = 129,600$$

You can verify the price of the house in 2000 by a similar calculation.

Year	Price of House)	(t, P)
1990	92,000	(0, 92,000)
1994	110,800	(4, 110,800)
1998	129,600	(8, 129,600)
2000	139,000	(10, 139,000)

b We let t stand for the number of years since 1990, so that $t = 0$ in 1990, $t = 4$ in 1994, and so on. To choose scales for the axes, we look at the values in the table. For this graph we scale the horizontal axis, or t-axis, in 1-year intervals and the vertical axis, or P-axis, for $90,000 to $140,000 in intervals of $5,000. The points lie on a straight line, as shown in the figure.



c Look back at the calculations in part (a). The price of the house started at \$92,000 in 1990 and increased by $t \times 4700$ dollars after t years. Thus,

$$P = 92,000 + 4700t$$

d We find the points on the graph for 1990 and 1996. These points lie above $t = 0$ and $t = 6$ on the t-axis. Next we find the values on the P-axis corresponding to the two points. The values are $P = 92,000$ in 1990 and $P = 120,200$ in 1996. The increase in price is the difference of the two P-values.

$$\begin{aligned}\text{increase in price} &= 120,200 - 92,000\\ &= 28,200\end{aligned}$$

The price of the home increased \$28,200 between 1990 and 1996. This increase is indicated by the arrows in the figure.

□

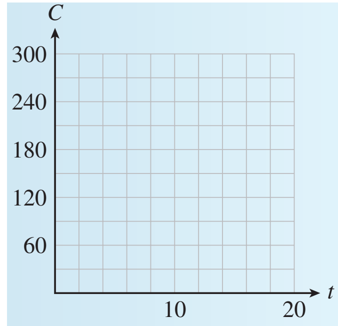
Note 1.1.7 The graphs in the preceding examples are **increasing graphs** . As we move along the graph from left to right (in the direction of increasing t), the second coordinate increases as well. Try Checkpoint 1.1.8, p. 7, which illustrates a **decreasing graph**.

Checkpoint 1.1.8 Silver Lake has been polluted by industrial waste products. The concentration of toxic chemicals in the water is currently 285 parts per million (ppm). Environmental officials would like to reduce the concentration by 15 ppm each year.

a Complete the table of values showing the desired concentration, C, of toxic chemicals t years from now. For each t-value, calculate the corresponding value for C. Write your answers as ordered pairs.

t	C		(t, C)
0		$C = 285 - 15(\mathbf{0})$	(0,)
5		$C = 285 - 15(\mathbf{5})$	(5,)
10		$C = 285 - 15(\mathbf{10})$	(10,)
15		$C = 285 - 15(\mathbf{15})$	(15,)

b To choose scales for the axes, notice that the value of C starts at 285 and decreases from there. We'll scale the vertical axis up to 300, and use 10 tick marks at intervals of 30. Graph the ordered pairs on the grid, and connect them with a straight line. Extend the graph until it reaches the horizontal axis, but no farther. Points with negative C-coordinates have no meaning for the problem.



c Write an equation for the concentration, C, of toxic chemicals t years from now.

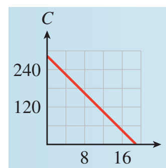
Hint. For part (c): The concentration is initially 285 ppm, and we subtract 15 ppm for each year that passes, or $15 \times t$.

Answer.

a

(t, C)
$(0, 285)$
$(5, 210)$
$(10, 135)$
$(15, 60)$

b



c $C = 285 - 15t$

Note 1.1.9 In the previous Checkpoint, we extend the graph until it reaches the horizontal axis, but no farther. Points with negative C-coordinates have no meaning for the problem.

Example 1.1.10 Using a Graphing Calculator. In Example 1.1.6, p. 6, we found the equation

$$P = 92,000 + 4700t$$

for the median price of a house t years after 1990. Graph this equation on a calculator.

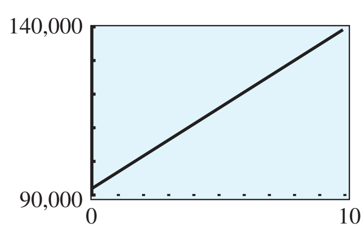
Solution. To begin, we press `Y=` and enter

$$Y1 = 92,000 + 4700X$$

For this graph, we'll use the grid in Example 1.1.6, p. 6 for our window settings, so we press `WINDOW` and enter

Xmin= 0	Xmax= 10
Ymin= 90,000	Ymax= 140,000

Finally, we press `GRAPH`. The calculator's graph is shown in the figure.



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Checkpoint 1.1.11

a Solve the equation $2y - 1575 = 45x$ for y in terms of x.

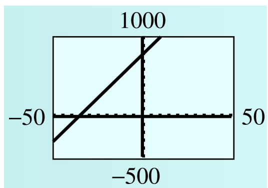
b Graph the equation on a graphing calculator. Use the window

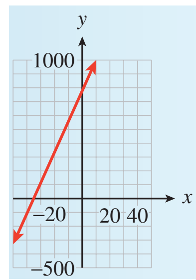
Xmin= -50	Xmax= 50	Xscl= 5
Ymin= -500	Ymax= 1000	Yscl= 100

c Sketch the graph on paper. Use the window settings to choose appropriate scales for the axes.

Answer.

a $y = (1575 + 45x)/2$

b 

c 

1.1.3 Linear Equations

All the models in the preceding examples have equations with a similar form:

$$\boldsymbol{y} = (\text{starting value}) + (\text{rate of change}) \cdot \boldsymbol{x}$$

(We'll talk more about rate of change in Section 1.4, p. 82.) Their graphs were all portions of straight lines. For this reason such equations are called **linear equations**. The order of the terms in the equation does not matter. For example, the equation in Example 1.1.1, p. 3,

$$C = 5 + 3t$$

can be written equivalently as

$$-3t + C = 5$$

and the equation in Example 1.1.6, p. 6,

$$P = 92,000 + 4700t$$

can be written as

$$-4700t + P = 92,000$$

This form of a linear equation, $Ax + By = C$, is called the **general form**.

General Form for a Linear Equation.

The graph of any equation

$$Ax + By = C$$

where A and B are not both equal to zero, is a straight line.

Example 1.1.12 The manager at Albert's Appliances has \$3000 to spend on advertising for the next fiscal quarter. A 30-second spot on television costs \$150 per broadcast, and a 30-second radio ad costs \$50.

a The manager decides to buy x television ads and y radio ads. Write an equation relating x and y.

b Make a table of values showing several choices for x and y.

c Plot the points from your table, and graph the equation.

Solution.

a Each television ad costs \$150, so x ads will cost \$150x. Similarly, y radio ads will cost \$50y. The manager has \$3000 to spend, so the sum of the costs must be \$3000. Thus,

$$150x + 50y = 3000$$

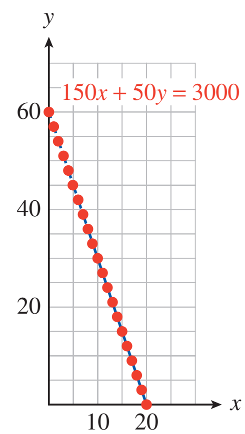
b We choose some values of x, and solve the equation for the corresponding value of y. For example, if $x = \mathbf{10}$ then

$$\begin{aligned}150(\mathbf{10}) + 50y &= 300\\ 1500 + 50y &= 3000\\ 50y &= 1500\\ y &= 30\end{aligned}$$

If the manager buys 10 television ads, she can also buy 30 radio ads. You can verify the other entries in the table.

x	8	10	12	14
y	36	30	24	18

c We plot the points from the table. All the solutions lie on a straight line, as shown in the figure.



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Checkpoint 1.1.13 In central Nebraska, each acre of corn requires 25 acre-inches of water per year, and each acre of winter wheat requires 18 acre-inches of water. (An acre-inch is the amount of water needed to cover one acre of land to a depth of one inch.) A farmer can count on 9000 acre-inches of water for the coming year. (Source: Institute of Agriculture and Natural Resources, University of Nebraska)

a Write an equation relating the number of acres of corn, x, and the number of acres of wheat, y, that the farmer can plant.

b Complete the table.

x	50	100	150	200
y				

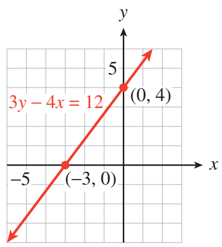
Answer.

a $25x + 18y = 9000$

b

x	50	100	150	200
y	430.6	361.1	291.7	222.2

1.1.4 Intercepts



Consider the graph of the equation

$$3x - 4y = 12$$

shown at left. The points where the graph crosses the axes are called the **intercepts** of the graph. The coordinates of these points are easy to find.

The y-coordinate of the x-intercept is zero, so we set $y = \mathbf{0}$ in the equation to get

$$3(\mathbf{0}) - 4x = 12$$
$$x = -3$$

The x-intercept is the point $(-3, 0)$. Also, the x-coordinate of the y-intercept is zero, so we set $x = \mathbf{0}$ in the equation to get

$$3y - 4(\mathbf{0}) = 12$$
$$y = 4$$

The y-intercept is $(0, 4)$.

Intercepts of a Graph.

The points where a graph crosses the axes are called the **intercepts of the graph**.

1. To find the y-intercept, set $x = 0$ and solve for y.
2. To find the x-intercept, set $y = 0$ and solve for x

The intercepts of a graph tell us something about the situation it models.

Example 1.1.14

a Find the intercepts of the graph in Checkpoint 1.1.8, p. 7, about the pollution in Silver Lake.

b What do the intercepts tell us about the problem?

Solution.

a An equation for the concentration of toxic chemicals is

$$C = 285 - 15t$$

To find the C-intercept, set t equal to zero.

$$C = 285 - 15(\mathbf{0}) = 285$$

The C-intercept is the point $(0, 285)$, or simply 285.

To find the t-intercept, set C equal to zero and solve for t.

$$\begin{aligned} \mathbf{0} &= 285 - 15t && \text{Add } \mathbf{15t} \text{ to both sides.} \\ 15t &= 285 && \text{Divide both sides by 15.} \\ t &= 19 \end{aligned}$$

The t-intercept is the point $(19, 0)$, or simply 19.

b The C-intercept represents the concentration of toxic chemicals in Silver Lake now: When $t = 0$, $C = 285$, so the concentration is currently 285 ppm.

The t-intercept represents the number of years it will take for the concentration of toxic chemicals to drop to zero: When $C = 0$, $t = 19$, so it will take 19 years for the pollution to be eliminated entirely.

□

Checkpoint 1.1.15

a Find the intercepts of the graph in Example 1.1.12, p. 9, about the advertising budget for Albert's Appliances: $150x + 50y = 3000$.

b What do the intercepts tell us about the problem?

Answer. $(20, 0)$: The manager can buy 20 television ads if she buys no radio ads. $(0, 60)$: The manager can buy 60 radio ads if she buys no television ads.

1.1.5 Intercept Method for Graphing Lines

Because we really only need two points to graph a linear equation, we might as well find the intercepts first and use them to draw the graph. The values of the intercepts will also help us choose suitable scales for the axes. It is always a good idea to find a third point as a check.

Example 1.1.16

a Find the x- and y-intercepts of the graph of $150x - 180y = 9000$.

b Use the intercepts to graph the equation. Find a third point as a check.

Solution.

a To find the x-intercept, we set $y = \mathbf{0}$.

$$\begin{aligned} 150x - 18(\mathbf{0}) &= 9000 && \text{Simpify.} \\ 150x &= 9000 && \text{Divide both sides by 150.} \\ x &= 60 \end{aligned}$$

The x-intercept is the point $(60, 0)$. To find the y-intercept, we set $x = \mathbf{0}$.

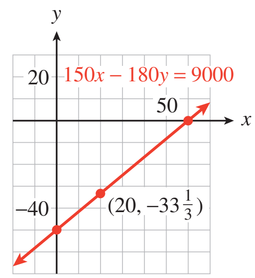
$$\begin{aligned} 150(\mathbf{0}) - 18y &= 9000 && \text{Simpify.} \\ -180y &= 9000 && \text{Divide both sides by } \mathbf{-180}. \\ y &= -50 \end{aligned}$$

The y-intercept is the point $(0, -50)$.

b We scale both axes in intervals of 10 and then plot the two intercepts, $(60, 0)$ and $(0, -50)$. We draw the line through them, as shown below. Finally, we find another point and check that it lies on this line. We choose $x = \mathbf{20}$ and solve for y.

$$\begin{aligned} 150(\mathbf{20}) - 180y &= 9000 \\ 3000 - 180y &= 9000 \\ -180y &= 6000 \\ y &= -33.\overline{3} \end{aligned}$$

We plot the point $(20, -33\frac{1}{3})$. Because this point lies on the line, we can be reasonably confident that our graph is correct.



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To Graph a Line Using the Intercept Method:.

1 Find the intercepts of the line.

  a To find the x-intercept, set $y = 0$ and solve for x.

  b To find the y-intercept, set $x = 0$ and solve for y.

2 Plot the intercepts.

3 Choose a value for x and find a third point on the line.

4 Draw a line through the points.

Checkpoint 1.1.17

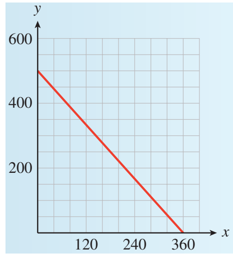
a In Checkpoint 1.1.13, p. 10, you wrote an equation about crops in Nebraska. Find the intercepts of the graph.

b Use the intercepts to help you choose appropriate scales for the axes, and then graph the equation.

c What do the intercepts tell us about the problem?

Answer. a., c. $(360, 0)$: If he plants no wheat, the farmer can plant 360 acres of corn. $(0, 500)$: If he plants no corn, the farmer can plant 500 acres of wheat.

b.



Note 1.1.18 The examples in this section model simple linear relationships between two variables. Such relationships, in which the value of one variable is determined by the value of the other, are called **functions**. We will study various kinds of functions throughout the course.

1.1.6 Section Summary

1.1.6.1 Vocabulary

Look up the definitions of new terms in the Glossary.

- Variable
- Solve an equation
- Evaluate an expression
- Linear equation
- Increasing graph
- Decreasing graph
- Intercept
- Mathematical model

1.1.6.2 CONCEPTS

1 We can describe a relationship between variables with a table of values, a graph, or an equation.

2 Linear models have equations of the following form:

$$y = (\text{starting value}) + (\text{rate of change}) \cdot x$$

3 To make a useful graph, we must choose appropriate scales for the axes.

4 **General Form for a Linear Equation.**

The graph of any equation

$$Ax + By = C$$

where A and B are not both equal to zero, is a straight line.

5 The intercepts of a graph are the points where the graph crosses the axes.

6 We can use the intercepts to graph a line.

To Graph a Line Using the Intercept Method:.

1 Find the intercepts of the line.

  a To find the x-intercept, set $y = 0$ and solve for x.

  b To find the y-intercept, set $x = 0$ and solve for y.

2 Plot the intercepts.

> 3 Choose a value for x and find a third point on the line.
>
> 4 Draw a line through the points.

7 The intercepts are also useful for interpreting a model.

1.1.6.3 STUDY QUESTIONS

1 Name three ways to represent a relationship between two variables.

2 If C is expressed in terms of H, which variable goes on the horizontal axis?

3 Explain the difference between evaluating an expression and solving an equation.

4 How many points do you need to graph a linear equation?

5 Explain how the words **intercept** and **intersect** are related; explain how they are different.

6 Delbert says that the intercepts of the line $3x + 5y = 30$ are $(10, 6)$. What is wrong with his answer?

1.1.6.4 SKILLS

Practice each skill in the Homework 1.1.7, p. 15 problems listed.

1 Make a table of values: #1–4, 7 and 8

2 Plot points and draw a graph: #1–4, 7 and 8

3 Choose appropriate scales for the axes: #5–12

4 Write a linear model of the form $y = (\text{starting value}) + (\text{rate of change}) \cdot x$: #1–8

5 Write a linear model in general form: #25–28, 33–36

6 Evaluate a linear expression, algebraically and graphically: #1–4

7 Solve a linear equation, algebraically and graphically: #1–4

8 Find the intercepts of a graph: #5 and 6, 13–24, 45–52

9 Graph a line by the intercept method: #5 and 6, 13–24

10 Interpret the meaning of the intercepts: #5 and 6, 25–28

11 Use a graphing calculator to graph a line: #37–52

12 Sketch on paper a graph obtained on a calculator: #37–44

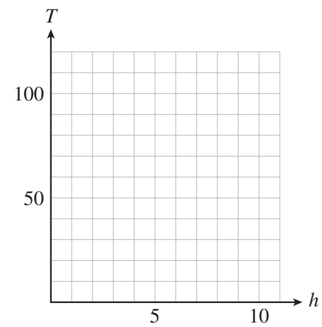
1.1.7 Linear Models (Homework 1.1)

1. The temperature in the desert at 6 a.m., just before sunrise, was 65°F. The temperature rose 5 degrees every hour until it reached its maximum value at about 5 p.m. Complete the table of values for the temperature, T, at h hours after 6 a.m.

h	0	3	6	9	10
T					

a Write an equation for the temperature, T, in terms of h.

b Graph the equation.



c How hot is it at noon? Illustrate the answer on your graph.

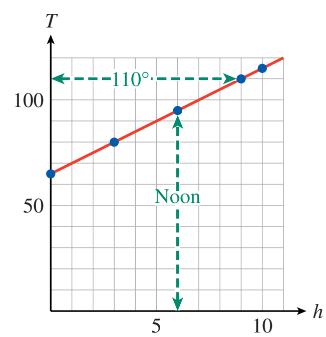
d When will the temperature be 110°F? Illustrate the answer on your graph.

Answer.

h	0	3	6	9	10
T	65	80	95	110	115

a $T = 65 + 5h$

b



c 95°

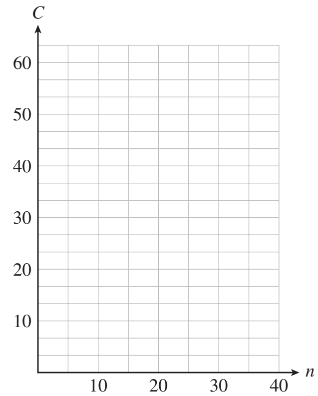
d 3 p.m.

2. The taxi out of Dulles Airport charges a traveler with one suitcase an initial fee of \$2.00, plus \$1.50 for each mile traveled. Complete the table of values showing the charge, C, for a trip of n miles.

n	0	5	10	15	20	25
C						

a Write an equation for the charge, C, in terms of the number of miles traveled, n.

b Graph the equation.



c What is the charge for a trip to Mount Vernon, 40 miles from the airport? Illustrate the answer on your graph.

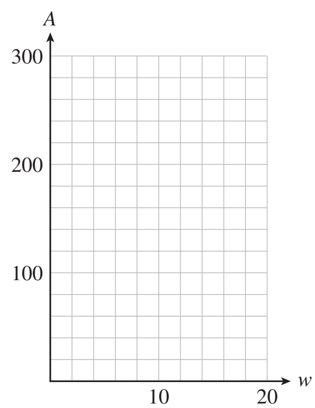
d If a ride to the National Institutes of Health (NIH) costs \$39.50, how far is it from the airport to the NIH? Illustrate the answer on your graph.

3. On October 31, Betty and Paul fill their 250-gallon oil tank for their heater. Beginning in November, they use an average of 15 gallons of oil per week. Complete the table of values for the amount of oil, A, left in the tank after w weeks.

w	0	4	8	12	16
A					

a Write an equation that expresses the amount of oil, A, in the tank in terms of the number of weeks, w, since October 31.

b Graph the equation.



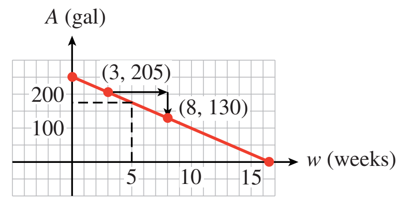
c How much did the amount of fuel oil in the tank decrease between the third week and the eighth week? Illustrate this amount on the graph.

d When will the tank contain more than 175 gallons of fuel oil? Illustrate on the graph.

Answer.

w	0	4	8	12	16
A	250	190	130	70	10

a $A = 250 - 15w$

b 

c 75 gallons

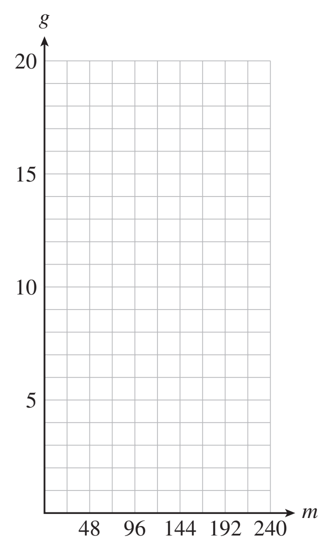
d Until the fifth week

4. Leon's camper has a 20-gallon gas tank, and he gets 12 miles to the gallon. (That is, he uses $\frac{1}{12}$ gallon per mile.) Complete the table of values for the amount of gas, g, left in Leon's tank after driving m miles.

m	0	48	96	144	192
g					

a Write an equation that expresses the amount of gas, g, in Leon's fuel tank in terms of the number of miles, m, he has driven.

b Graph the equation.



c How much gas will Leon use between 8 a.m., when his odometer reads 96 miles, and 9 a.m., when the odometer reads 144 miles? Illustrate on the graph.

d If Leon has less than 5 gallons of gas left, how many miles has he driven? Illustrate on the graph.

5. Phil and Ernie buy a used photocopier for \$800 and set up a copy service on their campus. For each hour that the copier runs, Phil and Ernie make \$40.

a Write an equation that expresses Phil and Ernie's profit (or loss), P, in terms of the number of hours, t, they run the copier.

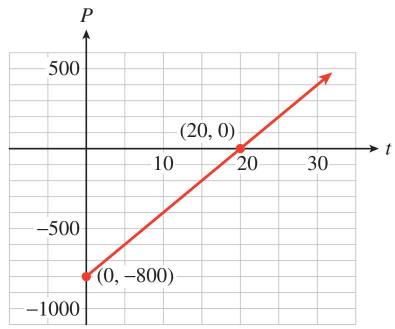
b Find the intercepts and sketch the graph. (Suggestion: Scale the horizontal axis from 0 to 40 in increments of 5, and scale the vertical axis from -1000 to 400 in increments of 100.)

c What do the intercepts tell us about the profit?

Answer.

a $P = -800 + 40t$

b $(0, -800)$, $(20, 0)$



c The P-intercept, -800, is the initial ($t = 0$) value of the profit. Phil and Ernie start out \$800 in debt. The t-intercept, 20, is the number of hours required for Phil and Ernie to break even.

6. A deep-sea diver is taking some readings at a depth of 400 feet. He begins rising at 20 feet per minute.

a Write an equation that expresses the diver's altitude, h, in terms of the number of minutes, m, elapsed. (Consider a depth of 400 feet as an altitude of -400 feet.)

b Find the intercepts and sketch the graph. (Suggestion: Scale the horizontal axis from 0 to 24 in increments of 2, and scale the vertical axis from -500 to 100 in increments of 50.)

c What do the intercepts tell us about the diver's depth?

7. There are many formulas for estimating the annual cost of driving. The Automobile Club estimates that fixed costs for a small car -- including insurance, registration, depreciation, and financing -- total about \$5000 per year. The operating costs for gasoline, oil, maintenance, tires, and so forth are about 12.5 cents per mile. (Source: Automobile Association of America)

a Write an equation for the annual driving cost, C, in terms of d, the number of miles driven.

b Complete the table of values.

Miles Driven	4000	8000	$12,000$	$16,000$	$20,000$
Cost (\$)					

c Choose scales for the axes and graph the equation.

d How much does the annual cost of driving increase when the mileage increases from 8000 to $12,000$ miles? Illustrate this amount on the graph.

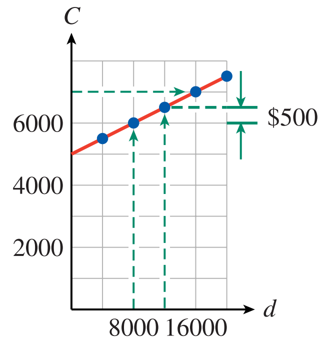
e How much mileage will cause the annual cost to exceed \$7000? Illustrate on the graph.

Answer.

a $C = 5000 + 0.125d$

b Complete the table of values.

Miles Driven	4000	8000	12,000	16,000	20,000
Cost ($)	5500	6000	6500	7000	7500

c 

d $500

e More than 16,000 miles

8. The boiling point of water changes with altitude. At sea level, water boils at 212°F, and the boiling point diminishes by approximately 0.002°F for each 1-foot increase in altitude.

a Write an equation for the boiling point, B, in terms of a, the altitude in feet.

b Complete the table of values.

Altitude (ft)	−500	0	1000	2000	3000	4000	5000
Boiling point (°F)							

c Choose scales for the axes and graph the equation.

d How much does the boiling point decrease when the altitude increases from 1000 to 3000 feet? Illustrate this amount on the graph.

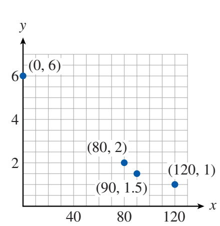
e At what altitudes is the boiling point less than 204°F? Illustrate on the graph.

For each table, choose appropriate scales for the axes and plot the given points.

9.

x	0	80	90	120
y	6	2	1.5	1

Answer.



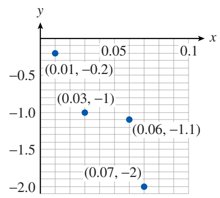
10.

x	300	500	800	1100
y	1.2	1.3	1.5	1.9

11.

x	0.01	0.03	0.06	0.07
y	-0.2	-1	-1.1	-2

Answer.



12.

x	0.003	0.005	0.008	0.011
y	6	2	1.5	1

For Problems 13-18,

(a) Find the intercepts of the graph.

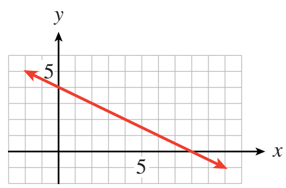
(b) Graph the equation by the intercept method.

13. $x + 2y = 8$

Answer.

a $(8, 0), (0, 4)$

b



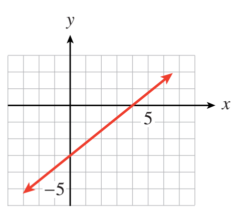
14. $2x - y = 6$

15. $3x - 4y = 12$

Answer.

a $(4, 0), (0, -3)$

b



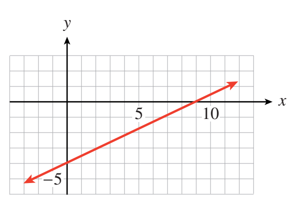
16. $2x + 6y = 6$

17. $\dfrac{x}{9} - \dfrac{y}{4} = 1$

Answer.

a $(9, 0), (0, -4)$

b



18. $\dfrac{x}{5} + \dfrac{y}{8} = 1$

For Problems 19-24,

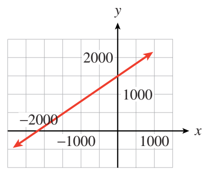
(a) Find the intercepts of the graph.

(b) Use the intercepts to choose scales for the axes, and then graph the equation by the intercept method.

19. $20x = 30y - 45,000$

Answer.

a $(-2250, 0), (0, 1500)$

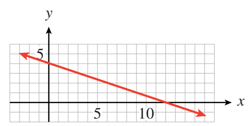
b 

20. $30x = 45y + 60,000$

21. $0.4x + 1.2y = 4.8$

Answer.

a $(12, 0), (0, 4)$

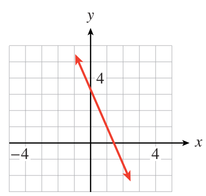
b 

22. $3.2x - 0.8y = 12.8$

23. $\dfrac{2x}{3} + \dfrac{3y}{11} = 1$

Answer.

a $\left(\dfrac{3}{2}, 0\right), \left(0, \dfrac{11}{3}\right)$

b 

24. $\dfrac{8x}{7} - \dfrac{2y}{7} = 1$

25. The owner of a gas station has \$19,200 to spend on unleaded gas this month. Regular unleaded costs him \$2.40 per gallon, and premium unleaded costs \$3.20 per gallon.

a How much do x gallons of regular cost? How much do y gallons of premium cost?

b Write an equation in general form that relates the amount of regular unleaded gasoline, x, the owner can buy and the amount of premium unleaded, y.

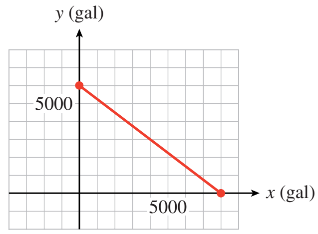
c Find the intercepts and sketch the graph.

d What do the intercepts tell us about the amount of gasoline the owner can purchase?

Answer.

a $\$2.40x$, $\$3.20y$

b $2.40x + 3.20y = 19,200$

c 

d The y-intercept, 6000 gallons, is the amount of premium that the gas station owner can buy if he buys no regular. The x-intercept, 8000 gallons, is the amount of regular he can buy if he buys no premium.

26. Five pounds of body fat is equivalent to $16,000$ calories. Carol can burn 600 calories per hour bicycling and 400 calories per hour swimming.

a How many calories will Carol burn in x hours of cycling? How many calories will she burn in y hours of swimming?

b Write an equation in general form that relates the number of hours, x, of cycling and the number of hours, y, of swimming Carol needs to perform in order to lose 5 pounds.

c Find the intercepts and sketch the graph.

d What do the intercepts tell us about Carol's exercise program?

27. Delbert must increase his daily potassium intake by 1800 mg. He decides to eat a combination of figs and bananas, which are both low in sodium. There are 9 mg potassium per gram of fig, and 4 mg potassium per gram of banana.

a How much potassium is in x grams of fig? How much potassium is in y grams of banana?

b Write an equation in general form that relates the number of grams, x, of fig and the number of grams, y, of banana Delbert needs to get 1800 mg of potassium.

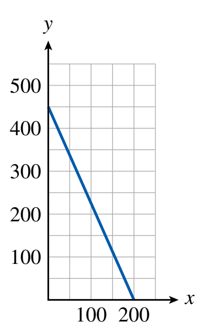
c Find the intercepts and sketch the graph.

d What do the intercepts tell us about Delbert's diet?

Answer.

a $9x$ mg, $4y$ mg

b $9x + 4y = 1800$

c 

d The x-intercept, 200 grams, tells how much fig Delbert should eat if he has no bananas, and the y-intercept, 450 grams, tells how much banana he should eat if he has no figs.

28. Leslie plans to invest some money in two CD accounts. The first account pays 3.6% interest per year, and the second account pays 2.8% interest per year. Leslie would like to earn \$500 per year on her investment.

a If Leslie invests x dollars in the first account, how much interest will she earn? How much interest will she earn if she invests y dollars in the second account?

b Write an equation in general form that relates x and y if Leslie earns \$500 interest.

c Find the intercepts and sketch the graph.

d What do the intercepts tell us about Leslie's investments?

29. Find the intercepts of the graph for each equation.

a $\dfrac{x}{3}+\dfrac{y}{5}=1$

b $2x-4y=1$

c $\dfrac{2x}{5}-\dfrac{2y}{3}=1$

d $\dfrac{x}{p}+\dfrac{y}{q}=1$

e. Why is the equation $\dfrac{x}{a}+\dfrac{y}{b}=1$ called the **intercept form** for a line?

Answer.

a $(3,0),(0,5)$

b $\left(\dfrac{1}{2},0\right),\left(0,\dfrac{-1}{4}\right)$

c $\left(\dfrac{5}{2},0\right),\left(0,\dfrac{-3}{2}\right)$

d $(p,0),(0,q)$

e The value of a is the x-intercept, and the value of b is the y-intercept.

30. Write an equation in intercept form (see Problem 29) for the line with the given intercepts. Then write the equation in general form.

a $(6,0),(0,2)$

b $(-3,0),(0,8)$

c $\left(\dfrac{3}{4},0\right),\left(0,\dfrac{-1}{4}\right)$

d $(v,0),(0,-w)$

e $\left(\dfrac{1}{H},0\right),\left(0,\dfrac{1}{T}\right)$

31.

a Find the y-intercept of the line $y=mx+b$.

b Find the x-intercept of the line $y=mx+b$.

Answer.

a $(0,b)$

b $\left(\dfrac{-b}{m},0\right)$, if $m\neq 0$

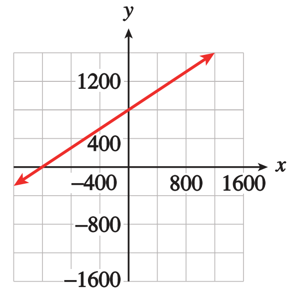
32.

a Find the y-intercept of the line $Ax+By=C$.

b Find the x-intercept of the line $Ax+By=C$.

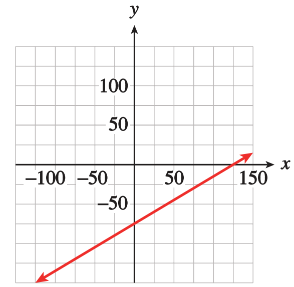
Write an equation in general form for each line.

33.

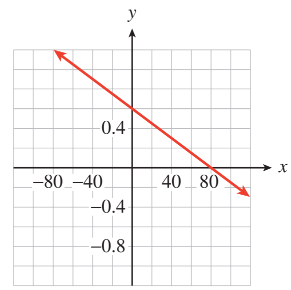


Answer. $-2x + 3y = 2400$

34.

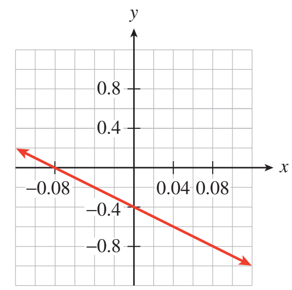


35.



Answer. $3x + 400y = 240$

36.



For Problems 37–44,

a Solve each equation for y in terms of x. (See the Algebra Skills Refresher Section A.2, p. 859 to review this skill.)

b Graph the equation on your calculator in the specified window.

c Make a pencil and paper sketch of the graph. Label the scales on your axes, and the coordinates of the intercepts.

37. $2 + y = 6$

Xmin = −10	Ymin = −10
Xmax = 10	Ymax = 10
Xscl = 1	Yscl = 1

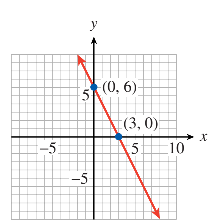
38. $8 - y + 3x = 0$

Xmin = −10	Ymin = −10
Xmax = 10	Ymax = 10
Xscl = 1	Yscl = 1

Answer.

a $y = 6 - 2x$

b

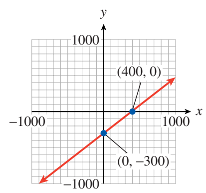


39. $3x - 4y = 1200$

$$\begin{aligned} \text{Xmin} &= -1000 & \text{Ymin} &= -1000 \\ \text{Xmax} &= 1000 & \text{Ymax} &= 1000 \\ \text{Xscl} &= 100 & \text{Yscl} &= 100 \end{aligned}$$

Answer.

a $y = \dfrac{3}{4}x - 300$

b 

40. $x + 2y = 500$

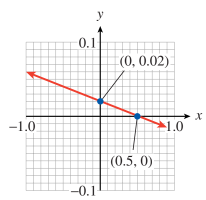
$$\begin{aligned} \text{Xmin} &= -1000 & \text{Ymin} &= -1000 \\ \text{Xmax} &= 1000 & \text{Ymax} &= 1000 \\ \text{Xscl} &= 100 & \text{Yscl} &= 100 \end{aligned}$$

41. $0.2x + 5y = 0.1$

$$\begin{aligned} \text{Xmin} &= -1 & \text{Ymin} &= -0.1 \\ \text{Xmax} &= 1 & \text{Ymax} &= 0.1 \\ \text{Xscl} &= 0.1 & \text{Yscl} &= 0.01 \end{aligned}$$

Answer.

a $y = 0.02 - 0.04x$

b 

42. $1.2x - 4.2y = 3.6$

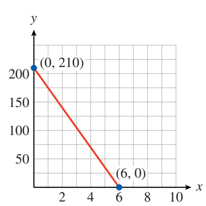
$$\begin{aligned} \text{Xmin} &= -1 & \text{Ymin} &= -1 \\ \text{Xmax} &= 4 & \text{Ymax} &= 1 \\ \text{Xscl} &= 0.2 & \text{Yscl} &= 0.1 \end{aligned}$$

43. $70x + 3y = y + 420$

$$\begin{aligned} \text{Xmin} &= 0 & \text{Ymin} &= 0 \\ \text{Xmax} &= 10 & \text{Ymax} &= 250 \\ \text{Xscl} &= 1 & \text{Yscl} &= 25 \end{aligned}$$

Answer.

a $y = 210 - 35x$

b 

44. $40y - 5x = 780 - 20y$

$$\begin{aligned} \text{Xmin} &= -200 & \text{Ymin} &= 0 \\ \text{Xmax} &= 0 & \text{Ymax} &= 20 \\ \text{Xscl} &= 20 & \text{Yscl} &= 2 \end{aligned}$$

For Problems 45–52,

a Find the x- and y-intercepts.

b Solve the equation for y.

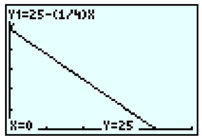
c Choose a graphing window in which both intercepts are visible, and graph the equation on your calculator.

45. $x + 4y = 100$

Answer.

a $(100, 0), (0, 25)$

b $y = 25 - \frac{1}{4}x$

c 

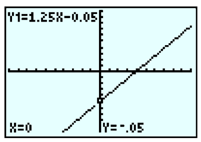
46. $2x - 3y = -72$

47. $25x - 20y = 1$

Answer.

a $(0.04, 0), (0, -0.05)$

b $y = 1.25x - 0.05$

c 

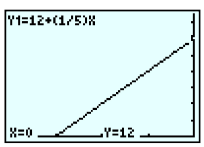
48. $4x + 75y = 60,000$

49. $\frac{y}{12} - \frac{x}{60} = 1$

Answer.

a $(-60, 0), (0, 12)$

b $y = 12 + \frac{1}{5}x$

c 

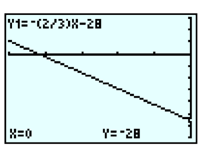
50. $\frac{x}{80} + \frac{y}{400} = 1$

51. $-2x = 3y + 84$

Answer.

a $(-42, 0), (0, -28)$

b $y = \frac{-2}{3}x - 28$

c 

52. $7x = 91 - 13y$

1.2 Functions

1.2.1 Definition of Function

We often want to predict values of one variable from the values of a related variable. For example, when a physician prescribes a drug in a certain dosage, she needs to know how long the dose will remain in the bloodstream. A sales manager needs to know how the price of his product will affect its sales. A **function** is a special type of relationship between variables that allows us to make such predictions.

Suppose it costs \$800 for flying lessons, plus \$30 per hour to rent a plane. If we let C represent the total cost for t hours of flying lessons, then

$$C = 800 + 30t \quad (t \geq 0)$$

Thus, for example

$$\begin{array}{lll} \text{when} & t = \mathbf{0}, & C = 800 + 30(\mathbf{0}) = 800 \\ \text{when} & t = \mathbf{4}, & C = 800 + 30(\mathbf{4}) = 920 \\ \text{when} & t = \mathbf{10}, & C = 800 + 30(\mathbf{10}) = 1100 \end{array}$$

The variable t is called the **input** or **independent** variable, and C is the **output** or **dependent** variable, because its values are determined by the value of t. We can display the relationship between two variables by a table or by ordered pairs. The input variable is the first component of the ordered pair, and the output variable is the second component.

t	C	(t, C)
0	800	$(0, 800)$
4	920	$(4, 920)$
10	1100	$(10, 1100)$

For this relationship, we can find the value of C for any given value of t. All we have to do is substitute the value of t into the equation and solve for C. Note that there can be only one value of C for each value of t.

Definition of Function.

A **function** is a relationship between two variables for which a unique value of the **output** variable can be determined from a value of the **input** variable.

Note 1.2.1 What distinguishes functions from other variable relationships? The definition of a function calls for a *unique value* -- that is, *exactly one value* of the output variable corresponding to each value of the input variable. This property makes functions useful in applications because they can often be used to make predictions.

Example 1.2.2

a The distance, d, traveled by a car in 2 hours is a function of its speed, r. If we know the speed of the car, we can determine the distance it travels by the formula $d = r \cdot 2$.

b The cost of a fill-up with unleaded gasoline is a function of the number of gallons purchased. The gas pump represents the function by displaying the corresponding values of the input variable (number of gallons) and the output variable (cost).

c Score on the Scholastic Aptitude Test (SAT) is not a function of score on an IQ test, because two people with the same score on an IQ test may score differently on the SAT; that is, a person's score on the SAT is not uniquely determined by his or her score on an IQ test.

□

Checkpoint 1.2.3

a As part of a project to improve the success rate of freshmen, the counseling department studied the grades earned by a group of students in English and algebra. Do you think that a student's grade in algebra is a function of his or her grade in English? Explain why or why not.

b Phatburger features a soda bar, where you can serve your own soft drinks in any size. Do you think that the number of calories in a serving of Zap Kola is a function of the number of fluid ounces? Explain why or why not.

Answer.

a No, students with the same grade in English can have different grades in algebra.

b Yes, the number of calories is proportional to the number of fluid ounces.

A function can be described in several different ways. In the following examples, we consider functions defined by tables, by graphs, and by equations.

1.2.2 Functions Defined by Tables

When we use a table to describe a function, the first variable in the table (the left column of a vertical table or the top row of a horizontal table) is the input variable, and the second variable is the output. We say that the output variable *is a function of* the input.

Example 1.2.4

a The table below shows data on sales compiled over several years by the accounting office for Eau Claire Auto Parts, a division of Major Motors. In this example, the year is the input variable, and total sales is the output. We say that total sales, S, *is a function of* t.

Year (t)	Total sales (S)
2000	\$612,000
2001	\$663,000
2002	\$692,000
2003	\$749,000
2004	\$904,000

b The table below gives the cost of sending printed material by first-class mail in 2016.

Weight in ounces (w)	Postage (P)
$0 < w \le 1$	\$0.47
$1 < w \le 2$	\$0.68
$2 < w \le 3$	\$0.89
$3 < w \le 4$	\$1.10
$4 < w \le 5$	\$1.31
$5 < w \le 6$	\$1.52
$6 < w \le 7$	\$1.73

If we know the weight of the article being shipped, we can determine the required postage from the table. For instance, a catalog weighing 4.5 ounces would require \$1.31 in postage. In this example, w is the input variable and p is the output variable. We say that p *is a function of* w.

c The table below records the age and cholesterol count for 20 patients tested in a hospital survey.

Age	Cholesterol count		Age	Cholesterol count
53	217		**51**	**209**
48	232		53	241
55	198		49	186
56	238		**51**	**216**
51	**227**		57	208
52	264		52	248
53	195		50	214
47	203		56	271
48	212		53	193
50	234		48	172

According to these data, cholesterol count is *not* a function of age, because several patients who are the same age have different cholesterol levels. For example, three different patients are 51 years old but have cholesterol counts of 227, 209, and 216, respectively. Thus, we cannot determine a *unique* value of the output variable (cholesterol count) from the value of the input variable (age). Other factors besides age must influence a person's cholesterol count.

□

Checkpoint 1.2.5 Decide whether each table describes y as a function of x. Explain your choice.

a

x	3.5	2.0	2.5	3.5	2.5	4.0	2.5	3.0
y	2.5	3.0	2.5	4.0	3.5	4.0	2.0	2.5

b

x	-3	-2	-1	0	1	2	3
y	17	3	0	-1	0	3	17

Answer.

a No, for example, $x = 3.5$ corresponds both to $y = 2.5$ and also to $y = 4.0$.

b Yes, each value of x has exactly one value of y associated with it.

1.2.3 Functions Defined by Graphs

A graph may also be used to define one variable as a function of another. The input variable is displayed on the horizontal axis, and the output variable on the vertical axis.

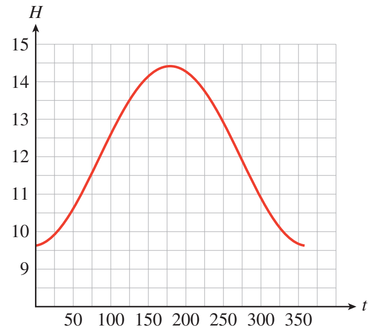
Example 1.2.6 The graph shows the number of hours, H, that the sun is above the horizon in Peoria, Illinois, on day t, where January 1 corresponds to $t = 0$.

a Which variable is the input, and which is the output?

b Approximately how many hours of sunlight are there in Peoria on day 150?

c On which days are there 12 hours of sunlight?

d What are the maximum and minimum values of H, and when do these values occur?



Solution.

a The input variable, t, appears on the horizontal axis. The number of daylight hours, H, is a function of the date. The output variable appears on the vertical axis.

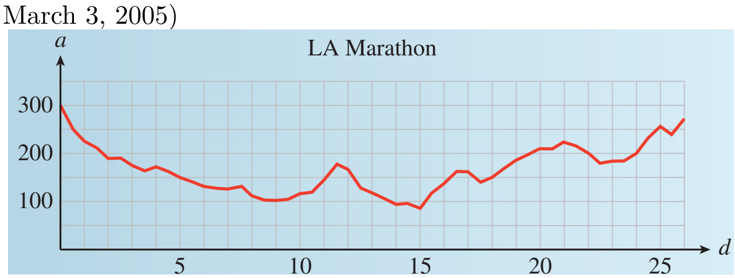
b The point on the curve where $t = 150$ has $H \approx 14.1$, so Peoria gets about 14.1 hours of daylight when $t = 150$, which is at the end of May.

c $H = 12$ at the two points where $t \approx 85$ (in late March) and $t \approx 270$ (late September).

d The maximum value of 14.4 hours occurs on the longest day of the year, when $t \approx 170$, about three weeks into June. The minimum of 9.6 hours occurs on the shortest day, when $t \approx 355$, about three weeks into December.

□

Checkpoint 1.2.7 The graph shows the elevation in feet, a, of the Los Angeles Marathon course at a distance d miles into the race. (Source: *Los Angeles Times*, March 3, 2005)



a Which variable is the input, and which is the output?

b What is the elevation at mile 20?

c At what distances is the elevation 150 feet?

d What are the maximum and minimum values of a, and when do these values occur?

e The runners pass by the Los Angeles Coliseum at about 4.2 miles into the race. What is the elevation there?

Answer.

a The input variable is d, and the output variable is a.

b Approximately 210 feet

c Approximately where $d \approx 5$, $d \approx 11$, $d \approx 12$, $d \approx 16$, $d \approx 17.5$, and $d \approx 18$

d The maximum value of 300 feet occurs at the start, when $d = 0$. The minimum of 85 feet occurs when $d \approx 15$.

e Approximately 165 feet

1.2.4 Functions Defined by Equations

Example 1.2.8, p. 32 illustrates a function defined by an equation.

Example 1.2.8 As of 2016, One World Trade Center in New York City is the nation's tallest building, at 1776 feet. If an algebra book is dropped from the top of One World Trade Center, its height above the ground after t seconds is given by the equation

$$h = 1776 - 16t^2$$

Thus, after **1** second the book's height is

$$h = 1776 - 16(\mathbf{1})^2 = 1760 \text{ feet}$$

After **2** seconds its height is

$$h = 1776 - 16(\mathbf{2})^2 = 1712 \text{ feet}$$

For this function, t is the input variable and h is the output variable. For any value of t, a unique value of h can be determined from the equation for h. We say that *h is a function of t*. □

Checkpoint 1.2.9 Write an equation that gives the volume, V, of a sphere as a function of its radius, r.

Answer. $V = \dfrac{4}{3}\pi r^3$

1.2.5 Function Notation

There is a convenient notation for discussing functions. First, we choose a letter, such as f, g, or h (or F, G, or H), to name a particular function. (We can use any letter, but these are the most common choices.)

For instance, in Example 1.2.8, p. 32, the height, h, of a falling algebra book is a function of the elapsed time, t. We might call this function f. In other words, f is the name of the relationship between the variables h and t. We write

$$h = f(t)$$

which means " h is a function of t, and f is the name of the function."

Caution 1.2.10 The new symbol $f(t)$, read " f of t," is another name for the height, h. The parentheses in the symbol $f(t)$ do not indicate multiplication. (It would not make sense to multiply the name of a function by a variable.) Think of the symbol $f(t)$ as a single variable that represents the output value of the function.

With this new notation we may write

$$h = f(t) = 1776 - 16t^2$$

or just

$$f(t) = 1776 - 16t^2$$

instead of

$$h = 1776 - 16t^2$$

to describe the function.

Note 1.2.11 Perhaps it seems complicated to introduce a new symbol for h, but the notation $f(t)$ is very useful for showing the correspondence between specific values of the variables h and t.

Example 1.2.12 In Example 1.2.8, p. 32, the height of an algebra book dropped from the top of One World Trade Center is given by the equation

$$h = 1776 - 16t^2$$

We see that

when $t = 1$	$h = 1760$
when $t = 2$	$h = 1712$

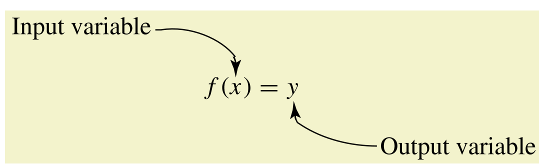
Using function notation, these relationships can be expressed more concisely as

$$f(1) = 1760 \quad \text{and} \quad f(2) = 1712$$

which we read as " f of 1 equals 1760" and " f of 2 equals 1712." The values for the input variable, t, appear *inside* the parentheses, and the values for the output variable, h, appear on the other side of the equation. ☐

Remember that when we write $y = f(x)$, the symbol $f(x)$ is just another name for the output variable.

Function Notation.



Checkpoint 1.2.13 Let F be the name of the function defined by the graph in Example 1.2.6, p. 30, the number of hours of daylight in Peoria.

a Use function notation to state that H is a function of t.

b What does the statement $F(15) = 9.7$ mean in the context of the problem?

Answer.

a $H = F(t)$

b The sun is above the horizon in Peoria for 9.7 hours on January 16.

1.2.6 Evaluating a Function

Finding the value of the output variable that corresponds to a particular value of the input variable is called **evaluating the function**.

Example 1.2.14 Let g be the name of the postage function defined by the table in Example 1.2.2, p. 28 b. Find $g(1)$, $g(3)$, and $g(6.75)$.

Solution. According to the table,

when $w = 1$, $p = 0.47$ so $g(1) = 0.47$
when $w = 3$, $p = 0.89$ so $g(3) = 0.89$
when $w = 6.75$, $p = 1.73$ so $g(6.75) = 1.73$

Thus, a letter weighing 1 ounce costs \$0.47 to mail, a letter weighing 3 ounces costs \$0.89, and a letter weighing 6.75 ounces costs \$1.73. □

Checkpoint 1.2.15 When you exercise, your heart rate should increase until it reaches your target heart rate. The table shows target heart rate, $r = f(a)$, as a function of age.

a	20	25	30	35	40	45	50	55	60	65	70
r	150	146	142	139	135	131	127	124	120	116	112

a Find $f(25)$ and $f(50)$.

b Find a value of a for which $f(a) = 135$.

Answer.

a $f(25) = 146$, $f(50) = 127$

b $a = 40$

If a function is described by an equation, we simply substitute the given input value into the equation to find the corresponding output, or function value.

Example 1.2.16 The function H is defined by $H = f(s) = \dfrac{\sqrt{s+3}}{s}$. Evaluate the function at the following values.

a $s = 6$

b $s = -1$

Solution.

a $f(\mathbf{6}) = \dfrac{\sqrt{\mathbf{6}+3}}{\mathbf{6}} = \dfrac{\sqrt{9}}{6} = \dfrac{3}{6} = \dfrac{1}{2}$. Thus, $f(6) = \dfrac{1}{2}$.

b $f(\mathbf{-1}) = \dfrac{\sqrt{\mathbf{-1}+3}}{\mathbf{-1}} = \dfrac{\sqrt{2}}{-1} = -\sqrt{2}$. Thus, $f(-1) = -\sqrt{2}$.

□

Checkpoint 1.2.17 Complete the table displaying ordered pairs for the function $f(x) = 5 - x^3$. Evaluate the function to find the corresponding $f(x)$-value for each value of x.

x	$f(x)$
-2	
0	
1	
3	

$f(\mathbf{-2}) = 5 - (\mathbf{-2})^3 =$
$f(\mathbf{0}) = 5 - \mathbf{0}^3 =$
$f(\mathbf{1}) = 5 - \mathbf{1}^3 =$
$f(\mathbf{3}) = 5 - \mathbf{3}^3 =$

Answer.

x	$f(x)$
-2	13
0	5
1	4
3	-22

To simplify the notation, we sometimes use the same letter for the output variable and for the name of the function. In the next example, C is used in this way.

Example 1.2.18 TrailGear decides to market a line of backpacks. The cost, C, of manufacturing backpacks is a function of the number, x, of backpacks produced, given by the equation

$$C(x) = 3000 + 20x$$

where $C(x)$ is measured in dollars. Find the cost of producing 500 backpacks.

Solution. To find the value of C that corresponds to $x = \mathbf{500}$, evaluate $C(500)$.

$$C(\mathbf{500}) = 3000 + 20(\mathbf{500}) = 13,000$$

The cost of producing 500 backpacks is $13,000. □

Checkpoint 1.2.19 The volume of a sphere of radius r centimeters is given by

$$V = V(r) = \frac{4}{3}\pi r^3$$

Evaluate $V(10)$ and explain what it means.

Answer. $V(10) = 4000\pi/3 \approx 4188.79$ cm^3 is the volume of a sphere whose radius is 10 cm.

1.2.7 Operations with Function Notation

Sometimes we need to evaluate a function at an algebraic expression rather than at aspecific number.

Example 1.2.20 TrailGear manufactures backpacks at a cost of

$$C(x) = 3000 + 20x$$

for x backpacks. The company finds that the monthly demand for backpacks increases by 50% during the summer. The backpacks are produced at several small co-ops in different states.

a If each co-op usually produces b backpacks per month, how many should it produce during the summer months?

b What costs for producing backpacks should the company expect during the summer?

Solution.

a An increase of 50% means an additional 50% of the current production level, b. Therefore, a co-op that produced b backpacks per month during the winter should increase production to $b + 0.5b$, or $1.5b$ backpacks per month in the summer.

b The cost of producing $1.5b$ backpacks will be

$$C(\mathbf{1.5b}) = 3000 + 20(\mathbf{1.5b}) = 3000 + 30b$$

□

Checkpoint 1.2.21 A spherical balloon has a radius of 10 centimeters.

a If we increase the radius by h centimeters, what will the new volume be?

b If $h = 2$, how much did the volume increase?

Answer.

a $V(10 + h) = \dfrac{4}{3}\pi(10 + h)^3$ cm^3

b By 3049.44 cm^3

Example 1.2.22 Evaluate the function $f(x) = 4x^2 - x + 5$ for the following expressions.

a $x = 2h$

b $x = a + 3$

Solution.

a

$$\begin{aligned} f(\mathbf{2h}) &= 4(\mathbf{2h})^2 - (\mathbf{2h}) + 5 \\ &= 4(4h^2) - 2h + 5 \\ &= 16h^2 - 2h + 5 \end{aligned}$$

b

$$\begin{aligned} f(\boldsymbol{a+3}) &= 4(\boldsymbol{a+3})^2 - (\boldsymbol{a+3}) + 5 \\ &= 4(a^2 + 6a + 9) - a - 3 + 5 \\ &= 4a^2 + 24a + 36 - a + 2 \\ &= 4a^2 + 23a + 38 \end{aligned}$$

□

Caution 1.2.23 In Example 1.2.22, p. 36, notice that

$$f(2h) \neq 2f(h)$$

and

$$f(a + 3) \neq f(a) + f(3)$$

To compute $f(a) + f(3)$, we must first compute $f(a)$ and $f(3)$, then add them:

$$\begin{aligned} f(a) + f(3) &= (4a^2 - a + 5) + (4 \cdot 3^2 - 3 + 5) \\ &= 4a^2 - a + 43 \end{aligned}$$

In general, it is not true that $f(a + b) = f(a) + f(b)$. Remember that the parentheses in the expression $f(x)$ do not indicate multiplication, so the distributive law does not apply to the expression $f(a + b)$.

Checkpoint 1.2.24 Let $f(x) = x^3 - 1$ and evaluate each expression.

a $f(2) + f(3)$

b $f(2 + 3)$

c $2f(x) + 3$

Answer.

a 33

b 124

c $2x^3 + 1$

1.2.8 Compostion and other Operations with Functions

(Note, some information found in this section has come from an OpenStax textbook titled "College Algebra". Access for free at this link.)

In Example 1.2.22, p. 36 you were asked to evaluate the function $f(x) = 4x^2 - x + 5$ at $x = 2h$. We can ask this question using composition notation. Let $f(x) = 4x^2 - x + 5$ and $g(h) = 2h$, and find $(f \circ g)(h)$, or, $f(g(h))$. The idea with composition is that you do not only evaluate a function at a point, but at another function. The end result is a new function.

The process of combining functions so that the output of one function becomes the input of another is known as a *composition of functions*. The resulting function is known as a composite function. We represent this combination by the following notation:

$$(f \circ g)(x) = f(g(x)).$$

We read the left-hand side as " f composed with g at x," and the right-hand side as " f of g of x". The two sides of the equation have the same mathematical meaning and are equal. The open circle symbol $\circ$ is called the composition operator. We use this operator mainly when we wish to emphasize the relationship between the functions themselves without referring to any particular input value. Composition is a binary operation that takes two functions and forms a new function, much as addition or multiplication takes two numbers and gives a new number. However, it is important not to confuse function composition with multiplication, because in most cases $f(g(x)) \neq f(x)g(x)$.

It is also important to understand the order of operations in evaluating a composite function. We follow the usual convention with parentheses by starting with the innermost parentheses first, and then working to the outside. In the equation above, the function g takes the input, x, first and yields an output, $g(x)$. Then the function f takes $g(x)$ as an input and yields an output, $f(g(x))$.

Just like subtraction, order matters. Typically, $f(g(x)) \neq g(f(x))$, so it is important to understand which is the "inner" function and which is the "outer" function.

Example 1.2.25 Let $f(x) = x^2 - 3x$ and $g(x) = x + 2$.

a Find $f(g(x))$.

b Find $g(f(x))$.

Solution.

a
$$\begin{aligned} f(g(x)) &= f(x+2) \\ &= (x+2)^2 - 3(x+2) \\ &= (x^2+4x+4) - (3x+6) \\ &= x^2 + 4x + 4 - 3x - 6 \\ &= x^2 + x - 2 \end{aligned}$$

b
$$\begin{aligned} g(f(x)) &= g(x^2 - 3x) \\ &= (x^2 - 3x) + 2 \\ &= x^2 - 3x + 2 \end{aligned}$$

□

Example 1.2.26 Let $f(x) = -2x^2 + 3x$ and $g(x) = 5x + 9$.

a Find $f(g(x))$.

b Find $g(f(x))$.

c Find $g(g(x))$.

Solution.

a $f(g(x)) = -50x^2 - 165x - 135$.

b $g(f(x)) = -10x^2 + 15x + 9$.

c $g(g(x)) = 25x + 54$.

□

The same process holds true if you want to evalute a composition of functions at a given point.

Example 1.2.27 Let $f(x) = \sqrt{4+x}$ and $g(x) = x + 5$.

a Find $f(g(0))$.

b Find $g(f(0))$.

c Find $f(g(5))$.

d Find $g(f(5))$.

Solution.

a $f(g(0))$. Using the order of operations, we want to start inside and work our way out. $g(0) = 0 + 5 = 5$. Now let us look at the original problem and replace $g(0)$ with 5. So we can evaluate $f(g(0)) = f(5) = \sqrt{4+5} = \sqrt{9} = 3$.

b $g(f(0))$. First, evaluate $f(0) = \sqrt{4+0} = \sqrt{4} = 2$. Finally, $g(f(0)) = g(2) = 2 + 5 = 7$.

c Another way that you can write the solution, all at once, is as follows. $f(g(5)) = f(5+5) = f(10) = \sqrt{4+10} = \sqrt{14}$. (Notice, we always want exact solutions unless stated otherwise. Often in applications problems it is sufficient to use rounded solutions.)

d $g(f(5)) = g(\sqrt{4+5}) = g(\sqrt{9}) = g(3) = 3 + 5 = 8$.

□

Example 1.2.28 Let $h(x) = \frac{2}{3}x + 6$ and $j(x) = x^2 + 2x$.

a Find $h(j(2))$.

b Find $j(h(2))$.

c Find $h(j(3))$.

d Find $j(j(1))$.

Solution.

a $h(j(2)) = \frac{34}{3}$.

b $j(h(2)) = \frac{616}{9}$.

c $h(j(3)) = 16$.

d $j(j(1)) = 15$.

□

Composition of functions has very useful applications, and more than likely you have done it yourself. This will appear in "day-to-day" life when you have to evaulate a function and use the output you just found to evaulate a different function.

Example 1.2.29 Suppose there is a function, $A(d)$, which gives the pain level on a scale of 0 to 10 experienced by a patient with d milligrams of a pain-reducing drug in her system. There is another function $m(t)$ which gives the number of milligrams of the drug in the patient's system after t minutes.

A nurse is asked how long it will be until her patient will be at pain level of 3. Which of the 4 options below indicate how she would evaluate this?

a $A(m(t)) = 3$

b $A(m(3))$

c $m(A(d)) = 3$

d $m(A(3))$

Solution. Let us first look at what order you evaluate the functions: $A(m(t))$ or $m(A(d))$. A nice way to determine order is to examine input and output variables. $A(d)$ has the input of milligrams and output of pain scale. $m(t)$ has the input of time and output of milligrams. This means we have to match up the output of $m(t)$ with the input of $A(d)$. Right away, this means we want option (a) or (b), since the order of composition is right. Notice, if we are asked "how long", this means we want to solve for the variable t. The pain level is at 3, which means $A(d) = 3$. The only way both of these is satisfied is in option (a). □

Composition of functions is not the only way that we can combine two functions: we can also add, subtract, multiply and divide. Note, mathematicians like to simplify as much as possible, and have created shorthand notation for various concepts. For instance:

$$\begin{aligned}(f+g)(x) =& f(x)+g(x),\\ (f-g)(x) =& f(x)-g(x),\\ (fg)(x) =& f(x)g(x),\\ \left(\frac{f}{g}\right)(x) =& \frac{f(x)}{g(x)}, \qquad \text{provided } g(x)\neq 0.\end{aligned}$$

Note that multiplication and composition are different function operations. Multiplication is the product of two functions. We notate multiplication of functions in the following ways: $(fg)(x) = f(x)g(x) = (f \cdot g)(x) = f(x) \cdot g(x)$.

Example 1.2.30 Let $f(x) = -x^2 + 8x - 15$ and $g(x) = x^2 - 2x - 15$.

a Find $(f + g)(x)$.

b Find $(f - g)(x)$.

c Find $(fg)(x)$.

d Find $\left(\frac{f}{g}\right)(x)$.

Solution.

a
$$\begin{aligned}(f+g)(x) =& f(x)+g(x)\\ =& (-x^2+8x-15)+(x^2-2x-15)\\ =& -x^2+8x-15+x^2-2x-15\\ =& 6x-30\end{aligned}$$

b When subtracting functions be sure to subtract \emph{all} of the second function. Putting parentheses around the second function will help you remember to do this.
$$\begin{aligned}(f-g)(x) =& f(x)-g(x)\\ =&(-x^2+8x-15)-(x^2-2x-15)\\ =&-x^2+8x-15-x^2+2x+15\\ =&-2x^2+10x\end{aligned}$$

c
$$\begin{aligned}(fg)(x) =& f(x)g(x)\\ =&(-x^2+8x-15)(x^2-2x-15)\\ =&-x^4+2x^3+15x^2+8x^3-16x^2-120x-15x^2+30x+225\\ =&-x^4+10x^3-16x^2-90x+225\end{aligned}$$

d
$$\begin{aligned}\left(\frac{f}{g}\right)(x) =& \frac{-x^2+8x-15}{x^2-2x-15}\\ =&\frac{(-x+3)(x-5)}{(x-5)(x+3)}\\ =&\frac{-x+3}{x+3}\end{aligned}$$

□

Example 1.2.31 Let $f(x) = x^2 + 3x + 2$ and $g(x) = x^2 + 6x + 8$.

a Find $(f + f)(x)$.

b Find $(g - f)(x)$.

c Find $(fg)(x)$.

d Find $\left(\frac{f}{g}\right)(x)$.

Solution.

a $(f + f)(x) = 2x^2 + 6x + 4$

b $(g - f)(x) = 3x + 6$

c $(fg)(x) = x^4 + 9x^3 + 28x^2 + 36x + 16$

d $\left(\frac{f}{g}\right)(x) = \frac{x+1}{x+4}$

□

1.2.9 Section Summary

1.2.9.1 Vocabulary

Look up the definitions of new terms in the Glossary.

- Function
- Independent variable
- Dependent variable
- Input variable
- Function value
- Output variable

1.2.9.2 CONCEPTS

1 A function is a rule that assigns to each value of the input variable a unique value of the output variable.

2 Functions may be defined by words, tables, graphs, or equations.

3 Function notation: $y = f(x)$, where x is the input and y is the output.

1.2.9.3 STUDY QUESTIONS

1 What property makes a relation between two variables a function?

2 Name three ways to define a function.

3 Give an example of a function in which two distinct values of the input variable correspond to the same value of the output variable.

4 Use function notation to write the statement "G defines w as a function of p."

5 Give an example of a function for which $f(2+3) \neq f(2) + f(3)$.

1.2.9.4 SKILLS

Practice each skill in the Homework 1.2.10, p. 41 problems listed.

1 Decide whether a relationship between two variables is a function: #1–26

2 Evaluate a function defined by a table, a graph, or an equation: #27–54

3 Choose appropriate scales for the axes: #5–12

4 Interpret function notation: #31–34, 49–54

5 Simplify expressions involving function notation: #59–76

1.2.10 Functions (Homework 1.2)

For which of Problems 1-6 is the second quantity a function of the first? Explain your answers.

1. Price of an item; sales tax on the item at 4%

Answer. Function; the tax is determined by the price of the item.

2. Time traveled at constant speed; distance traveled

3. Number of years of education; annual income

Answer. Not a function; incomes may differ for same number of years of education.

4. Distance flown in an airplane; price of the ticket

5. Volume of a container of water; the weight of the water

Answer. Function; weight is determined by volume.

6. Amount of a paycheck; amount of Social Security tax withheld

Each of the objects in Problems 7-14 establishes a correspondence between two variables. Suggest appropriate input and output variables and decide whether the relationship is a function.

7. An itemized grocery receipt

Answer. Input: items purchased; output: price of item. Yes, a function because each item has only one price.

8. An inventory list

9. An index

Answer. Input: topics; output: page or pages on which topic occurs. No, not a function because the same topic may appear in more than one page.

10. A will

11. An instructor's grade book

Answer. Input: students' names; output: students' scores on quizzes, tests, etc. No, not a function because the same student can have different grades on different tests.

12. An address book

13. A bathroom scale

Answer. Input: person stepping on scales; output: person's weight. Yes, a function because a person cannot have two different weights at the same time.

14. A radio dial

Which of the tables in Problems 15-26 define the second variable as a function of the first variable? Explain why or why not.

15.

x	t
-1	2
0	9
1	-2
0	-3
-1	5

Answer.
No

16.

y	w
0	8
1	12
3	7
5	-3
7	4

17.

x	y
-3	8
-2	3
-1	0
0	-1
1	0
2	3
3	8

Answer.
Yes

18.

s	t
2	5
4	10
6	15
8	20
6	25
4	30
2	35

19.

r	-4	-2	0	2	4
v	6	6	3	6	8

Answer. Yes

20.

p	-5	-4	-3	-2	-1
d	-5	-4	-3	-2	-1

21.

Pressure (p)	Volume (v)
15	100.0
20	75.0
25	60.0
30	50.0
35	42.8
40	37.5
45	33.3
50	30.0

Answer. Yes

22.

Frequency (f)	Wavelength (w)
5	60.0
10	30.0
20	15.0
30	10.0
40	7.5
50	6.0
60	5.0
70	4.3

23.

Temperature (T)		Humidity (h)
Jan. 1	34°F	42%
Jan. 2	36°F	44%
Jan. 3	35°F	47%
Jan. 4	29°F	50%
Jan. 5	31°F	52%
Jan. 6	35°F	51%
Jan. 7	34°F	49%

Answer. No

24.

Inflation rate (I)		Unemployment rate (U)
1972	5.6%	5.1%
1973	6.2%	4.5%
1974	10.1%	4.9%
1975	9.2%	7.4%
1976	5.8%	6.7%
1977	5.6%	6.8%
1978	6.7%	7.4%

25.

Adjusted gross income (I)	Tax bracket (T)
\$0 – 2479	0%
\$2480 – 3669	4.5%
\$3670 – 4749	12%
\$4750 – 7009	14%
\$7010 – 9169	15%
\$9170 – 11,649	16%
\$11,650 – 13,919	18%

Answer. Yes

26.

Cost of merchandise (M)	Shipping charge (C)
\$0.01 – 10.00	\$2.50
10.01 – 20.00	3.75
20.01 – 35.00	4.85
35.01 – 50.00	5.95
50.01 – 75.00	6.95
75.01 – 100.00	7.95
Over 100.00	8.95

27. The function described in Problem 21 is called g, so that $v = g(p)$. Find the following:

a $g(25)$

b $g(40)$

c x so that $g(x) = 50$

Answer.

a 60

b 37.5

c 30

28. The function described in Problem 22 is called h, so that $w = h(f)$. Find the following:

a $h(20)$

b $h(60)$

c x so that $h(x) = 10$

29. The function described in Problem 25 is called T, so that $T = T(I)$. Find the following:

a $T(8750)$

b $T(6249)$

c x so that $T(x) = 15\%$

Answer.

a 15%

b 14%

c \$7010–\$9169

30. The function described in Problem 26 is called C, so that $C = C(M)$. Find the following:

a $C(11.50)$

b $C(47.24)$

c x so that $C(x) = 7.95$

31. Data indicate that U.S. women are delaying having children longer than their counterparts 50 years ago. The table shows $f(t)$ the percent of 20–24-year-old women in year t who had not yet had children. (Source: U.S. Dept of Health and Human Services)

Year (t)	1960	1965	1970	1975	1980	1985	1990	1995	2000
Percent of women	47.5	51.4	47.0	62.5	66.2	67.7	68.3	65.5	66.0

a Evaluate $f(1985)$ and explain what it means.

b Estimate a solution to the equation $f(t) = 68$ and explain what it means.

c In 1997, 64.9% of 20–24-year-old women had not yet had children. Write an equation with function notation that states this fact.

Answer.

a 67.7: In 1985, 67.7% of 20–24 year old women had not yet had children.

b 1987: Approximately 68% of 20–24 year old women had not yet had children in 1987.

c $f(1997) = 64.9$

32. The table shows $f(t)$, the death rate (per 100,000 people) from HIV among 15–24-year-olds, and $g(t)$, the death rate from HIV among 25–34-year-olds, for selected years from 1997 to 2002. (Source: U.S. Dept of Health and Human Services)

Year	1987	1988	1989	1990	1992	1994	1996	1998	2000	2002
15–24-year-olds	1.3	1.4	1.6	1.5	1.6	1.8	1.1	0.6	0.5	0.4
25–34-year-olds	11.7	14.0	17.9	19.7	24.2	28.6	19.2	8.1	6.1	4.6

a Evaluate $f(1995)$ and explain what it means.

b Find a solution to the equation $g(t) = 28.6$ and explain what it means.

c In 1988, the death rate from HIV for 25–34-year-olds was 10 times the corresponding rate for 15–24-year-olds. Write an equation with function notation that states this fact.

33. When you exercise, your heart rate should increase until it reaches your target heart rate. The table shows target heart rate, $r = f(a)$, as a function of age.

a	20	25	30	35	40	45	50	55	60	65	70
r	150	146	142	139	135	131	127	124	120	116	112

a Does $f(50) = 2f(25)$?

b Find a value of a for which $f(a) = 2a$. Is $f(a) = 2a$ for all values of a?

c Is $r = f(a)$ an increasing function or a decreasing function?

Answer.

a No

b 60; no

c Decreasing

34. The table shows $M = f(d)$, the men's Olympic record time, and $W = g(d)$, the women's Olympic record time, as a function of the length, d, of the race. For example, the women's record in the 100 meters is 10.62 seconds, and the men's record in the 800 meters is 1 minute, 42.58 seconds. (Source: www.hickoksports.com)

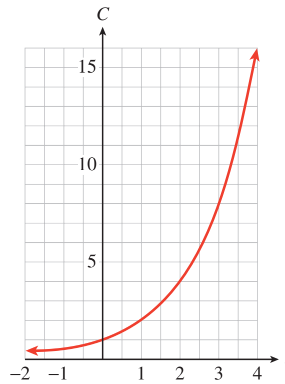
Distance (meters)	100	200	400	800	1500	5000	10,000
Men	9.63	19.30	43.03	1 : 40.91	3 : 32.07	12 : 57.82	27 : 01.17
Women	10.62	21.34	48.25	1 : 53.43	3 : 53.96	14 : 26.17	29 : 17.45

a Does $f(800) = 2f(400)$? Does $g(400) = 2g(200)$?

b Find a value of d for which $f(2d) < 2f(d)$. Is there a value of d for which $g(2d) < 2g(d)$?

In Problems 35—40, use the graph of the function to answer the questions.

35. The graph shows C as a function of t. C stands for the number of students (in thousands) at State University who consider themselves computer literate, and t represents time, measured in years since 1990.



a When did 2000 students consider themselves computer literate?

b How long did it take that number to double?

c How long did it take for the number to double again?

d How many students became computer literate between January 1992 and June 1993?

Answer.

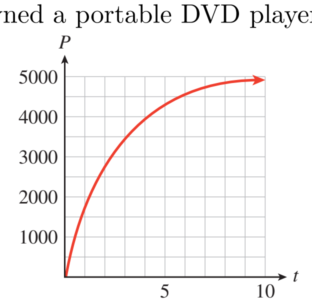
a 1991

b 1 yr

c 1 yr

d About 7300

36. The graph shows P as a function of t. P is the number of people in Cedar Grove who owned a portable DVD player t years after 2000.



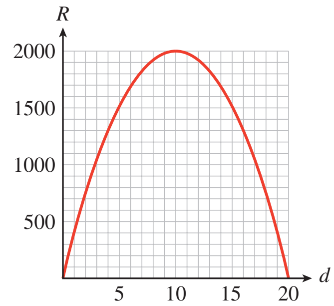
a When did 3500 people own portable DVD players?

b How many people owned portable DVD players in 2005?

c The number of owners of portable DVD players in Cedar Grove seems to be leveling off at what number?

d How many people acquired portable DVD players between 2001 and 2004?

37. The graph shows the revenue, R, a movie theater collects as a function of the price, d, it charges for a ticket.



a What is the revenue if the theater charges $12.00 for a ticket?

b What should the theater charge for a ticket in order to collect $1500 in revenue?

c For what values of d is $R > 1875$?

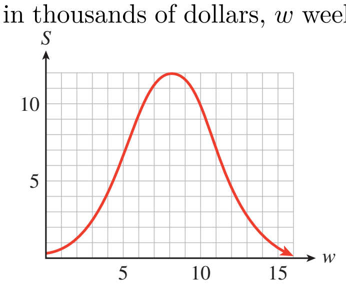
Answer.

a Approximately $1920

b $5 or $15

c $7.50 < d < 12.50$

38. The graph shows S as a function of w. S represents the weekly sales of a best-selling book, in thousands of dollars, w weeks after it is released.

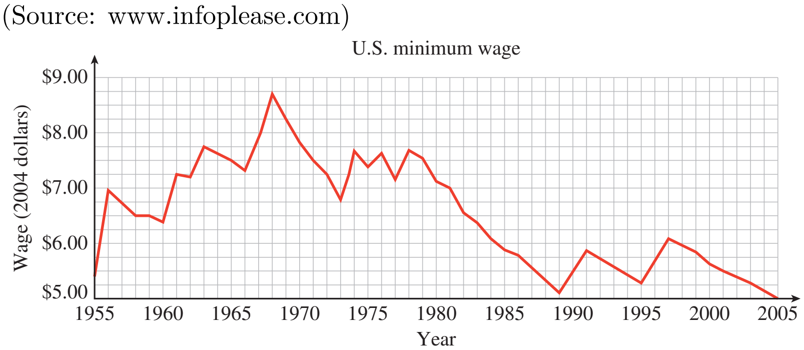


a In which weeks were sales over $7000?

b In which week did sales fall below $5000 on their way down?

c For what values of w is $S > 3.4$?

39. The graph shows the federal minimum wage, M, as a function of time, t, adjusted for inflation to reflect its buying power in 2004 dollars. (Source: www.infoplease.com)



a When did the minimum wage reach its highest buying power, and what was it worth in 2004 dollars?

b When did the minimum wage fall to its lowest buying power after its peak, and what was its worth at that time?

c Give two years in which the minimum wage was worth \$8 in 2004 dollars.

Answer.

a 1968, about \$8.70

b 1989, about \$5.10

c 1967, approximately 1970

40. The graph shows the U.S. unemployment rate, U, as a function of time, t, for the years 1985–2004. (Source: U.S. Bureau of Labor Statistics)

a When did the unemployment rate reach its highest value, and what was its highest value?

b When did the unemployment rate fall to its lowest value, and what was its lowest value?

c Give two years in which the unemployment rate was 4.5%.

In Problems 41–48, evaluate each function for the given values, if possible. If not, state why.

41. $f(x) = 6 - 2x$

a $f(3)$

b $f(-2)$

c $f(12.7)$

d $f\left(\frac{2}{3}\right)$

42. $g(t) = 5t - 3$

a $g(1)$

b $g(-4)$

c $g(14.1)$

d $g\left(\frac{3}{4}\right)$

Answer.

a 0

b 10

c -19.4

d $\frac{14}{3}$

43. $h(v) = 2v^2 - 3v + 1$

a $h(0)$

b $h(-1)$

c $h\left(\frac{1}{4}\right)$

d $h(-6.2)$

44. $r(s) = 2s - s^2$

a $r(2)$

b $r(-4)$

c $r\left(\frac{1}{3}\right)$

d $r(-1.3)$

Answer.

a 1

b 6

c $\frac{3}{8}$

d 96.48

45. $H(z) = \dfrac{2z-3}{z+2}$

a $H(4)$

b $H(-3)$

c $H\left(\frac{4}{3}\right)$

d $H(4.5)$

46. $F(x) = \dfrac{1-x}{2x-3}$

a $F(0)$

b $F(-3)$

c $F\left(\frac{5}{2}\right)$

d $F(\frac{3}{2})$

Answer.

a $\frac{5}{6}$

b 9

c $\frac{-1}{10}$

d $\frac{12}{13} \approx 0.923$

47. $E(t) = \sqrt{t-4}$

a $E(16)$

b $E(4)$

c $E(7)$

d $E(4.2)$

48. $D(r) = \sqrt{5-r}$

a $D(4)$

b $D(-3)$

c $D(9)$

d $D(4.6)$

Answer.

a $\sqrt{12}$

b 0

c $\sqrt{3}$

d $\sqrt{0.2} \approx 0.447$

49. A sport utility vehicle costs \$28,000 and depreciates according to the formula

$$V(t) = 28,000(1 - 0.08t)$$

where V is the value of the vehicle after t years.

a Evaluate $V(12)$ and explain what it means.

b Solve the equation $V(t) = 0$ and explain what it means.

c If this year is $t = n$, what does $V(n+2)$ mean?

Answer.

a $V(12) = 1120$: After 12 years, the SUV is worth \$1120.

b $t = 12.5$: The SUV has zero value after $12\frac{1}{2}$ years.

c The value 2 years later

50. In a profit-sharing plan, an employee receives a salary of

$$S(x) = 20,000 + 0.01x$$

where x represents the company's profit for the year.

a Evaluate $S(850,000)$ and explain what it means.

b Solve the equation $S(x) = 30,000$ and explain what it means.

c If the company made a profit of p dollars this year, what does $S(2p)$ mean?

51. The number of compact cars that a large dealership can sell at price p is given by

$$N(p) = \frac{12,000,000}{p}$$

a Evaluate $N(6000)$ and explain what it means.

b As p increases, does $N(p)$ increase or decrease? Why is this reasonable?

c If the current price for a compact car is D, what does $2N(D)$ mean?

Answer.

a $N(6000) = 2000$: 2000 cars will be sold at a price of \$6000.

b $N(p)$ decreases with increasing p because fewer cars will be sold when the price increases.

c $2N(D)$ represents twice the number of cars that can be sold at the current price.

52. A department store finds that the market value of its Christmas-related merchandise is given by

$$M(t) = \frac{600,000}{t}, \quad t \leq 30$$

where t is the number of weeks after Christmas.

a Evaluate $M(2)$ and explain what it means.

b As t increases, does $M(t)$ increase or decrease? Why is this reasonable?

c If this week $t = n$, what does $M(n + 1)$ mean?

53. The velocity of a car that brakes suddenly can be determined from the length of its skid marks, d, by

$$v(d) = \sqrt{12d}$$

where d is in feet and v is in miles per hour.

a Evaluate $v(250)$ and explain what it means.

b Estimate the length of the skid marks left by a car traveling at 100 miles per hour.

c Write your answer to part (b) with function notation.

Answer.

a $v(250) = 54.8$ is the speed of a car that left 250-foot skid marks.

b $833\frac{1}{3}$ feet

c $v\left(833\frac{1}{3}\right) = 100$

54. The distance, d, in miles that a person can see on a clear day from a height, h, in feet is given by

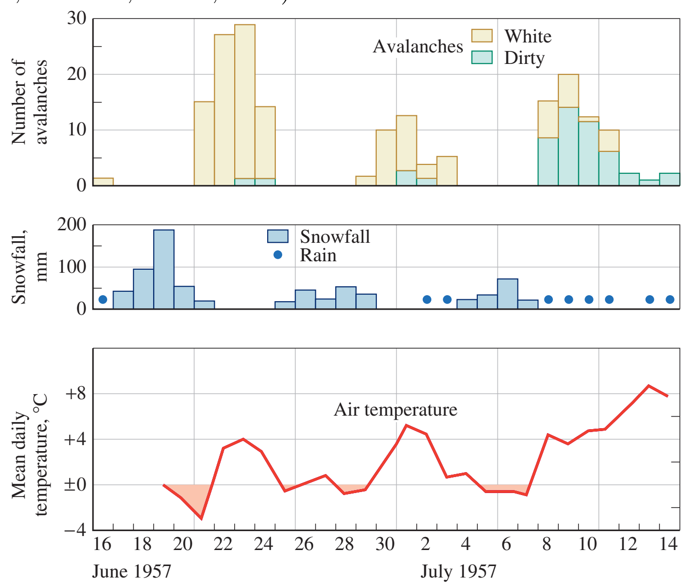
$$d(h) = 1.22\sqrt{h}$$

a Evaluate $d(20,320)$ and explain what it means.

b Estimate the height you need in order to see 100 miles.

c Write your answer to part (b) with function notation.

55. The figure gives data about snowfall, air temperature, and number of avalanches on the Mikka glacier in Sarek, Lapland, in 1957. (Source: Leopold, Wolman, Miller, 1992)



a During June and July, avalanches occurred over three separate time intervals. What were they?

b Over what three time intervals did snow fall?

c When was the temperature above freezing (0°C)?

d Using your answers to parts (a)–(c), make a conjecture about the conditions that encourage avalanches.

Answer.

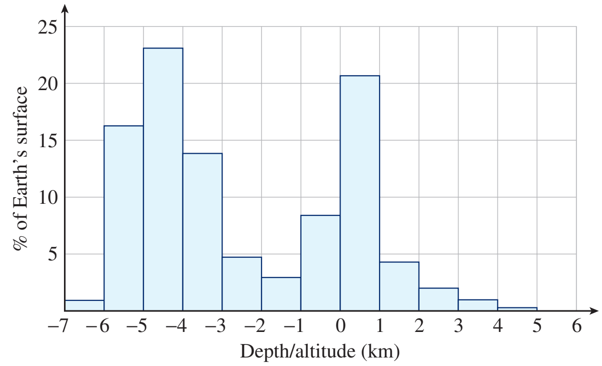
a June 21–24, June 29–July 3, July 8–14

b June 17–21, June 25–29, July 4–7

c June 22–24, June 27, June 29–July 4, July 8–14

d Avalanches occur when temperatures rise above freezing immediately after snowfall.

56. The bar graph shows the percent of Earth's surface that lies at various altitudes or depths below the surface of the oceans. (Depths are given as negative altitudes.) (Source: Open University)



a Read the graph and complete the table.

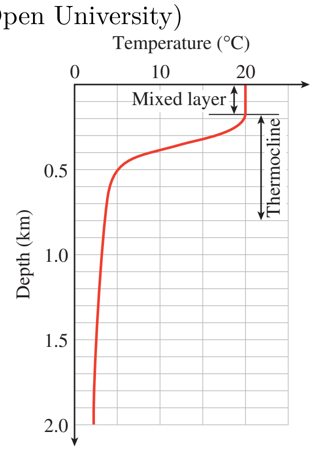
Altitude (km)	Percent of Earth's surface
-7 to -6	
-6 to -5	
-5 to -4	
-4 to -3	
-3 to -2	
-2 to -1	
-1 to 0	
0 to 1	
1 to 2	
2 to 3	
3 to 4	
4 to 5	

b What is the most common altitude? What is the second most common altitude??

c Approximately what percent of the Earth's surface is below sea level?

d The height of Mt. Everest is 8.85 kilometers. Can you think of a reason why it is not included in the graph?

57. The graph shows the temperature of the ocean at various depths. (Source: Open University)



a Is depth a function of temperature?

b Is temperature a function of depth?

c The axes are scaled in an unusual way. Why is it useful to present the graph in this way?

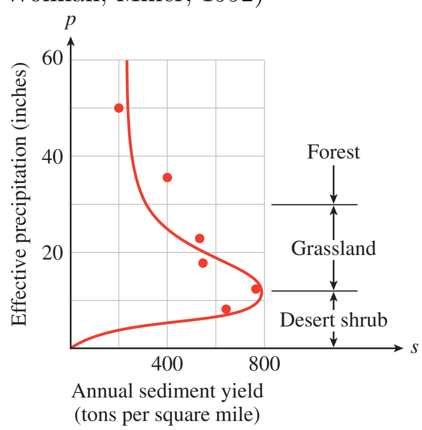
Answer.

a No

b Yes

c Moving downwards on the graph corresponds to moving downwards in the ocean.

58. The graph shows the relationship between annual precipitation, p, in a region and the amount of erosion, measured in tons per square mile, s. (Source: Leopold, Wolman, Miller, 1992)



a Is the amount of erosion a function of the amount of precipitation?

b At what annual precipitation is erosion at a maximum, and what is that maximum?

c Over what interval of annual precipitation does erosion decrease?

d An increase in vegetation inhibits erosion, and precipitation encourages vegetation. What happens to the amount of erosion as precipitation increases in each of these three environments?

desert shrub:	$0 < p < 12$
grassland:	$12 < p < 30$
forest:	$30 < p < 60$

In Problems 59—64, evaluate the function and simplify.

59. $G(s) = 3s^2 - 6s$

a $G(3a)$ c $G(a)+2$

b $G(a+2)$ d $G(-a)$

60. $h(x) = 2x^2 + 6x - 3$

a $h(2a)$ c $h(a)+3$

b $h(a+3)$ d $h(-a)$

Answer.

a $27a^2 - 18a$ c $3a^2 - 6a + 2$

b $3a^2 + 6a$ d $3a^2 + 6a$

61. $g(x) = 8$

a $g(2)$
b $g(8)$
c $g(a+1)$
d $g(-x)$

Answer.

a 8
b 8
c 8
d 8

62. $f(t) = -3$

a $f(4)$
b $f(-3)$
c $f(b-2)$
d $f(-t)$

63. $P(x) = x^3 - 1$

a $P(2x)$
b $2P(x)$
c $P(x^2)$
d $[P(x)]^2$

Answer.

a $8x^3 - 1$
b $2x^3 - 2$
c $x^6 - 1$
d $x^6 - 2x^3 + 1$

64. $Q(t) = 5t^3$

a $Q(2t)$
b $2Q(t)$
c $Q(t^2)$
d $[Q(t)]^2$

In Problems 65—68, evaluate the function for the given expressions and simplify.

65. $f(x) = x^3$

a $f(a^2)$
b $a^3 \cdot f(a^3)$
c $f(ab)$
d $f(a+b)$

Answer.

a a^6
b a^{12}
c a^3b^3
d $a^3 + 3a^2b + 3ab^2 + b^3$

66. $g(x) = x^4$

a $g(a^3)$
b $a^4 \cdot g(a^4)$
c $g(ab)$
d $g(a+b)$

67. $F(x) = 3x^5$

a $F(2a)$
b $2F(a)$
c $F(a^2)$
d $[F(a)]^2$

Answer.

a $96a^5$
b $6a^5$
c $3a^{10}$
d $9a^{10}$

68. $G(x) = 4x^3$

a $G(3a)$
b $3G(a)$
c $G(a^4)$
d $[G(a)]^4$

For the functions in Problems 69–76, compute the following:

a $f(2) + f(3)$
b $f(2+3)$
c $f(a) + f(b)$
d $f(a+b)$

For which functions does $f(a+b) = f(a) + f(b)$ for all values of a and b?

69. $f(x) = 3x - 2$

Answer.

a
11

b
13

c
$3a+$
$3b-$
4

d
$3a+$
$3b-$
2

This function does NOT satisfy $f(a+b) = f(a) + f(b)$.

70. $f(x) = 1 - 4x$

71. $f(x) = x^2 + 3$

Answer.

a
19

b
28

c
a^2+
b^2+
6

d
a^2+
$2ab+$
b^2+
3

This function does NOT satisfy $f(a+b) = f(a) + f(b)$.

72. $f(x) = x^2 - 1$

73. $f(x) = \sqrt{x+1}$

Answer.

a
$\sqrt{3}+$
2

b
$\sqrt{6}$

c
$\sqrt{a+1}+$
$\sqrt{b+1}$

d
$\sqrt{a+b+1}$

This function does NOT satisfy $f(a+b) = f(a) + f(b)$.

74. $f(x) = \sqrt{6-x}$

75. $f(x) = \dfrac{-2}{x}$

Answer.

a
$\dfrac{-5}{3}$

b
$\dfrac{-2}{5}$

c
$\dfrac{-2}{a} -$
$\dfrac{-2}{b}$

d
$\dfrac{-2}{a+b}$

This function does NOT satisfy $f(a+b) = f(a) + f(b)$.

76. $f(x) = \dfrac{3}{x}$

77. Use a table of values to estimate a solution to

$$f(x) = 800 + 6x - 0.2x^2 = 500$$

as follows:

a Make a table starting at $x = 0$ and increasing by $\Delta x = 10$, as shown in the accompanying tables. Find two x-values a and b so that $f(a) > 500 > f(b)$.

x	0	10	20	30	40	50	60	70	80	90	100
$f(x)$											

b Make a new table starting at $x = a$ and increasing by $\Delta x = 1$. Find two x-values, c and d, so that $f(c) > 500 > f(d)$.

c Make a new table starting at $x = c$ and increasing by $\Delta x = 0.1$. Find two x-values, p and q, so that $f(p) > 500 > f(q)$.

d Take the average of p and q, that is, set $s = \dfrac{p+q}{2}$. Then s is an approximate solution that is off by at most 0.05.

e Evaluate $f(s)$ to check that the output is approximately 500.

Answer.

a

x	0	10	20	30	40	50	60	70	80	90	100
$f(x)$	800	840	840	800	720	600	440	240	0	−280	−600

$a = 50$ and $b = 60$

b

x	50	51	52	53	54	55	56	57	58
$f(x)$	600	585.8	571.2	556.2	540.8	525	508.8	492.2	475.2

$c = 56$ and $d = 57$

c

x	56	56.1	56.2	56.3	56.4	56.5	56.6
$f(x)$	508.8	507.158	505.512	503.862	502.208	500.55	498.888

$p = 56.5$ and $q = 56.6$

d $s = 56.55$

e $f(56.55) = 499.7195$

78. Use a table of values to estimate a solution to

$$f(x) = x^3 - 4x^2 + 5x = 18,000$$

as follows:

a Make a table starting at $x = 0$ and increasing by $\Delta x = 10$, as shown in the accompanying tables. Find two x-values a and b so that $f(a) < 18,000 < f(b)$.

x	0	10	20	30	40	50	60	70	80	90	100
$f(x)$											

b Make a new table starting at $x = a$ and increasing by $\Delta x = 1$. Find two x-values, c and d, so that $f(c) < 18,000 < f(d)$.

c Make a new table starting at $x = c$ and increasing by $\Delta x = 0.1$. Find two x-values, p and q, so that $f(p) < 18,000 < f(q)$.

d Take the average of p and q, that is, set $s = \dfrac{p+q}{2}$. Then s is an approximate solution that is off by at most 0.05.

e Evaluate $f(s)$ to check that the output is approximately 18,000.

79. Use tables of values to estimate the positive solution to

$$f(x) = x^2 - \frac{1}{x} = 9000,$$

accurate to within 0.05.

Answer. 94.85

80. Use tables of values to estimate the positive solution to

$$f(x) = \frac{8}{x} + 500 - \frac{x^2}{9} = 300,$$

accurate to within 0.05.

81. Let $f(x) = -5x - 5$ and $g(x) = 2x^2 + 1$. Evaluate each of the following.

a $g(f(4))$

b $f(g(-4))$

c $f(f(2))$

d $g(g(4))$

e $(g - f)(2)$

f $(fg)(-4)$

82. Answer "True" or "False": $f(g(x))$ must always equal $g(f(x))$.

83. Suppose $f(x) = 3x + 1$ and $g(x) = |x|$. Evaluate each of the following.

a $f(x) + g(x)$

b $g(x) - f(x)$

c $f(x)g(x)$

d $f(x)/g(x)$

e $f(g(x))$

f $g(f(x))$

84. Let $f(x) = x^2 - 2$ and $g(x) = \sqrt{x} + 6$. Find $f(g(x))$ and $g(f(x))$.

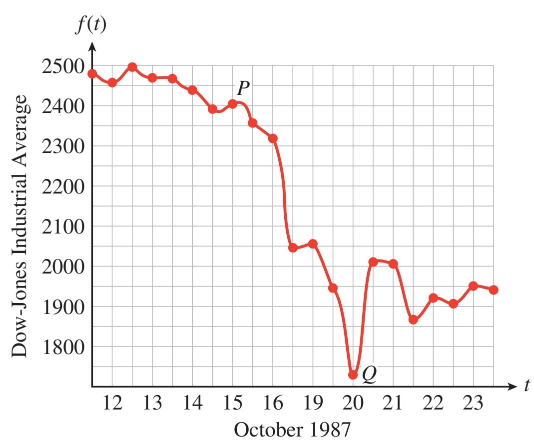
a $f(g(x))$

b $g(f(x))$

1.3 Graphs of Functions

1.3.1 Reading Function Values from a Graph

The Dow-Jones Industrial Average (DJIA) gives the average of the stock prices of 30 major companies. The graph below shows the DJIA as a function of time during the stock market correction of October 1987. The DJIA is thus $f(t)$, recorded at noon on day t of October.



The values of the input variable, time, are displayed on the horizontal axis, and the values of the output variable, DJIA, are displayed on the vertical axis. There is no formula that gives the DJIA for a particular day; but it is still a function, defined by its graph. The value of $f(t)$ is specified by the vertical coordinate of the point with the given t-coordinate.

Example 1.3.1

a The coordinates of point P on the DJIA graph are $(15, 2412)$. What do the coordinates tell you about the function f?

b If the DJIA was 1726 at noon on October 20, what can you say about the graph of f?

Solution.

a The coordinates of point P tell us that $f(15) = 2412$, so the DJIA was 2412 at noon on October 15.

b We can say that $f(20) = 1726$, so the point $(20, 1726)$ lies on the graph of f. This point is labeled Q in the figure above.

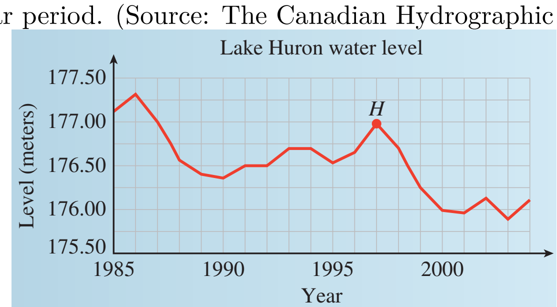
□

Thus, the coordinates of each point on the graph of the function represent a pair of corresponding values of the two variables.

Graph of a Function.

The point (a, b) lies on the graph of the function f if and only if $f(a) = b$.

Checkpoint 1.3.2 The water level in Lake Huron alters unpredictably over time. The graph below gives the average water level, $L(t)$, in meters in the year t over a 20-year period. (Source: The Canadian Hydrographic Service)



a The coordinates of point H on the graph are $(1997, 176.98)$. What do the coordinates tell you about the function L?

b The average water level in 2004 was 176.11 meters. Write this fact in function notation. What can you say about the graph of L?

Answer.

a $L(1997) = 176.98$; the average water level was 176.98 meters in 1997.

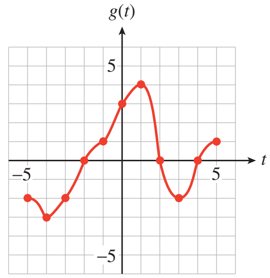
b $L(2004) = 176.11$. The point $(2004, 176.11)$ lies on the graph of L.

Here is another way of describing how a graph depicts a function.

Functions and Coordinates.

Each point on the graph of the function f has coordinates $(x, f(x))$ for some value of x.

Example 1.3.3 The figure shows the graph of a function g.



a Find $g(-2)$ and $g(5)$.

b For what value(s) of t is $g(t) = -2$?

c What is the largest, or maximum, value of $g(t)$? For what value of t does the function take on its maximum value?

d On what intervals is g increasing?

Solution.

a To find $g(-2)$, we look for the point with t-coordinate -2. The point $(-2, 0)$ lies on the graph of g, so $g(-2) = 0$. Similarly, the point $(5, 1)$ lies on the graph, so $g(5) = 1$.

b We look for points on the graph with y-coordinate -2. Because the points $(-5, -2)$, $(-3, -2)$, and $(3, -2)$ lie on the graph, we know that $g(-5) = -2$, $g(-3) = -2$, and $g(3) = -2$. Thus, the t-values we want are -5, -3, and 3.

c The highest point on the graph is $(1, 4)$, so the largest y-value is 4. Thus, the maximum value of $g(t)$ is 4, and it occurs when $t = 1$.

d A graph is increasing if the y-values get larger as we read from left to right. The graph of g is increasing for t-values between -4 and 1, and between 3 and 5. Thus, g is increasing on the intervals $(-4, 1)$ and $(3, 5)$.

□

Checkpoint 1.3.4 Refer to the graph of the function g shown in Example 1.3.3, p. 59.

a Find $g(0)$.

b For what value(s) of t is $g(t) = 0$?

c What is the smallest, or minimum, value of $g(t)$? For what value of t does the function take on its minimum value?

d On what intervals is g decreasing?

Answer.

a 3

b $-2, 2, 4$

c -3; $t = -4$

d $(-5, -4)$ and $(1, 3)$

1.3.2 Constructing the Graph of a Function

Although some functions are defined by their graphs, we can also construct graphs for functions described by tables or equations. We make these graphs the same way we graph equations in two variables: by plotting points whose coordinates satisfy the equation.

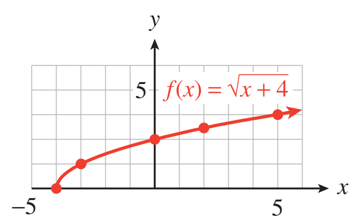
Example 1.3.5 Graph the function $f(x) = \sqrt{x+4}$

Solution. We choose several convenient values for x and evaluate the function to find the corresponding $f(x)$-values. For this function we cannot choose x-values less than -4, because the square root of a negative number is not a real number.

$$f(\mathbf{-4}) = \sqrt{\mathbf{-4}+4} = \sqrt{0} = 0$$
$$f(\mathbf{-3}) = \sqrt{\mathbf{-3}+4} = \sqrt{1} = 1$$
$$f(\mathbf{0}) = \sqrt{\mathbf{0}+4} = \sqrt{4} = 2$$
$$f(\mathbf{2}) = \sqrt{\mathbf{2}+4} = \sqrt{6} \approx 2.45$$
$$f(\mathbf{5}) = \sqrt{\mathbf{5}+4} = \sqrt{9} = 3$$

The results are shown in the table.

x	$f(x)$
-4	0
-3	1
0	2
2	$\sqrt{6}$
5	3



Points on the graph have coordinates $(x, f(x))$, so the vertical coordinate of each popint is given by the value of $f(x)$. We plot the points and connect them with a smooth curve, as shown in the figure. Notice that no points on the graph have x-coordinates less than -4. □

Checkpoint 1.3.6 $f(x) = x^3 - 2$

a Complete the table of values and sketch a graph of the function.

x	-2	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	2
$f(x)$							

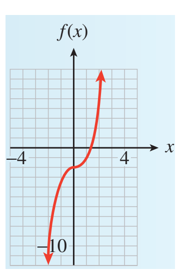
b Use your calculator to make a table of values and graph the function.

Answer.

a

x	-2	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	2
$f(x)$	-10	-3	$\frac{-17}{8}$	-2	$\frac{-15}{8}$	-1	6

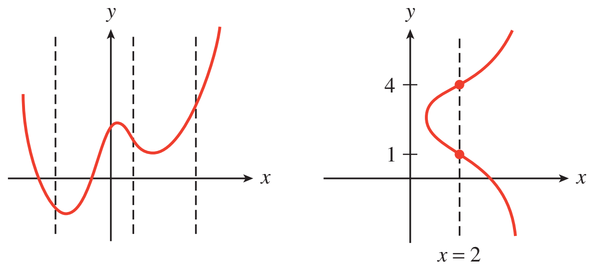
b



1.3.3 The Vertical Line Test

In a function, two different outputs cannot be related to the same input. This restriction means that two different ordered pairs cannot have the same first coordinate. What does it mean for the graph of the function?

Consider the graph shown in figure (a) below. Every vertical line intersects the graph in at most one point, so there is only one point on the graph for each x-value. This graph represents a function.



(a) **(b)**

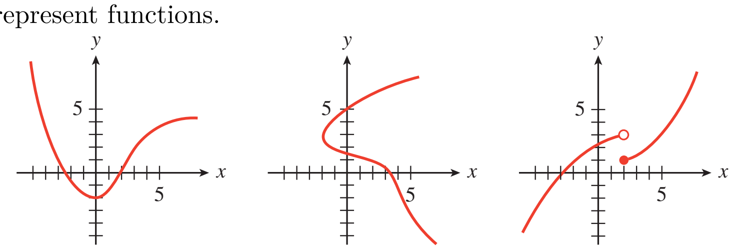
In figure (b), however, the line $x = 2$ intersects the graph at two points, $(2, 1)$ and $(2, 4)$. Two different y-values, 1 and 4, are related to the same x-value, 2. This graph cannot be the graph of a function.

We summarize these observations as follows.

The Vertical Line Test.

A graph represents a function if and only if every vertical line intersects the graph in at most one point.

Example 1.3.7 Use the vertical line test to decide which of the graphs in the figure represent functions.



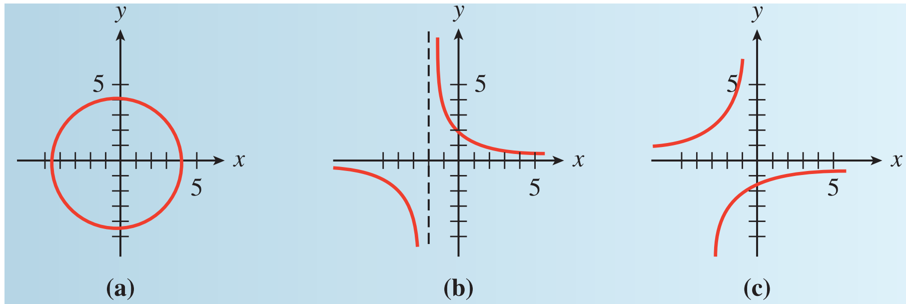
(a) **(b)** **(c)**

Solution.

- Graph (a) represents a function, because it passes the vertical line test.
- Graph (b) is not the graph of a function, because the vertical line at (for example) $x = 1$ intersects the graph at two points.
- For graph (c), notice the break in the curve at $x = 2$: The solid dot at $(2, 1)$ is the only point on the graph with $x = 2$; the open circle at $(2, 3)$ indicates that $(2, 3)$ is not a point on the graph. Thus, graph (c) is a function, with $f(2) = 1$.

□

Checkpoint 1.3.8 Use the vertical line test to determine which of the graphs below represent functions.

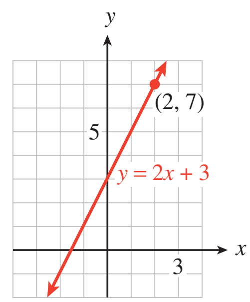


Answer. Only (b) is a function.

1.3.4 Graphical Solution of Equations and Inequalities

The graph of an equation in two variables is just a picture of its solutions. When we read the coordinates of a point on the graph, we are reading a pair of x- and y-values that make the equation true.

For example, the point $(2, 7)$ lies on the graph of $y = 2x + 3$ shown at right, so we know that the ordered pair $(2, 7)$ is a solution of the equation $y = 2x + 3$. You can verify algebraically that $x = \mathbf{2}$ and $y = \mathbf{7}$ satisfy the equation:



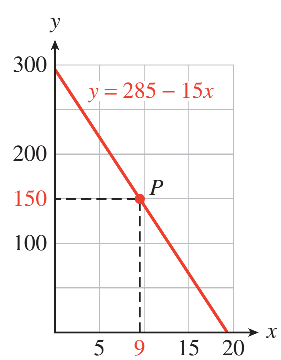
Does $\mathbf{7} = 2(\mathbf{2}) + 3$? Yes

We can also say that $x = 2$ is a solution of the one-variable equation $2x + 3 = 7$. In fact, we can use the graph of $y = 2x + 3$ to solve the equation $2x + 3 = k$ for any value of k. Thus, we can use graphs to find solutions to equations in one variable.

Example 1.3.9 Use the graph of $y = 285 - 15x$ to solve the equation $150 = 285 - 15x$.

Solution.

We begin by locating the point P on the graph for which $y = 150$, as shown in the figure. Next we find the x-coordinate of point P by drawing an imaginary line from P straight down to the x-axis. The x-coordinate of P is $x = 9$. Thus, P is the point $(9, 150)$, and $x = 9$ when $y = 150$. The solution of the equation $150 = 285 - 15x$ is $x = 9$.



You can verify the solution algebraically by substituting $x = \mathbf{9}$ into the equation:

Does $150 = 285 - 15(\mathbf{9})$?

$$285 - 15(\mathbf{9}) = 285 - 135 = 150. \qquad \text{Yes}$$

□

Note 1.3.10 The relationship between an equation and its graph is an important one. For the previous example, make sure you understand that the following three statements are equivalent:

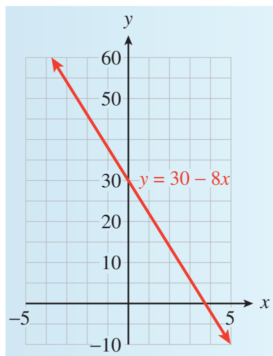
- The point $(9, 150)$ lies on the graph of $y = 285 - 15x$.
- The ordered pair $(9, 150)$ is a solution of the equation $y = 285 - 15x$.
- $x = 9$ is a solution of the equation $150 = 285 - 15x$.

Checkpoint 1.3.11

a Use the graph of $y = 30 - 8x$ shown in the figure to solve the equation

$$30 - 8x = 50$$

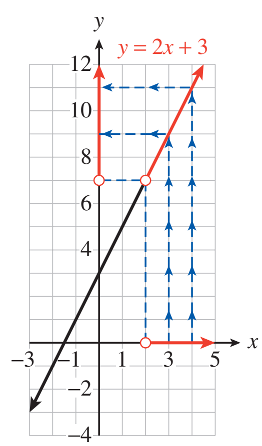
b Verify your solution algebraically.



Answer. -2.5

In a similar fashion, we can solve inequalities with a graph.

Consider again the graph of $y = 2x+3$, shown at right. We saw that $x = 2$ is the solution of the equation $2x + 3 = 7$. When we use $x = 2$ as the input for the function $f(x) = 2x + 3$, the output is $y = 7$. Which input values for x produce output values greater than 7?
You can see that x-values greater than 2 produce y-values greater than 7, because points on the graph with x-values greater than 2 have y-values greater than 7. Thus, the solutions of the inequality $2x + 3 > 7$ are $x > 2$. You can verify this result by solving the inequality algebraically.



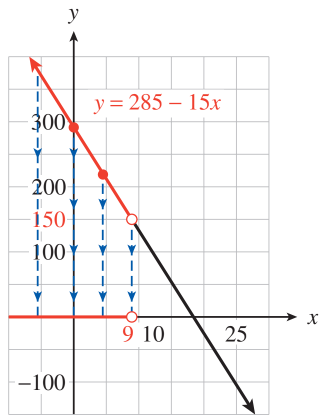
Example 1.3.12 Use the graph of $y = 285 - 15x$ to solve the inequality

$$285 - 15x > 150$$

Solution. We begin by locating the point P on the graph for which $y = 150$. Its x-coordinate is $x = 9$. Now, because $y = 285 - 15x$ for points on the graph, the inequality

$$285 - 15x > 150$$

is equivalent to $y > 150$.



So we are looking for points on the graph with y-coordinate greater than 150. These points are shown in red on the graph. The x-coordinates of these points are the x-values that satisfy the inequality. From the graph, we see that the solutions are $x < 9$. □

Checkpoint 1.3.13

a Use the graph of $y = 30 - 8x$ in the previous Checkpoint to solve the inequality

$$30 - 8x \leq 50$$

b Solve the inequality algebraically.

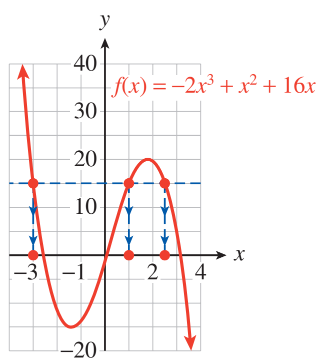
Answer. $x \geq -2.5$

We can also use this graphical technique to solve nonlinear equations and inequalities.

Example 1.3.14 Use a graph of $f(x) = -2x^3 + x^2 + 16x$ to solve the equation

$$-2x^3 + x^2 + 16x = 15$$

Solution. If we sketch in the horizontal line $y = 15$, we can see that there are three points on the graph of f that have y-coordinate 15, as shown below. The x-coordinates of these points are the solutions of the equation.



From the graph, we see that the solutions are $x = -3$, $x = 1$, and approximately $x = 2.5$. We can verify each solution algebraically.

For example, if $x = \mathbf{-3}$, we have

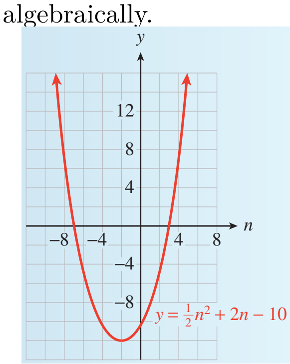
$$\begin{aligned} f(\mathbf{-3}) &= -2(\mathbf{-3})3 + (\mathbf{-3})^2 + 16(\mathbf{-3}) \\ &= -2(-27) + 9 - 48 \\ &= 54 + 9 - 48 = 15 \end{aligned}$$

so -3 is a solution. Similarly, you can check that $x = 1$ and $x = 2.5$ are solutions. □

Checkpoint 1.3.15 Use the graph of $y = \frac{1}{2}n^2 + 2n - 10$ shown below to solve

$$\frac{1}{2}n^2 + 2n - 10 = 6$$

and verify your solutions algebraically.



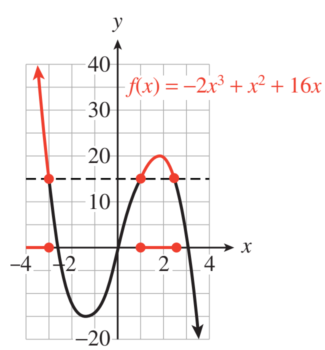
Answer. $-8, 4$

Example 1.3.16 Use the graph in Example 1.3.14, p. 64 to solve the inequality

$$-2x^3 + x^2 + 16x \geq 15$$

Solution. We first locate all points on the graph that have y-coordinates greater than or equal to 15. The x-coordinates of these points are the solutions of the inequality.

The figure below shows the points in red, and their x-coordinates as intervals on the x-axis. The solutions are $x \leq -3$ and $1 \leq x \leq 2.5$, or in interval notation, $(-\infty, -3] \cup [1, 2.5]$.



□

Checkpoint 1.3.17 Use the graph in Checkpoint 1.3.15, p. 65 to solve the inequality

$$\frac{1}{2}n^2 + 2n - 10 < 6$$

Answer. $(-8, 4)$

1.3.5 Section Summary

1.3.5.1 Vocabulary

Look up the definitions of new terms in the Glossary.

- Coordinates
- Maximum
- Minimum
- Interval
- Vertical line test
- Inequality
- Algebraic solution
- Graphical solution

1.3.5.2 CONCEPTS

1 The point (a, b) lies on the graph of the function f if and only if $f(a) = b$.

2 Each point on the graph of the function f has coordinates $(x, f(x))$ for some value of x.

3 The vertical line test tells us whether a graph represents a function.

4 We can use a graph to solve equations and inequalities in one variable.

1.3.5.3 STUDY QUESTIONS

1 How can you find the value of $f(3)$ from a graph of f?

2 If $f(8) = 2$, what point lies on the graph of f?

3 Explain how to construct the graph of a function from its equation.

4 Explain how to use the vertical line test.

5 How can you solve the equation $x+\sqrt{x} = 56$ using the graph of $y = x+\sqrt{x}$?

1.3.5.4 SKILLS

Practice each skill in the Homework 1.3.6, p. 66 problems listed.

1 Read function values from a graph: #1–8, 17–20, 33–36

2 Recognize the graph of a function: #9–10, 31 and 32

3 Construct a table of values and a graph of a function: #11–16

4 Solve equations and inequalities graphically: #21–30, 41–50

1.3.6 Graphs of Functions (Homework 1.3)

In Problems 1–8, use the graphs to answer the questions about the functions.

1.

a Find $h(-3)$, $h(1)$, and $h(3)$.

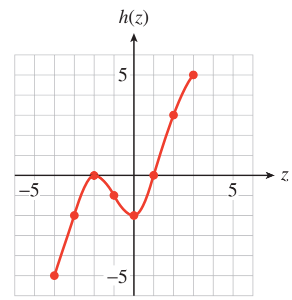
b For what value(s) of z is $h(z) = 3$?

c Find the intercepts of the graph. List the function values given by the intercepts.

d What is the maximum value of $h(z)$?

e For what value(s) of z does h take on its maximum value?

f On what intervals is the function increasing? Decreasing?



Answer.

a $-2, 0, 5$

b 2

c $h(-2) = 0,\ h(1) = 0,\ h(0) = -2$

d 5

e 3

f Increasing: $(-3, 0)$ and $(1, 3)$; decreasing: $(0, 1)$ and $(3, 5)$

2.

a Find $G(-3)$, $G(-1)$, and $G(2)$.

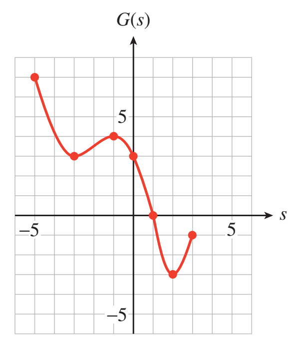
b For what value(s) of s is $G(s) = 3$?

c Find the intercepts of the graph. List the function values given by the intercepts.

d What is the minimum value of $G(s)$?

e For what value(s) of s does G take on its minimum value?

f On what intervals is the function increasing? Decreasing?



3.

a Find $R(1)$ and $R(3)$.

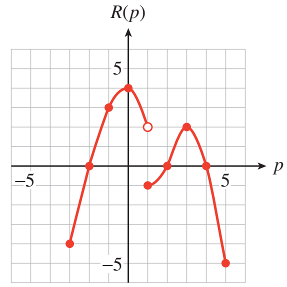
b For what value(s) of p is $R(p) = 2$?

c Find the intercepts of the graph. List the function values given by the intercepts.

d Find the maximum and minimum values of $R(p)$.

e For what value(s) of p does R take on its maximum and minimum values?

f On what intervals is the function increasing? Decreasing?



Answer.

a $-1, 2$

b $3, -1.3$

c $R(-2) = 0,\ R(2) = 0,\ R(4) = 0,\ R(0) = 4$

d Max: 4; min: -5

e Max at $p = 0$; min at $p = 5$

f Increasing: $(-3, 0)$ and $(1, 3)$; decreasing: $(0, 1)$ and $(3, 5)$

4.

a Find $f(-1)$ and $f(3)$.

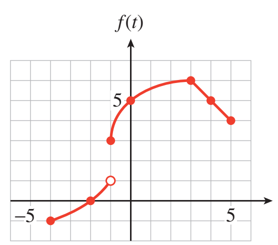
b For what value(s) of t is $f(t) = 5$?

c Find the intercepts of the graph. List the function values given by the intercepts.

d Find the maximum and minimum values of $f(t)$.

e For what value(s) of t does f take on its maximum and minimum values?

f On what intervals is the function increasing? Decreasing?



5.

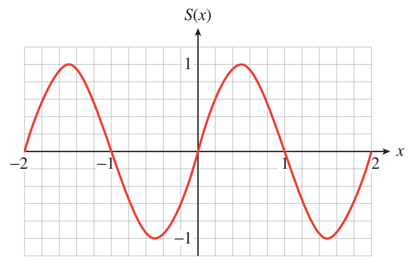
a Find $S(0)$, $S\left(\dfrac{1}{6}\right)$, and $S(-1)$.

b Estimate the value of $S\left(\dfrac{1}{3}\right)$ from the graph.

c For what value(s) of x is $S(x) = -\dfrac{1}{2}$?

d Find the maximum and minimum values of $S(x)$.

e For what value(s) of x does S take on its maximum and minimum values?



Answer.

a $0, \frac{1}{2}, 0$

b 0.9

c $\frac{-5}{6}, \frac{-1}{6}, \frac{7}{6}, \frac{11}{6}$

d Max: 1; min: −1

e Max at $x = -1.5, 0.5$; min at $x = -0.5, 1.5$

6.

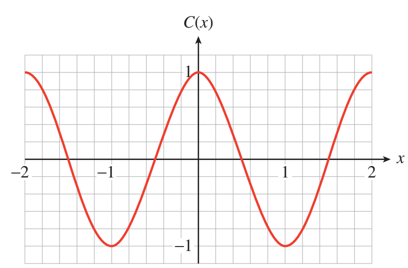
a Find $C(0)$, $C\left(-\frac{1}{3}\right)$, and $C(1)$.

b Estimate the value of $C\left(\frac{1}{6}\right)$ from the graph.

c For what value(s) of x is $C(x) = \frac{1}{2}$?

d Find the maximum and minimum values of $C(x)$.

e For what value(s) of x does C take on its maximum and minimum values?



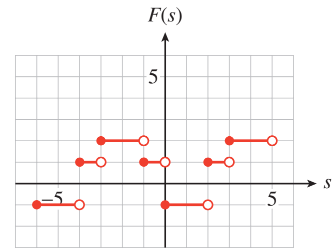
7.

a Find $F(-3)$, $F(-2)$, and $F(2)$.

b For what value(s) of s is $F(s) = -1$?

c Find the maximum and minimum values of $F(s)$.

d For what value(s) of s does F take on its maximum and minimum values?



Answer.

a $2, 2, 1$

b $-6 \leq s < -4$ or $0 \leq s < 2$

c Max: 2; min: -1

d Max for $-3 \leq s < -1$ or $3 \leq s < 5$; min for $-6 \leq s < -4$ or $0 \leq s < 2$

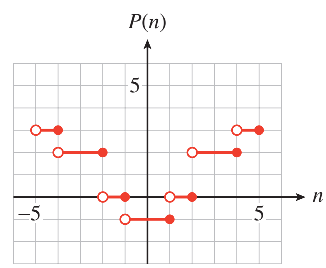
8.

a Find $P(-3)$, $P(-2)$, and $P(1)$.

b For what value(s) of n is $P(n) = 0$?

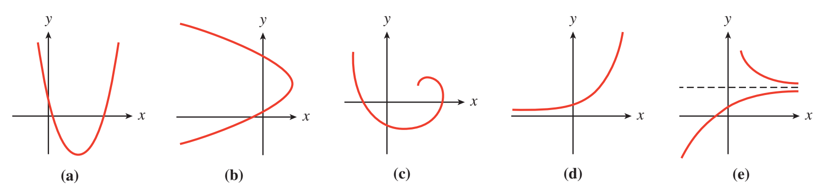
c Find the maximum and minimum values of $P(n)$.

d For what value(s) of n does P take on its maximum and minimum values?



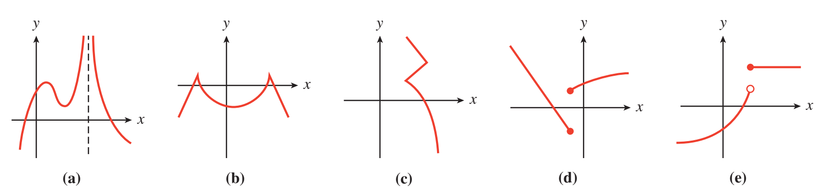
Which of the graphs in Problems 9 and 10 represent functions?

9.



Answer. (a) and (d)

10.



In Problems 11–16,

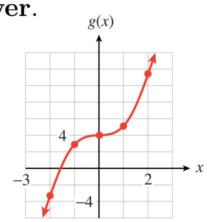
a Make a table of values and sketch a graph of the function by plotting points. (Use the suggested x-values.)

b Use your calculator to graph the function.

Compare the calculator's graph with your sketch.

11. $g(x) = x^3 + 4$;
$x = -2, -1, \ldots, 2$

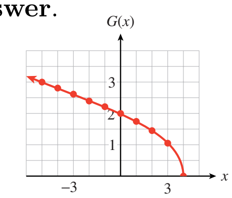
Answer.



12. $h(x) = 2 + \sqrt{x}$;
$x = 0, 1, \ldots, 9$

13. $G(x) = \sqrt{4 - x}$;
$x = -5, -4, \ldots, 4$

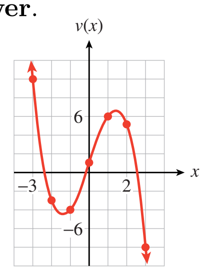
Answer.



14. $F(x) = \sqrt{x - 1}$;
$x = 1, 2, \ldots, 10$

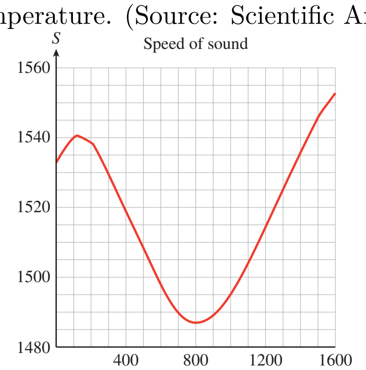
15. $v(x) = 1 + 6x - x^3$;
$x = -3, -2, \ldots, 3$

Answer.



16. $w(x) = x^3 - 8x$;
$x = -4, -3, \ldots, 4$

17. The graph shows the speed of sound in the ocean as a function of depth, $S = f(d)$. The speed of sound is affected both by increasing water pressure and by dropping temperature. (Source: Scientific American)



a Evaluate $f(1000)$ and explain its meaning.

b Solve $f(d) = 1500$ and explain its meaning.

c At what depth is the speed of sound the slowest, and what is the speed? Write your answer with function notation.

d Describe the behavior of $f(d)$ as d increases.

Answer.

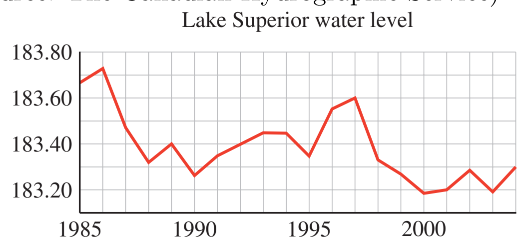
a $f(1000) = 1495$: The speed of sound at a depth of 1000 meters is approximately 1495 meters per second.

b $d = 570$ or $d = 1070$: The speed of sound is 1500 meters per second at both a depth of 570 meters and a depth of 1070 meters.

c The slowest speed occurs at a depth of about 810 meters and the speed is about 1487 meters per second, so $f(810) = 1487$.

d f increases from about 1533 to 1541 in the first 110 meters of depth, then drops to about 1487 at 810 meters, then rises again, passing 1553 at a depth of about 1600 meters.

18. The graph shows the water level in Lake Superior as a function of time, $L = f(t)$. (Source: The Canadian Hydrographic Service)



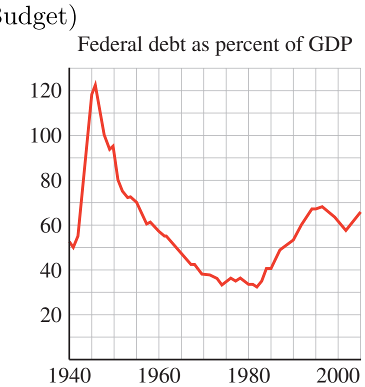
a Evaluate $f(1997)$ and explain its meaning.

b Solve $f(t) = 183.5$ and explain its meaning.

c In which two years did Lake Superior reach its highest levels, and what were those levels? Write your answers with function notation.

d Over which two-year period did the water level drop the most?

19. The graph shows the federal debt as a percentage of the gross domestic product (GDP), as a function of time, $D = f(t)$. (Source: Office of Management and Budget)



a Evaluate $f(1985)$ and explain its meaning.

b Solve $f(t) = 70$ and explain its meaning.

c When did the federal debt reach its highest level since 1960, and what was that level? Write your answer with function notation.

d What is the longest time interval over which the federal debt was decreasing?

Answer.

a $f(1985) = 41$: The federal debt in 1985 was about 41% of the gross domestic product.

b $t = 1942$ or $t = 1955$: The federal debt was 70% of the gross domestic product in 1942 and 1955.

c In about 1997, the debt was about 67% of the gross domestic product, so $f(1997) \approx 67.3$.

d The percentage basically dropped from 1946 to 1973, but there were small rises around 1950, 1954, 1958, and 1968, so the longest time interval was from 1958 to 1967.

20. The graph shows the elevation of the Los Angeles Marathon course as a function of the distance into the race, $a = f(t)$. (Source: Los Angeles Times, March 3, 2005)

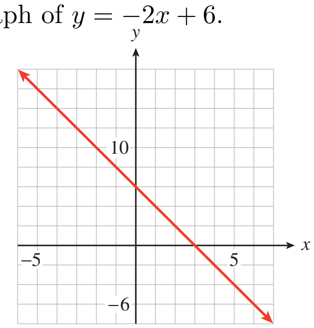
a Evaluate $f(5)$ and explain its meaning.

b Solve $f(d) = 200$ and explain its meaning.

c When does the marathon course reach its lowest elevation, and what is that elevation? Write your answer with function notation.

d Give three intervals over which the elevation is increasing.

21. The figure shows a graph of $y = -2x + 6$.



a Use the graph to find all values of x for which

I $y = 12$

II $y > 12$

III $y < 12$

b Use the graph to solve

I $-2x + 6 = 12$

II $-2x + 6 > 12$

III $-2x + 6 < 12$

c Explain why your answers to parts (a) and (b) are the same.

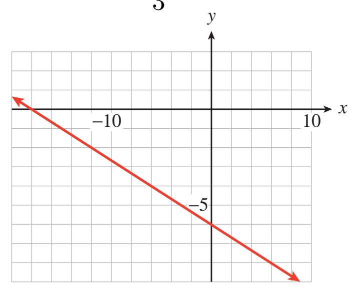
Answer.

a
   i $x = -3$
   ii $x < -3$
   iii $x > -3$

b
   I $x = -3$
   II $x < -3$
   III $x > -3$

c On the graph of $y = -2x + 6$, a value of y is the same as a value of $-2x + 6$, so parts (a) and (b) are asking for the same x's.

22. The figure shows a graph of $y = \dfrac{-x}{3} - 6$.



a Use the graph to find all values of x for which

   i $y = -4$
   ii $y > -4$
   iii $y < -4$

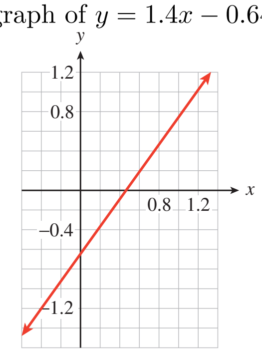
b Use the graph to solve

   i $\dfrac{-x}{3} - 6 = -4$
   ii $\dfrac{-x}{3} - 6 > -4$
   iii $\dfrac{-x}{3} - 6 < -4$

c Explain why your answers to parts (a) and (b) are the same.

In Problems 23 and 24, use the graph to solve the equation or inequality, and then solve algebraically. (To review solving linear inequalities algebraically, see Algebra Skills Refresher A.2, p. 859.)

23. The figure shows the graph of $y = 1.4x - 0.64$. Solve the following:



a $1.4x - 0.64 = 0.2$

b $-1.2 = 1.4x - 0.64$

c $1.4x - 0.64 > 0.2$

d $-1.2 > 1.4x - 0.64$

Answer.

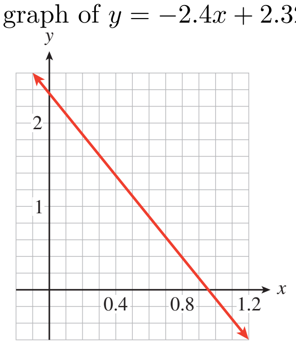
a $x = 0.6$

b $x = -0.4$

c $x > 0.6$

d $x < -0.4$

24. The figure shows the graph of $y = -2.4x + 2.32$. Solve the following:



a $1.6 = -2.4x + 2.32$

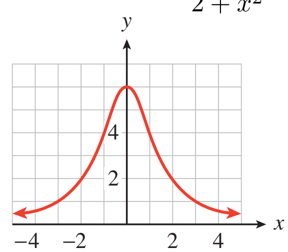
b $-2.4x + 2.32 = 0.4$

c $-2.4x + 2.32 \geq 1.6$

d $0.4 \geq -2.4x + 2.32$

For Problems 25–30, use the graphs to estimate solutions to the equations and inequalities.

25. The figure shows the graph of $g(x) = \dfrac{12}{2 + x^2}$.



a Solve $\dfrac{12}{2 + x^2} = 4$

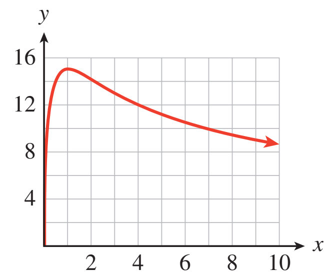
b Solve $1 \leq \dfrac{12}{2 + x^2} \leq 2$

Answer.

a $x = -1$ or $x = 1$

b Approximately $-3 \leq x \leq -2$ or $2 \leq x \leq 3$

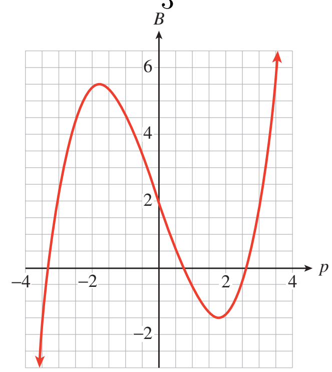
26. The figure shows the graph of $f(x) = \dfrac{30\sqrt{x}}{1 + x}$.



a Solve $\dfrac{30\sqrt{x}}{1+x} = 15$

b Solve $\dfrac{30\sqrt{x}}{1+x} < 12$

27. The figure shows a graph of $B = \dfrac{1}{3}p^3 - 3p + 2$.



a Solve $\dfrac{1}{3}p^3 - 3p + 2 = 6$

b Solve $\dfrac{1}{3}p^3 - 3p + 2 = 5$

c Solve $\dfrac{1}{3}p^3 - 3p + 2 < 1$

d What range of values does B have for p between -2.5 and 0.5?

e For what values of p is B increasing?

Answer.

a 3.5

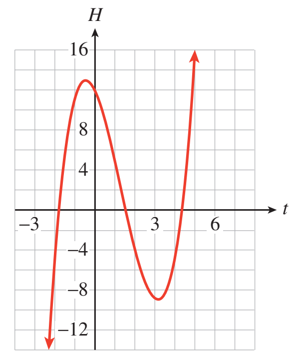
b $-2.2, -1.2, 3.4$

c $p < -3.1$ or $0.3 < p < 2.8$

d $0.5 < B < 5.5$

e $p < -1.7$ or $p > 1.7$

28. The figure shows a graph of $H = t^3 - 4t^2 - 4t + 12$.



a Solve $t^3 - 4t^2 - 4t + 12 = -4$

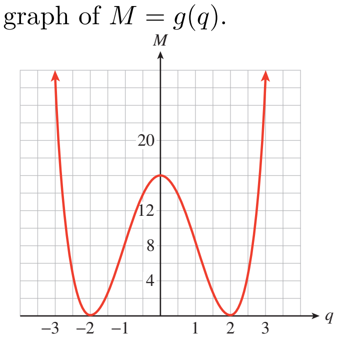
b Solve $t^3 - 4t^2 - 4t + 12 = 16$

c Solve $t^3 - 4t^2 - 4t + 12 > 6$

d Estimate the horizontal and vertical intercepts of the graph.

e For what values of t is H increasing?

29. The figure shows a graph of $M = g(q)$.



a Find all values of q for which

I $g(q) = 0$

II $g(q) = 16$

III $g(q) < 6$

b For what values of q is $g(q)$ increasing?

Answer.

a

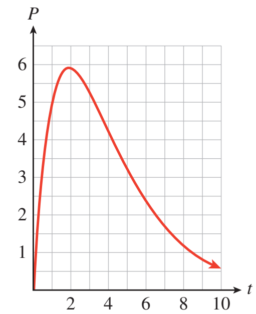
i $-2, 2$

ii $-2.8, 0, 2.8$

iii $-2.5 < q < -1.25$ or $1.25 < q < 2.5$

b $-2 < q < 0$ or $q > 2$

30. The figure shows a graph of $P = f(t)$.



a Find all values of t for which

I $f(t) = 3$

II $f(t) > 4.5$

III $2 \leq f(t) \leq 4$

b For what values of t is $f(t)$ decreasing?

31.

a Delbert reads the following values from the graph of a function:

$$f(-3) = 5,\ f(-1) = 2,\ f(1) = 0,$$

$$f(-1) = -4,\ f(-3) = -2$$

Can his readings be correct? Explain why or why not.

b Francine reads the following values from the graph of a function:

$$g(-2) = 6,\ g(0) = 0,\ g(2) = 6,$$

$$g(4) = 0,\ g(6) = 6$$

Can her readings be correct? Explain why or why not.

Answer.

a He has an error: $f(-3)$ cannot have both the value 5 and also the value -2, and $f(-1)$ cannot have both values 2 and -4.

b Her readings are possible for a function: each input has only one output.

32.

a Sketch the graph of a function that has the following values:

$$F(-2) = 3,\ F(-1) = 3,\ F(0) = 3,$$

$$F(1) = 3,\ F(2) = 3$$

b Sketch the graph of a function that has the following values:

$$G(-2) = 1,\ G(-1) = 0,\ G(0) = -1,$$

$$G(1) = 0,\ G(2) = 1$$

For Problems 33–36, graph each function in the friendly window

$$\text{Xmin} = -9.4 \qquad \text{Xmax} = 9.4$$

Ymin = −10 Ymax = 10

Then answer the questions about the graph. (See Appendix B, p. 977 for an explanation of friendly windows.)

33. $g(x) = \sqrt{36 - x^2}$

a Complete the table. (Round values to tenths.)

x	-4	-2	3	5
$g(x)$				

b Find all points on the graph for which $g(x) = 3.6$.

Answer.

a

x	-4	-2	3	5
$g(x)$	4.5	5.7	5.2	3.3

b $-4.8, 4.8$

34. $g(x) = \sqrt{x^2} - 6$

(a) Complete the table. (Round values to tenths.)

x	-8	-2	3	6
$f(x)$				

(b) Find all points on the graph for which $f(x) = -2$.

35. $F(x) = 0.5x^3 - 4x$

a Estimate the coordinates of the turning points of the graph, that is, where the graph changes from increasing to decreasing or vice versa.

b Write an equation of the form $F(a) = b$ for each turning point.

Answer.

a $(-1.6, 4.352), (1.6, -4.352)$

b $F(-1.6) = 4.352$; $F(1.6) = -4.352$

36. $G(x) = 2 + 4x - x^3$

a Estimate the coordinates of the turning points of the graph, that is, where the graph changes from increasing to decreasing or vice versa.

b Write an equation of the form $G(a) = b$ for each turning point.

For Problems 37–40, graph the function

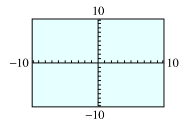
a First using the standard window.

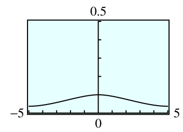
b Then using the suggested window. Explain how the window alters the appearance of the graph in each case.

37. $h(x) = \dfrac{1}{x^2 + 10}$

$\text{Xmin} = -5 \quad \text{Xmax} = 5$
$\text{Ymin} = 0 \quad \text{Ymax} = 0.5$

Answer.

a 

b 

The curve cannot be distinguished from the x-axis in the standard window because the values of y are closer to zero than the resolution of the calculator can display. The second window provides sufficient resolution to see the curve.

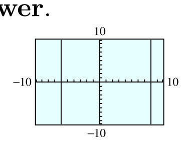
38. $H(x) = \sqrt{1 - x^2}$

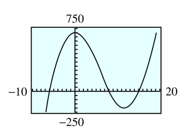
$\text{Xmin} = -2 \quad \text{Xmax} = 2$
$\text{Ymin} = -2 \quad \text{Ymax} = 2$

39. $P(x) = (x - 8)(x + 6)(x - 15)$

$\text{Xmin} = -10 \quad \text{Xmax} = 20$
$\text{Ymin} = -250 \quad \text{Ymax} = 750$

Answer.

a 

b 

The curve looks like two vertical lines in the standard window because that window covers too small a region of the plane. The second window allows us to see the turning points of the curve.

40. $p(x) = 200x^3$

$\text{Xmin} = -5 \quad \text{Xmax} = 5$
$\text{Ymin} = -10,000 \quad \text{Ymax} = 10,000$

For Problems 41–44, graph the equation with the ZInteger setting. (Press `ZOOM` 6,then `ZOOM` 8 `ENTER`.) Use the graph to answer each question. Use the equation to verify your answers.

41. Graph $y = 2x - 3$

a For what value of x is $y = 5$?

b For what value of x is $y = -13$?

c For what values of x is $y > -1$?

d For what values of x is $y < 25$?

Answer.

a $x = 4$ b $x = -5$ c $x > 1$ d $x < 14$

42. Graph $y = 4 - 2x$

a For what value of x is $y = 6$?

b For what value of x is $y = -4$?

c For what values of x is $y > -12$?

d For what values of x is $y < 18$?

43. Graph $y = 6.5 - 1.8x$

a For what value of x is $y = -13.3$?

b For what value of x is $y = 24.5$?

c For what values of x is $y \leq 15.5$?

d For what values of x is $y \geq -7.9$?

Answer.

a $x = 11$ b $x = -10$ c $x \geq -5$ d $x \leq 8$

44. Graph $y = 0.2x + 1.4$

a For what value of x is $y = -5.2$?

b For what value of x is $y = 2.8$?

c For what values of x is $y \leq -3.2$?

d For what values of x is $y \geq 4.4$?

For Problems 45–48, graph the equation with the ZInteger setting. Use the graph to solve each equation or inequality. Check your solutions algebraically.

45. Graph $y = -0.4x + 3.7$

a Solve $-0.4x + 3.7 = 2.1$

b Solve $-0.4x + 3.7 > -5.1$

Answer.

a $x = 4$ b $x < 22$

46. Graph $y = 0.4(x - 1.5)$

a Solve $0.4(x - 1.5) = -8.6$

b Solve $0.4(x - 1.5) < 8.6$

47. Graph $y = \frac{2}{3}x - 24$

a Solve $\frac{2}{3}x - 24 = -10\frac{2}{3}$

b Solve $\frac{2}{3}x - 24 \leq -19\frac{1}{3}$

Answer.

a $x = 20$

b $x \leq 7$

48. Graph $y = \dfrac{80 - 3x}{5}$.

a Solve $\dfrac{80 - 3x}{5} = 22\frac{3}{5}$.

b Solve $\dfrac{80 - 3x}{5} \leq -9\frac{2}{5}$.

49. Graph $y = 0.01x^3 - 0.1x^2 - 2.75x + 15$.

a Use your graph to solve $0.01x^3 - 0.1x^2 - 2.75x + 15 = 0$.

b Press `Y=` and enter $Y_2 = 10$. Press `GRAPH`, and you should see the horizontal line $y = 10$ superimposed on your previous graph. How many solutions does the equation

$$0.01x^3 - 0.1x^2 - 2.75x + 15 = 10$$

have? Estimate each solution to the nearest whole number.

Answer.

a $-15, 5, 20$

b $-13, 2, 22$

50. Graph $y = 2.5x - 0.025x^2 - 0.005x^3$.

a Use your graph to solve $2.5x - 0.025x^2 - 0.005x^3 = 0$.

b Press `Y=` and enter $Y_2 = -5$. Press `GRAPH`, and you should see the horizontal line $y = -5$ superimposed on your previous graph. How many solutions does the equation

$$2.5x - 0.025x^2 - 0.005x^3 = -5$$

have? Estimate each solution to the nearest whole number.

1.4 Slope and Rate of Change

1.4.1 Using Ratios for Comparison

Which is more expensive, a 64-ounce bottle of Velvolux dish soap that costs \$3.52, or a 60-ounce bottle of Rainfresh dish soap that costs \$3.36?

You are probably familiar with the notion of comparison shopping. To decide which dish soap is the better buy, we compute the unit price, or price per ounce, for each bottle. The unit price for Velvolux is

$$\frac{352 \text{ cents}}{64 \text{ ounces}} = 5.5 \text{ cents per ounce}$$

and the unit price for Rainfresh is

$$\frac{336 \text{ cents}}{60 \text{ ounces}} = 5.6 \text{ cents per ounce}$$

The Velvolux costs less per ounce, so it is the better buy. By computing the price of each brand for *the same amount of soap*, it is easy to compare them.

In many situations, a ratio, similar to a unit price, can provide a basis for comparison. Example 1.4.1, p. 83 uses a ratio to measure a rate of growth.

Example 1.4.1 Which grow faster, Hybrid A wheat seedlings, which grow 11.2 centimeters in 14 days, or Hybrid B seedlings, which grow 13.5 centimeters in 18 days?

Solution. We compute the growth rate for each strain of wheat. Growth rate is expressed as a ratio, $\frac{\text{centimeters}}{\text{days}}$, or centimeters per day. The growth rate for Hybrid A is

$$\frac{11.2 \text{ centimeters}}{14 \text{ days}} = 0.8 \text{ centimeters per day}$$

and the growth rate for Hybrid B is

$$\frac{13.5 \text{ centimeters}}{18 \text{ days}} = 0.75 \text{ centimeters per day}$$

Because their rate of growth is larger, the Hybrid A seedlings grow faster. □

By computing the growth of each strain of wheat seedling over the same unit of time, a single day, we have a basis for comparison. In this case, the ratio $\frac{\text{centimeters}}{\text{day}}$ measures the rate of growth of the wheat seedlings.

Checkpoint 1.4.2 Delbert traveled 258 miles on 12 gallons of gas, and Francine traveled 182 miles on 8 gallons of gas. Compute the ratio $\frac{\text{miles}}{\text{gallon}}$ for each car. Whose car gets the better gas mileage?

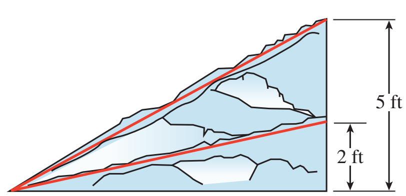
Answer. Delbert: 21.5 mpg, Francine: 22.75 mpg. Francine gets better mileage.

In Checkpoint 1.4.2, p. 83, the ratio $\frac{\text{miles}}{\text{gallon}}$ measures the rate at which each car uses gasoline. By computing the mileage for each car for the same amount of gas, we have a basis for comparison. We can use this same idea, finding a common basis for comparison, to measure the steepness of an incline.

1.4.2 Measuring Steepness

Imagine you are an ant carrying a heavy burden along one of the two paths shown below. Which path is more difficult? Most ants would agree that the steeper path is more difficult.

But what exactly is steepness? It is not merely the gain in altitude, because even a gentle incline will reach a great height eventually. Steepness measures how sharply the altitude increases. An ant finds the second path more difficult, or steeper, because it rises 5 feet while the first path rises only 2 feet over the same horizontal distance.



To compare the steepness of two inclined paths, we compute the ratio of change in altitude to change in horizontal distance for each path.

Example 1.4.3 Which is steeper, Stony Point trail, which climbs 400 feet over a horizontal distance of 2500 feet, or Lone Pine trail, which climbs 360 feet over a horizontal distance of 1800 feet?

Solution. For each trail, we compute the ratio of vertical gain to horizontal distance. For Stony Point trail, the ratio is

$$\frac{400 \text{ feet}}{2500 \text{ feet}} = 0.16$$

and for Lone Pine trail, the ratio is

$$\frac{360 \text{ feet}}{1800 \text{ feet}} = 0.20$$

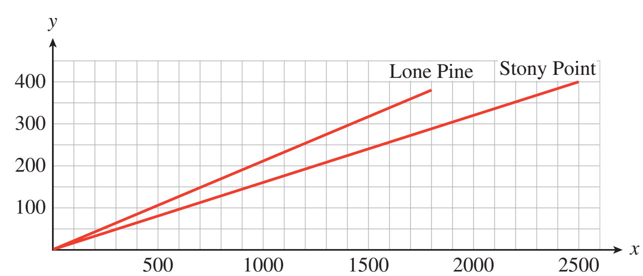
Lone Pine trail is steeper, because it has a vertical gain of 0.20 foot for every foot traveled horizontally. Or, in more practical units, Lone Pine trail rises 20 feet for every 100 feet of horizontal distance, whereas Stony Point trail rises only 16 feet over a horizontal distance of 100 feet. □

Checkpoint 1.4.4 Which is steeper, a staircase that rises 10 feet over a horizontal distance of 4 feet, or the steps in the football stadium, which rise 20 yards over a horizontal distance of 12 yards?

Answer. The staircase is steeper.

1.4.3 Definition of Slope

To compare the steepness of the two trails in Example 1.4.3, p. 84, it is not enough to know which trail has the greater gain in elevation overall. Instead, we compare their elevation gains over the same horizontal distance. Using the same horizontal distance provides a basis for comparison. The two trails are illustrated below as lines on a coordinate grid.



The ratio we computed in Example 1.4.3, p. 84,

$$\frac{\text{change in elevation}}{\text{change in horizontal position}}$$

appears on the graphs as

$$\frac{\text{change in } y\text{-coordinate}}{\text{change in } x\text{-coordinate}}$$

For example, as we travel along the line representing Stony Point trail, we move from the point $(0, 0)$ to the point $(2500, 400)$. The y-coordinate changes by 400 and the x-coordinate changes by 2500, giving the ratio 0.16 that we found in Example 1.4.3, p. 84. We call this ratio the **slope** of the line.

Definition of Slope.

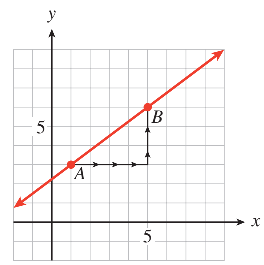
The **slope** of a line is the ratio

$$\frac{\text{change in } y\text{-coordinate}}{\text{change in } x\text{-coordinate}}$$

as we move from one point to another on the line.

Example 1.4.5

Compute the slope of the line that passes through points A and B on the graph at right.



Solution. As we move along the line from A to B, the y-coordinate changes by 3 units, and the x-coordinate changes by 4 units. The slope of the line is thus

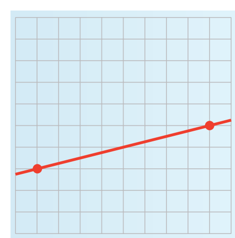
$$\frac{\text{change in } y\text{-coordinate}}{\text{change in } x\text{-coordinate}} = \frac{3}{4}$$

□

Checkpoint 1.4.6

Compute the slope of the line through the indicated points on the graph at right. On both axes, one square represents one unit.

$$\frac{\text{change in } y\text{-coordinate}}{\text{change in } x\text{-coordinate}} =$$



Answer. $\frac{1}{4}$

Note 1.4.7 The slope of a line is a *number*. It tells us how much the y-coordinates of points on the line increase when we increase their x-coordinates by 1 unit. For instance, the slope $\frac{3}{4}$ in Example 1.4.5, p. 85 means that the y-coordinate increases by $\frac{3}{4}$ unit when the x-coordinate increases by 1 unit. For increasing graphs, a larger slope indicates a greater increase in altitude, and hence a steeper line.

1.4.4 Notation for Slope

We use a shorthand notation for the ratio that defines slope,

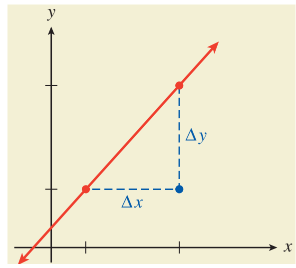
$$\frac{\text{change in } y\text{-coordinate}}{\text{change in } x\text{-coordinate}}$$

The symbol Δ (the Greek letter delta) is used in mathematics to denote *change in*. In particular, Δy means *change in y-coordinate*, and Δx means *change in x-coordinate*. We also use the letter m to stand for slope. With these symbols, we can write the definition of slope as follows.

Notation for Slope.

The **slope** of a line is given by

$$\frac{\Delta y}{\Delta x} = \frac{\text{change in } y\text{-coordinate}}{\text{change in } x\text{-coordinate}}, \quad \Delta x \neq 0$$



Example 1.4.8 The Great Pyramid of Khufu in Egypt was built around 2550 B.C. It is 147 meters tall and has a square base 229 meters on each side. Calculate the slope of the sides of the pyramid, rounded to two decimal places.

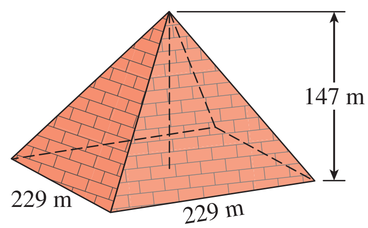
Solution.

From the figure, we see that Δx is only half the base of the Great Pyramid, so

$$\Delta x = 0.5(229) = 114.5$$

and the slope of the side is

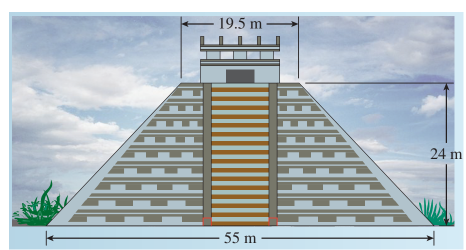
$$m = \frac{\Delta y}{\Delta x} = \frac{147}{114.5} = 1.28$$



□

Checkpoint 1.4.9 The Kukulcan Pyramid at Chichen Itza in Mexico was built around 800 A.D. It is 24 meters high, with a temple built on its top platform, as shown below.

The square base is 55 meters on each side, and the top platform is 19.5 meters on each side. Calculate the slope of the sides of the pyramid. Which pyramid is steeper, Kukulcan or the Great Pyramid?



Answer. 1.35; Kukulcan is steeper.

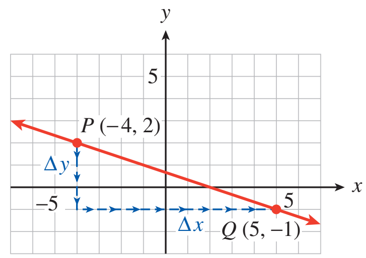
So far, we have only considered examples in which Δx and Δy are positive numbers, but they can also be negative.

$$\Delta x = \begin{cases} \text{positive if } x \text{ increases (move to the right)} \\ \text{negative if } x \text{ decreases (move to the left)} \end{cases}$$

$$\Delta y = \begin{cases} \text{positive if } y \text{ increases (move up)} \\ \text{negative if } y \text{ decreases (move down)} \end{cases}$$

Example 1.4.10

Compute the slope of the line that passes through the points $P(-4, 2)$ and $Q(5, -1)$ shown in at right. Illustrate Δy and Δx on the graph.



Solution. As we move from the point $P(-4, 2)$ to the point $Q(5, -1)$, we move 3 units *down*, so $\Delta y = -3$. We then move 9 units to the right, so $\Delta x = 9$.Thus, the slope is

$$m = \frac{\Delta y}{\Delta x} = \frac{-3}{9} = \frac{-1}{3}$$

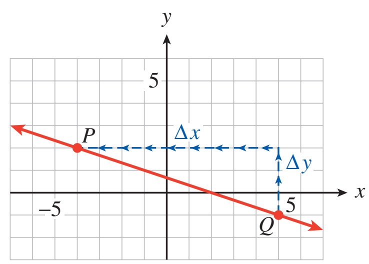
Δy and Δx are labeled on the graph. □

We can move from point to point in either direction to compute the slope. The line graphed in Example 1.4.10, p. 86 decreases as we move from left to right and hence has a negative slope.

The slope is the same if we move from point Q to point P instead of from P to Q, as shown at right. In that case, our computation looks like this:

$$m = \frac{\Delta y}{\Delta x} = \frac{3}{-9} = \frac{-1}{3}$$

Δy and Δx are labeled on the graph.



1.4.5 Lines Have Constant Slope

How do we know which two points to choose when we want to compute the slope of a line? It turns out that any two points on the line will do.

Checkpoint 1.4.11

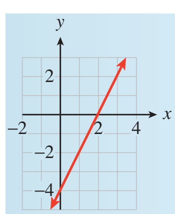
a Graph the line $4x - 2y = 8$ by finding the x- and y-intercepts

b Compute the slope of the line using the x-intercept and y-intercept.

c Compute the slope of the line using the points $(4, 4)$ and $(1, -2)$.

Answer.

a

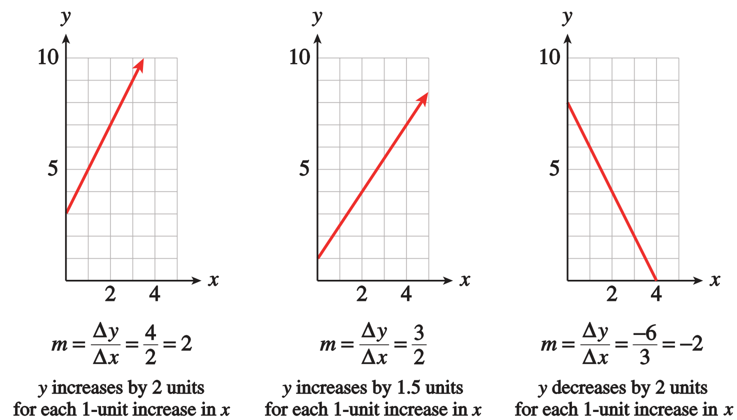


b 2

c 2

Note 1.4.12 Checkpoint 1.4.11, p. 87 illustrates an important property of lines: They have constant slope. No matter which two points we use to calculate the slope, we will always get the same result. We will see later that lines are the only graphs that have this property.

We can think of the slope as a scale factor that tells us how many units y increases (or decreases) for each unit of increase in x. Compare the lines shown below.



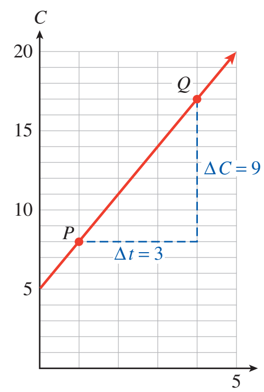
Observe that a line with positive slope increases from left to right, and one with negative slope decreases. What sort of line has slope $m = 0$?

1.4.6 Meaning of Slope

In Example 1 of Section 1.1, p. 3, we graphed the equation $C = 5+3t$ showing the cost of a bicycle rental in terms of the length of the rental. The graph is reproduced at right. We can choose any two points on the line to compute its slope. Using points P and Q as shown, we find that

$$m = \frac{\Delta C}{\Delta t} = \frac{9}{3} = 3$$

The slope of the line is 3.



What does this value mean for the cost of renting a bicycle? The expression

$$\frac{\Delta C}{\Delta t} = \frac{9}{3}$$

stands for

$$\frac{\text{change in cost}}{\text{change in time}} = \frac{9 \text{ dollars}}{3 \text{ hours}}$$

If we increase the length of the rental by 3 hours, the cost of the rental increases by 9 dollars. The slope gives the rate of increase in the rental fee, 3 dollars per hour. In general, we can make the following statement.

Rate of Change.

The slope of a line measures the **rate of change** of the output variable with respect to the input variable.

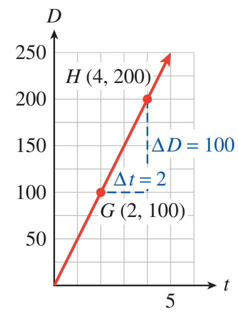
Depending on the variables involved, this rate might be interpreted as a rate of growth or a rate of speed. A negative slope might represent a rate of decrease or a rate of consumption. The slope of a graph can give us valuable information about the variables.

Example 1.4.13

The graph at right shows the distance in miles traveled by a big-rig truck driver after t hours on the road.

a Compute the slope of the graph.

b What does the slope tell us about the problem?



Solution.

a Choose any two points on the line, say $G(2, 100)$ and $H(4, 200)$, as shown. As we move from G to H, we find

$$m = \frac{\Delta D}{\Delta t} = \frac{100}{2} = 50$$

The slope of the line is 50.

b The best way to understand the slope is to include units in the calculation. For our example,

$$\frac{\Delta D}{\Delta t} \quad \text{means} \quad \frac{\text{change in distance}}{\text{change in time}}$$

or

$$\frac{\Delta D}{\Delta t} = \frac{100 \text{ miles}}{2 \text{ hours}} = 50 \text{ miles per hour}$$

The slope represents the trucker's average speed or velocity.

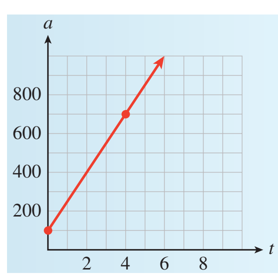
□

Checkpoint 1.4.14

The graph shows the altitude, a (in feet), of a skier t minutes after getting on a ski lift.

a Choose two points and compute the slope (including units).

b What does the slope tell us about the problem?



Answer.

a 150

b Altitude increases by 150 feet per minute.

1.4.7 A Formula for Slope

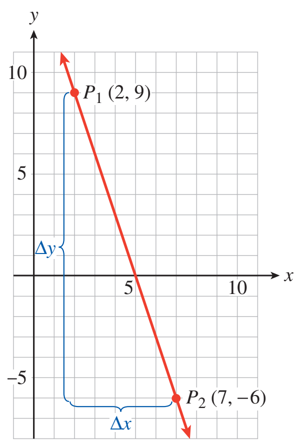
We have defined the slope of a line to be the ratio $m = \dfrac{\Delta y}{\Delta x}$ as we move from one point to another on the line. So far, we have computed Δy and Δx by counting squares on the graph, but this method is not always practical. All we really need are the coordinates of two points on the graph.

Subscripts.

We will use **subscripts** to distinguish the two points:

P_1 means "first point" and P_2 means "second point."

We denote the coordinates of P_1 by (x_1, y_1) and the coordinates of P_2 by (x_2, y_2).



Now consider a specific example. The line through the two points $P_1(2, 9)$ and $P_2(7, -6)$ is shown at left. We can find Δx by subtracting the x-coordinates of the points:

$$\Delta x = 7 - 2 = 5$$

In general, we have

$$\Delta x = x_2 - x_1$$

and similarly

$$\Delta y = y_2 - y_1$$

These formulas work even if some of the coordinates are negative; in our example

$$\Delta y = y_2 - y_1 = -6 - 9 = -15$$

By counting squares *down* from P_1 to P_2, we see that Δy is indeed -15. The slope of the line is

$$m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-15}{5} = -3$$

We now have a formula for the slope of a line that works even if we do not have a graph.

Two-Point Slope Formula.

The slope of the line passing through the points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ is given by

$$m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}, \quad x_2 \neq x_1$$

Example 1.4.15 Compute the slope of the line above using the points $Q_1(6, -3)$ and $Q_2(4, 3)$.

Solution. We substitute the coordinates of Q_1 and Q_2 into the slope formula to find

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{3 - (-3)}{4 - 6} = \frac{6}{-2} = -3$$

This value for the slope, -3, is the same value we found above. □

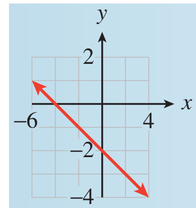
Checkpoint 1.4.16

a Find the slope of the line passing through the points $(2, -3)$ and $(-2, -1)$.

b Sketch a graph of the line by hand.

Answer.

a $\frac{-1}{2}$

b 

It will also be useful to write the slope formula with function notation. Recall that $f(x)$ is another symbol for y, and, in particular, that $y_1 = f(x_1)$ and $y_2 = f(x_2)$. Thus, if $x_2 \neq x_1$, we have this formula.

Slope Formula in Function Notation.

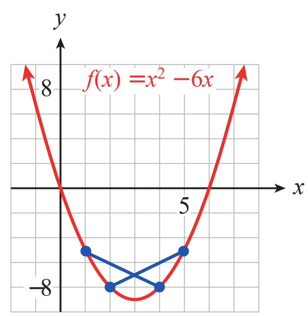
$$m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}, \qquad x_2 \neq x_1$$

Example 1.4.17
The figure shows a graph of

$$f(x) = x^2 - 6x$$

a Compute the slope of the line segment joining the points at $x = 1$ and $x = 4$.

b Compute the slope of the line segment joining the points at $x = 2$ and $x = 5$.



Solution.

a We set $x_1 = \mathbf{1}$ and $x_2 = \mathbf{4}$ and find the function values at each point.

$$f(x_1) = f(\mathbf{1}) = \mathbf{1}^2 - 6(\mathbf{1}) = -5$$

$$f(x_2) = f(\mathbf{4}) = \mathbf{4}^2 - 6(\mathbf{4}) = -8$$

Then

$$m = \frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{-8 - (-5)}{4 - 1} = \frac{-3}{3} = -1$$

b We set $x_1 = \mathbf{2}$ and $x_2 = \mathbf{5}$ and find the function values at each point.

$$f(x_1) = f(\mathbf{2}) = \mathbf{2}^2 - 6(\mathbf{2}) = -8$$

$$f(x_2) = f(\mathbf{5}) = \mathbf{5}^2 - 6(\mathbf{5}) = -5$$

Then

$$m = \frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{-5 - (-8)}{5 - 2} = \frac{3}{3} = 1$$

□

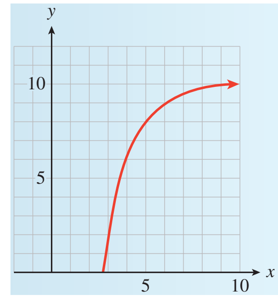
Note that the graph of f is not a straight line and that the slope is not constant.

Checkpoint 1.4.18 The figure shows the graph of a function f.

a Find $f(3)$ and $f(5)$.

b Compute the slope of the line segment joining the points at $x = 3$ and $x = 5$.

c Write an expression for the slope of the line segment joining the points at $x = a$ and $x = b$.



Answer.

a $f(3) = 2,\ f(5) = 8$

b 3

c $\dfrac{f(b) - f(a)}{b - a}$

1.4.8 Section Summary

1.4.8.1 Vocabulary

Look up the definitions of new terms in the Glossary.

- Ratio
- Rate of change
- Slope
- Scale factor

1.4.8.2 CONCEPTS

1 We can use ratios to compare quantities.

2 The slope ratio, $\dfrac{\text{change in y-coordinate}}{\text{change in } x\text{-coordinate}}$, measures the steepness of a line.

3 Notation for slope: $m = \dfrac{\Delta y}{\Delta x}$, $\Delta x \neq 0$.

4 Formula for slope: $m = \dfrac{y_2 - y_1}{x_2 - x_1}, \quad x_2 \neq x_1$

5 Formula for slope: $m = \dfrac{f(x_2) - f(x_1)}{x_2 - x_1}, \quad x_2 \neq x_1$

6 Lines have constant slope.

7 Slope is a scale factor that tells us how many units Δy increases for each unit increase in Δx as we move along the line.

8 The slope gives us the rate of change.

1.4.8.3 STUDY QUESTIONS

1 Explain how to compare prices with unit pricing.

2 Why is Δy the numerator of the slope ratio and Δx the denominator?

3 Which line is steeper, one with $m = -2$ or one with $m = -5$?

4 A classmate says that you must always use the intercepts to calculate the slope of a line. Do you agree? Explain.

5 In an application, what does the slope of the graph tell you about the situation?

1.4.8.4 SKILLS

Practice each skill in the Homework 1.4.9, p. 93 problems listed.

1 Use ratios for comparison: #1–4

2 Compute slope from a graph: #5–16, 23–26

3 Use slope to find Δy or Δx: #17–20, 27–30

4 Use slope to compare steepness: #21 and 22

5 Decide whether data points lie on a straight line: #41–46

6 Interpret slope as a rate of change: #31–40

7 Use function notation to discuss graphs and slope: #53–62

1.4.9 Slope and Rate of Change (Homework 1.4)

Compute ratios to answer the questions in Problems 1–4.

1. Carl runs 100 meters in 10 seconds. Anthony runs 200 meters in 19.6 seconds. Who has the faster average speed?

Answer. Anthony

2. On his 512-mile round trip to Las Vegas and back, Corey needed 16 gallons of gasoline. He used 13 gallons of gasoline on a 429-mile trip to Los Angeles. On which trip did he get better fuel economy?

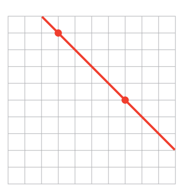
3. Grimy Gulch Pass rises 0.6 miles over a horizontal distance of 26 miles. Bob's driveway rises 12 feet over a horizontal distance of 150 feet. Which is steeper?

Answer. Bob's driveway

4. Which is steeper, the truck ramp for Acme Movers, which rises 4 feet over a horizontal distance of 9 feet, or a toy truck ramp, which rises 3 centimeters over a horizontal distance of 7 centimeters?

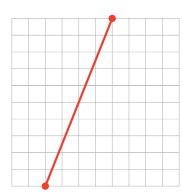
In Problems 5-8, compute the slope of the line through the indicated points. On both axes, one square represents one unit.

5.

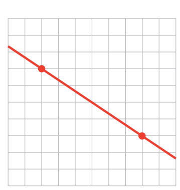


Answer. -1

6.

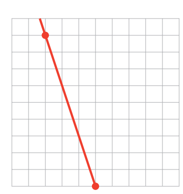


7.



Answer. $\frac{-2}{3}$

8.



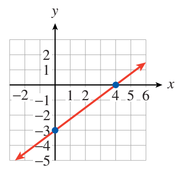
For Problems 9–14,

a Graph each line by the intercept method.

b Use the intercepts to compute the slope.

9. $3x - 4y = 12$

Answer.

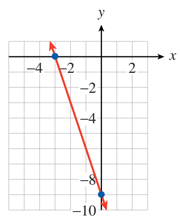
a 

b $\frac{3}{4}$

10. $2y - 5x = 10$

11. $2y + 6x = -18$

Answer.

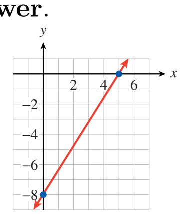
a 

b -3

12. $9x + 12y = 36$

13. $\frac{x}{5} - \frac{y}{8} = 1$

Answer.

a 

b $\frac{8}{5}$

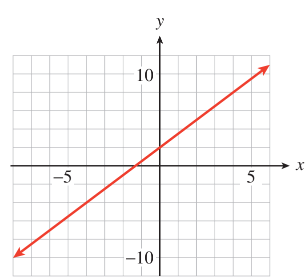
14. $\frac{x}{7} - \frac{y}{4} = 1$

15.

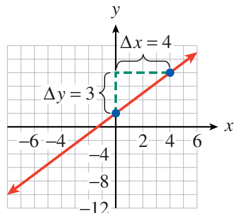
a Use the points $(0, 2)$ and $(4, 8)$ to compute the slope of the line. Illustrate Δy and Δx on the graph.

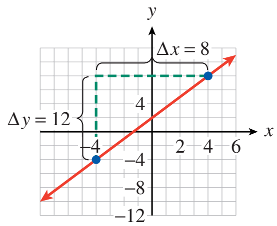
b Use the points $(-4, -4)$ and $(4, 8)$ to compute the slope of the line. Illustrate Δy and Δx on the graph.

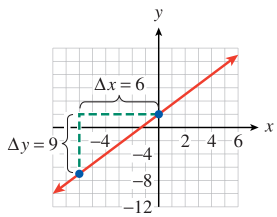
c Use the points $(0, 2)$ and $(-6, -7)$ to compute the slope of the line. Illustrate Δy and Δx on the graph.



Answer.

a 

b 

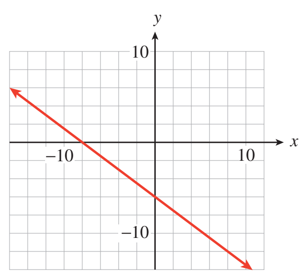
c 

16.

a Use the points $(0, -6)$ and $(8, -12)$ to compute the slope of the line. Illustrate Δy and Δx on the graph.

b Use the points $(-8, 0)$ and $(4, -9)$ to compute the slope of the line. Illustrate Δy and Δx on the graph.

c Use the points $(4, -9)$ and $(0, -6)$ to compute the slope of the line. Illustrate Δy and Δx on the graph.



For Problems 17–20, use the formula $m = \dfrac{\Delta y}{\Delta x}$

17. A line has slope $\dfrac{-3}{4}$.

a Find the vertical change associated with each horizontal change along the line.

i $\Delta x = 4$

ii $\Delta x = -8$

iii $\Delta x = 2$

iv $\Delta x = -6$

b Find the horizontal change associated with each vertical change along the line.

i $\Delta y = 3$
ii $\Delta y = -6$
iii $\Delta y = -2$
iv $\Delta y = 1$

Answer.

a i -3 ii 6 iii $\frac{-3}{2}$ iv $\frac{9}{2}$

b i -4 ii 8 iii $\frac{8}{3}$ iv $\frac{4}{3}$

18. A line has slope $\frac{5}{3}$.

a Find the vertical change associated with each horizontal change along the line.

i $\Delta x = 3$
ii $\Delta x = -6$
iii $\Delta x = 1$
iv $\Delta x = -24$

b Find the horizontal change associated with each vertical change along the line.

i $\Delta y = -5$
ii $\Delta y = -2.5$
iii $\Delta y = -1$
iv $\Delta y = 3$

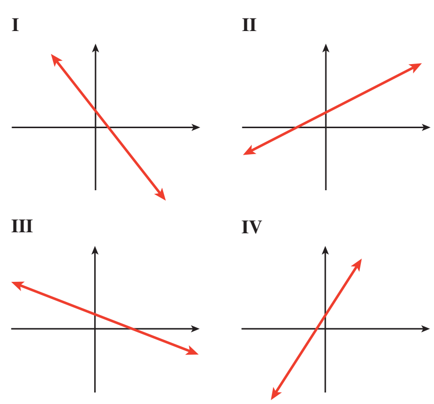
19. Residential staircases are usually built with a slope of 70%, or $\frac{7}{10}$. If the vertical distance between stories is 10 feet, how much horizontal space does the staircase require?

Answer. $\frac{100}{7}$ ft ≈ 14.286 ft ≈ 14 ft 3.4 in

20. A straight section of highway in the Midwest maintains a grade (slope) of 4%, or $\frac{1}{25}$, for 12 miles. How much does your elevation change as you travel the road?

21. Choose the line with the correct slope. The scales are the same on both axes.

a $m = 2$ b $m = -\frac{1}{2}$ c $m = \frac{2}{3}$ d $m = -\frac{5}{3}$

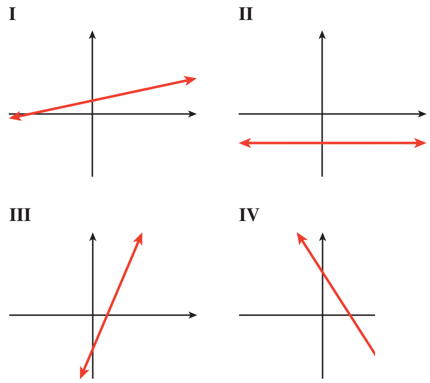


Answer.

a IV b III c II d I

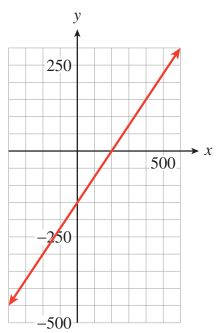
22. Choose the line with the correct slope. The scales are the same on both axes.

a $0 < m < 1$ b $m < -1$ c $m => 1$ d $m = 0$



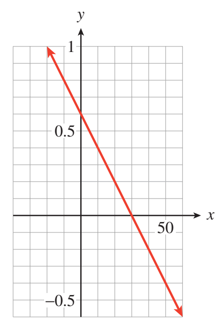
Compute the slope of the line in Problems 23-26. Note the scales on the axes.

23.

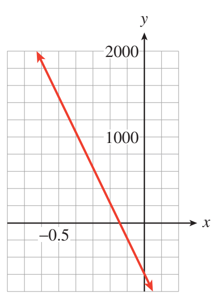


Answer. $\frac{3}{4}$

24.

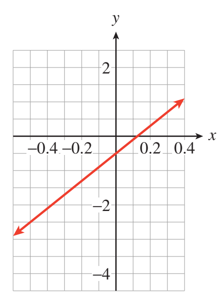


25.



Answer. -4000

26.



Each table in Problems 27–30 gives the coordinates of points on a line.

a Find the slope of the line.

b Fill in the missing table entries.

27.

x	y
-4	-14
-2	-9
2	1
3	
	11

28.

x	y
-5	-3.8
-1	-0.6
2	1.8
	4.2
7	

29.

x	y
-3	36
-1	
	12
6	9
10	-3

30.

x	y
-10	800
-2	
5	440
	368
16	176

Answer (27).

a $\frac{5}{2}$

b

x	y
3	$\frac{7}{2}$
6	11

Answer (29).

a -3

b

x	y
-1	**30**
5	12

31. A temporary typist's paycheck (before deductions) is given, in dollars, by $S = 8t$, where t is the number of hours she worked.

(a) Make a table of values for the function.

t	4	8	20	40
S				

(b) Graph the function.

(c) Using two points on the graph, compute the slope $\dfrac{\Delta S}{\Delta t}$, including units.

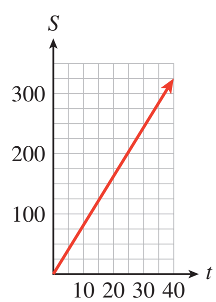
(d) What does the slope tell us about the typist's paycheck?

Answer.

a

t	4	8	20	40
S	32	64	160	320

b



c 8 dollars/hour

d The typist is paid \$8 per hour.

32. The distance (in miles) covered by a cross-country competitor is given by $d = 6t$, where t is the number of hours she runs.

(a) Make a table of values for the function.

t	2	4	6	8
d				

(b) Graph the function.

(c) Using two points on the graph, compute the slope $\dfrac{\Delta d}{\Delta t}$, including units.

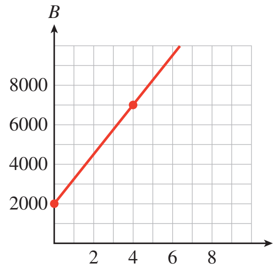
(d) What does the slope tell us about the cross-country runner?

In Problems 33–40,

a Choose two points and compute the slope of the graph (including units).

b Explain what the slope measures in the context of the problem.

33. The graph shows the number of barrels of oil, B, that has been pumped at a drill site t days after a new drill is installed.

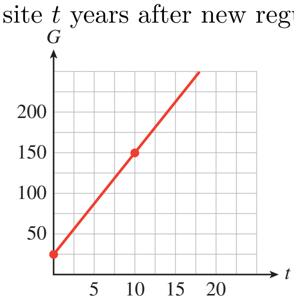


Answer.

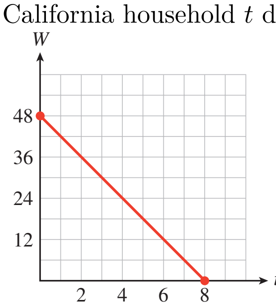
a 1250 barrels/day

b The slope indicates that oil is pumped at a rate of 1250 barrels per day.

34. The graph shows the amount of garbage, G (in tons), that has been deposited at a dump site t years after new regulations go into effect.



35. The graph shows the amount of emergency water, W (in liters), remaining in a southern California household t days after an earthquake.

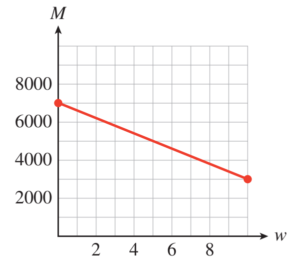


Answer.

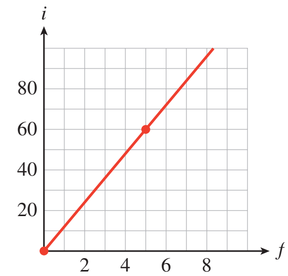
a -6 liters/day

b The slope indicates that the water is diminishing at a rate of 6 liters per day.

36. The graph shows the amount of money, M (in dollars), in Tammy's bank account w weeks after she loses all sources of income.



37. The graph shows the length in inches, i, corresponding to various lengths in feet f.

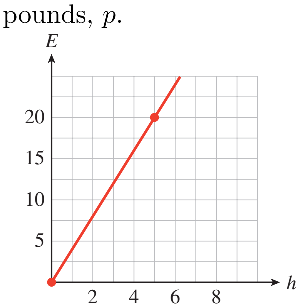


Answer.

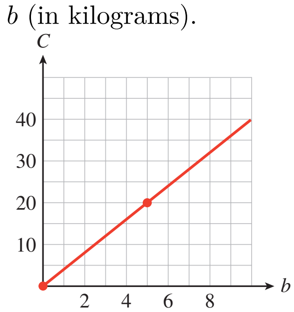
a 12 inches/foot

b The slope gives the conversion rate of 12 inches per foot.

38. The graph shows the number of ounces, z, that correspond to various weights measured in pounds, p.



39. The graph shows the cost, C (in dollars), of coffee beans in terms of the amount of coffee, b (in kilograms).

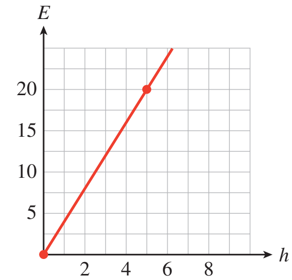


Answer.

a 4 dollars/kilogram

b The slope gives the unit price of \$4 per kilogram

40. The graph shows Tracey's earnings, E (in dollars), in terms of the number of hours, h, that she babysits.



Which of the tables in Problems 41 and 42 represent variables that are related by a linear function? (Hint: Which relationships have constant slope?)

41.

a

x	y
2	12
3	17
4	22
5	27
6	32

b

t	P
2	4
3	9
4	16
5	25
6	36

Answer. (a)

42.

a

h	w
−6	20
−3	18
0	16
3	14
6	12

b

t	d
5	0
10	3
15	6
20	12
25	24

43. The table shows the amount of ammonium chloride salt, in grams, that can be dissolved in 100 grams of water at different temperatures.

Temperature,°C	10	12	15	21	25	40	52
Grams of salt	33	34	35.5	38.5	40.5	48	54

a If you plot the data, will the points lie on a straight line? Why or why not?

b Calculate the rate of change of salt dissolved with respect to temperature.

Answer.

a Yes, the slope between any two points is $\frac{1}{2}$.

b 0.5 grams of salt per degree Celsius

44. A spring is suspended from the ceiling. The table shows the length of the spring, in centimeters, as it is stretched by hanging various weights from it.

Weight, kg	3	4	8	10	12	15	22
Length, cm	25.87	25.88	26.36	26.6	26.84	27.2	28.04

a If you plot the data, will the points lie on a straight line? Why or why not?

b Calculate the rate of change of length with respect to weight.

45. The table gives the radius and circumference of various circles, rounded to three decimal places.

r	C
4	25.133
6	37.699
10	62.832
15	94.248

a If we plot the data, will the points lie on a straight line?

b What familiar number does the slope turn out to be? (Hint: Recall a formula from geometry.)

Answer.

a Yes

b 2π

46. The table gives the side and the diagonal of various squares, rounded to three decimal places.

s	d
3	4.243
6	8.485
8	11.314
10	14.142

a If we plot the data, will the points lie on a straight line?

b What familiar number does the slope turn out to be? (Hint: Draw a picture of one of the squares and use the Pythagorean theorem to compute its diagonal.)

47. Geologists can measure the depth of the ocean at different points using a technique called echo-sounding. Scientists on board a ship send a pulse of sound toward the ocean floor and measure the time interval until the echo returns to the ship. The speed of sound in seawater is about 1500 meters per second.

a Write the speed of sound as a ratio.

b If the echo returns in 4.5 seconds, what is the depth of the ocean at that point?

Answer.

a $\dfrac{1500 \text{ meters}}{1 \text{ second}}$

b 3375 meters

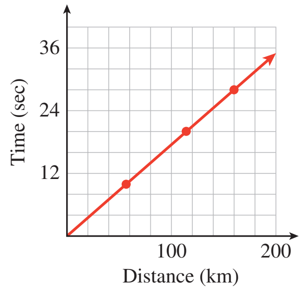
48. Niagara Falls was discovered by Father Louis Hennepin in 1682. In 1952, much of the water of the Niagara River was diverted for hydroelectric power, but until that time erosion caused the Falls to recede upstream by 3 feet per year.

a How far did the Falls recede from 1682 to 1952?

b The Falls were formed about $12,000$ years ago during the end of the last ice age. How far downstream from their current position were they then? (Give your answer in miles.)

49. Geologists calculate the speed of seismic waves by plotting the travel times for waves to reach seismometers at known distances from the epicenter.

The speed of the wave can help them determine the nature of the material it passes through. The graph shows a travel-time graph for P-waves from a shallow earthquake.



a Why do you think the graph is plotted with distance as the input variable?

b Use the graph to calculate the speed of the wave.

Answer.

a The distances are known.

b 5.7 km per second

50. Energy (supplied by heat) is required to raise the temperature of a substance, and it is also needed to melt a solid substance to a liquid. The table shows data from heating a solid sample of stearic acid. Heat was applied at a constant rate throughout the experiment. (Source: J. A. Hunt and A. Sykes, 1984)

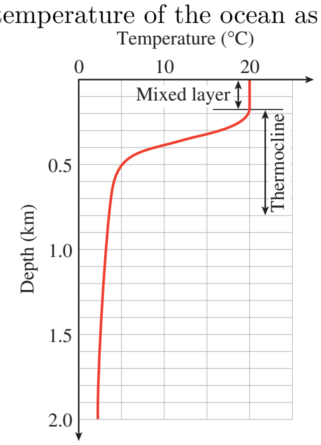
Time (minutes	0	0.5	1.5	2	2.5	3	4	5	6	7	8	8.5	9	9.5	10
Temperature, °C	19	29	40	48	53	55	55	55	55	55	55	64	70	73	74

a Did the temperature rise at a constant rate? Describe the temperature as a function of time.

b Graph temperature as a function of time.

c What is the melting point of stearic acid? How long did it take the sample to melt?

51. The graph shows the temperature of the ocean as a function of depth.



a What is the difference in temperature between the surface of the ocean and the deepest level shown?

b Over what depths does the temperature change most rapidly?

c What is the average rate of change of temperature with respect to depth in the region called the thermocline?

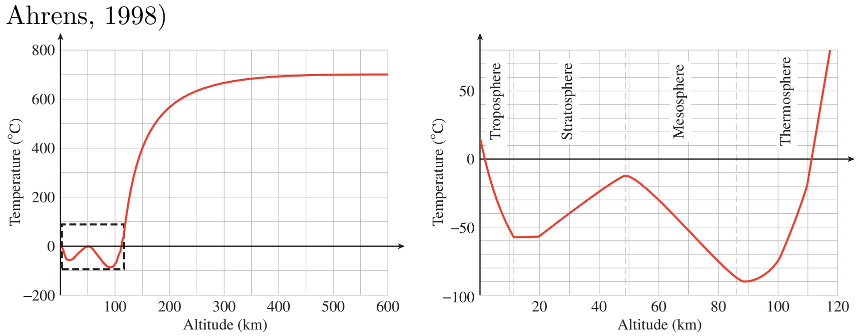
Answer.

a About 18°C

b 0.3 km to 0.4 km

c About −28°C per kilometer

52. The graph shows the average air temperature as a function of altitude. (Figure (b) is an enlargement of the indicated region of Figure (a).) (Source: Ahrens, 1998)



(a) (b)

a Is temperature a decreasing function of altitude?

b The **lapse rate** is the rate at which the temperature changes with altitude. In which regions of the atmosphere is the lapse rate positive?

c The region where the lapse rate is zero is called the isothermal zone. Give an interval of altitudes that describes the isothermal zone.

d What is the lapse rate in the mesosphere?

e Describe the temperature for altitudes greater than 90 kilometers.

In Problems 53–56, evaluate the function at $x = a$ and $x = b$, and then find the slope of the line segment joining the two corresponding points on the graph. Illustrate the line segment on a graph of the function.

53. $f(x) = x^2 - 2x - 8$

a $a = -2, b = 1$

b $a = -1, b = 5$

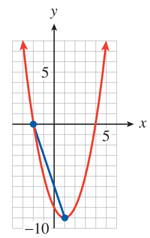
54. $g(x) = \sqrt{x+4}$

a $a = -2, b = 0$

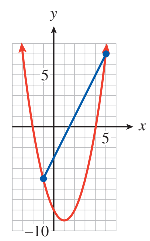
b $a = 0, b = 5$

Answer.

a -3



b 2



55. $h(x) = \dfrac{4}{x+2}$

a $a = 0, b = 6$

b $a = -1, b = 2$

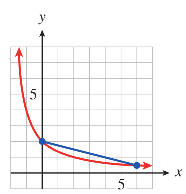
56. $q(x) = x^3 - 4x$

a $a = -1, b = 2$

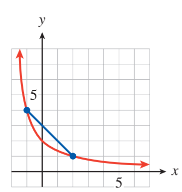
b $a = -1, b = 3$

Answer.

a $\dfrac{-1}{4}$



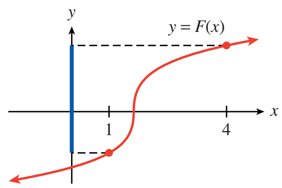
b -1



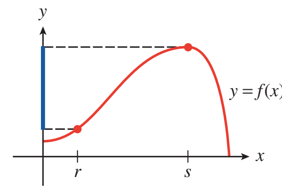
In Problems 57–62, find the coordinates of the indicated points, then write an algebraic expression using function notation for the indicated quantity.

57. The length of the vertical line segment on the y-axis

a



b



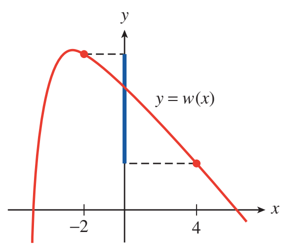
Answer.

a $(1, F(1)), (4, F(4));\qquad F(4) - F(1)$

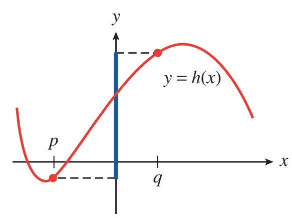
b $(r, f(r)), (s, f(s));\qquad f(s) - f(r)$

58. The length of the vertical line segment on the y-axis

a

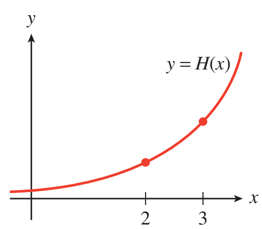


b

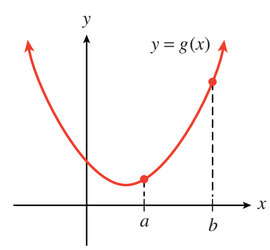


59.

a The increase in y as x increases from 2 to 3



b The increase in y as x increases from a to b



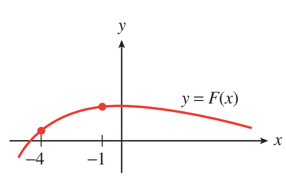
Answer.

a $(2, H(2)), (3, H(3));\qquad H(3) - H(2)$

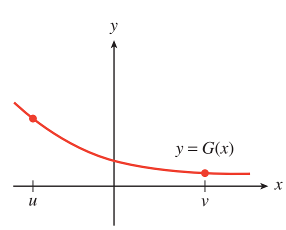
b $(a, g(a)), (b, g(b));\qquad g(b) - g(a)$

60.

a The increase in y as x increases from -4 to -1

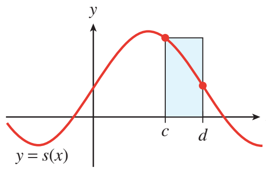


b The increase in y as x increases from u to v

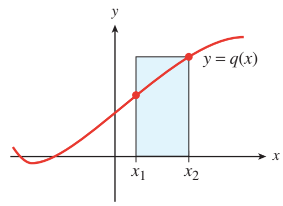


61. The shaded area

a



b



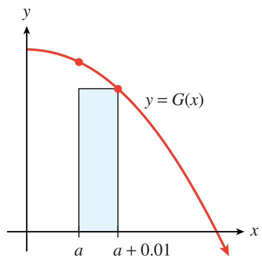
Answer.

a $(c, s(c)), (d, s(d));$ $\quad s(c)(d-c)$

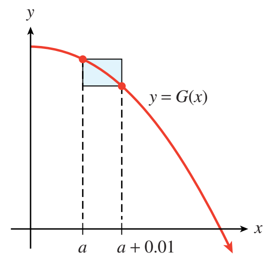
b $(x_1, q(x_1)), (x_2, q(x_2));$ $\quad q(x_2)(x_2-x_1)$

62. The shaded area

a



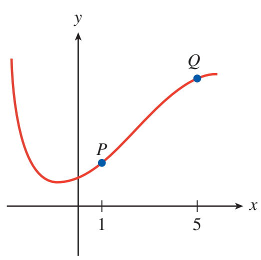
b



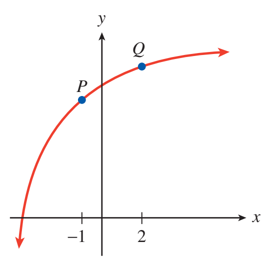
In Problems 63–66, find the coordinates of the indicated points on the graph of $y = f(x)$ and write an algebraic expression using function notation for the slope of the line segment joining points P and Q.

63.

a



b



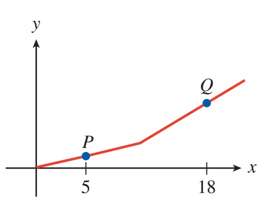
Answer.

a $(1, f(1)), (5, f(5));$ $\quad \dfrac{f(5)-f(1)}{4}$

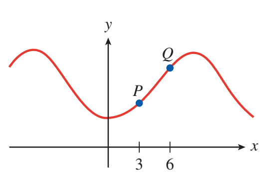
b $(-1, f(-1)), (2, f(2));$ $\quad \dfrac{f(2)-f(-1)}{3}$

64.

a

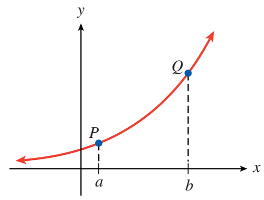


b

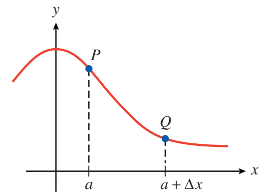


65.

a



b



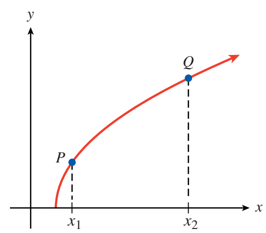
Answer.

a $(a, f(a)), (b, f(b));\quad \dfrac{f(b) - f(a)}{b - a}$

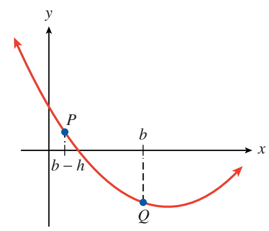
b $(a, f(a)), (a + \Delta x, f(a + \Delta x));\quad \dfrac{f(a + \Delta x) - f(a)}{\Delta x}$

66.

a



b



1.5 Linear Functions

1.5.1 Slope-Intercept Form

As we saw in Section 1.1, p. 2, many linear models $y = f(x)$ have equations of the form

$$\boldsymbol{f(x)} = (\text{starting value}) + (\text{rate of change}) \cdot \boldsymbol{x}$$

The starting value, or the value of y at $x = 0$, is the y-intercept of the graph, and the rate of change is the slope of the graph. Thus, we can write the equation of a line as

$$\boldsymbol{f(x) = b + mx}$$

where the constant term, b, is the y-intercept of the line, and m, the coefficient of x, is the slope of the line. This form for the equation of a line is called the **slope-intercept form**.

Slope-Intercept Form.

If we write the equation of a linear function in the form,

$$f(x) = b + mx$$

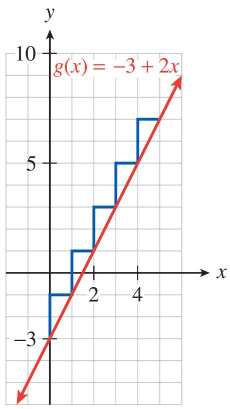
then m is the **slope** of the line, and b is the **y-intercept**.

(You may have encountered the slope-intercept equation in the equivalent form $y = mx + b$.)

For example, consider the two linear functions and their graphs shown below.

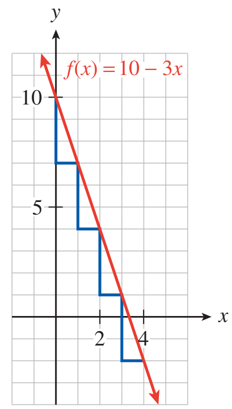
$f(x) = 10 - 3x$

x	$f(x)$
0	10
1	7
2	4
3	1
4	−2



$g(x) = -3 + 2x$

x	$f(x)$
0	−3
1	−1
2	1
3	3
4	5



Some observations:

- We can see that the y-intercept of each line is given by the constant term, b.
- By examining the table of values, we can also see why the coefficient of $x$gives the slope of the line:
- For $f(x)$, each time x increases by 1 unit, y decreases by 3 units.
- For $g(x)$, each time x increases by 1 unit, y increases by 2 units.

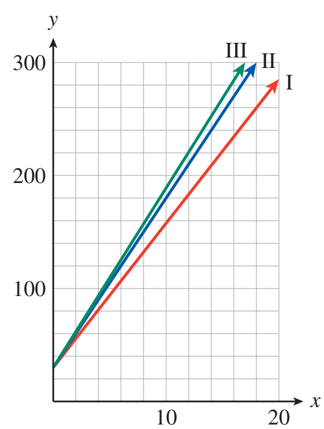
For each graph, the coefficient of x is a scale factor that tells us how many units y changes for 1 unit increase in x. But that is exactly what the slope tells us about a line.

Example 1.5.1

Francine is choosing an Internet service provider. She paid $30 for a modem, and she is considering three companies for service:

- Juno charges $14.95 per month,
- ISP.com charges $12.95 per month,
- and peoplepc charges $15.95 per month.

Match the graphs in the figure to Francine's Internet cost with each company.



Solution. Francine pays the same initial amount, $30 for the modem, under each plan. The monthly fee is the rate of change of her total cost, in dollars per month. We can write a formula for her cost under each plan.

$$\text{Juno: } f(x) = 30 + 14.95x$$

$$\text{ISP.com: } g(x) = 30 + 12.95x$$

$$\text{peoplepc: } h(x) = 30 + 15.95x$$

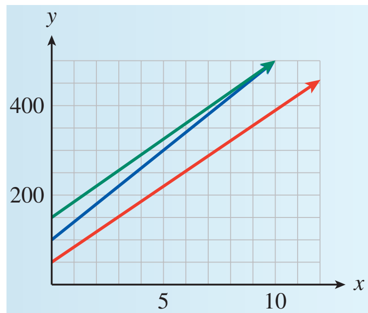
The graphs of these three functions all have the same y-intercept, but their slopes are determined by the monthly fees. The steepest graph, III, is the one with the largest monthly fee, peoplepc, and ISP.com, which has the lowest monthly fee, has the least steep graph, I. □

Checkpoint 1.5.2 Delbert decides to use DSL for his Internet service.

- Earthlink charges a \$99 activation fee and \$39.95 per month,
- DigitalRain charges \$50 for activation and \$34.95 per month,
- and FreeAmerica charges \$149 for activation and \$34.95 per month.

a Write a formula for Delbert's Internet costs under each plan.

b Match Delbert's Internet cost under each company with its graph shown below.

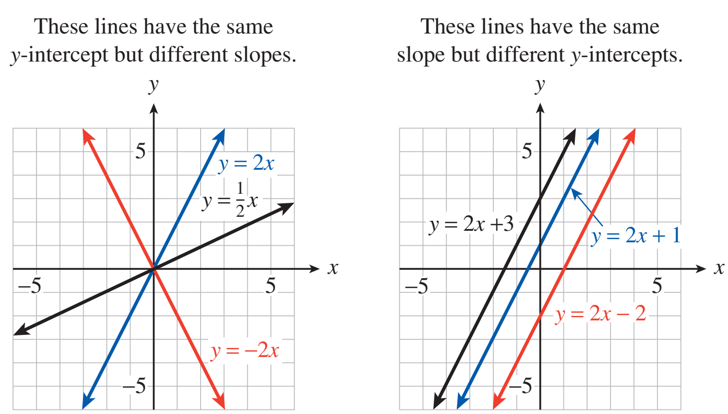


Answer.

a Earthlink: $f(x) = 99 + 39.95x$; DigitalRain: $g(x) = 50 + 34.95x$; FreeAmerica: $h(x) = 149 + 34.95x$

b DigitalRain: I; Earthlink: II; FreeAmerica: III

Note 1.5.3 In the equation $f(x) = b + mx$, we call m and b **parameters**. Their values are fixed for any particular linear equation; for example, in the equation $y = 3 + 2x$, $b = 3$ and $m = 2$, and the variables are x and y. By changing the values of m and b, we can write the equation for any line except a vertical line. The collection of all linear functions $f(x) = b + mx$ is called a **two-parameter** family of functions.



1.5.2 Slope-Intercept Method of Graphing

Look again at the lines in the previous figure: There is only one line that has a given slope and passes through a particular point. That is, the values of m and b determine the particular line. The value of b gives us a starting point, and the value of m tells us which direction to go to plot a second point. Thus, we can graph a line given in slope-intercept form without having to make a table of values.

Example 1.5.4

a Write the equation $4x - 3y = 6$ in slope-intercept form.

b Graph the line by hand.

Solution.

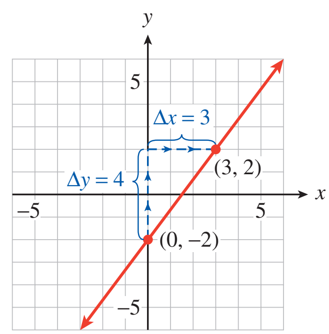
a We solve the equation for y in terms of x.

$$-3y = 6 - 4x \qquad \text{Divide both sides by } \mathbf{-3}$$
$$y = \frac{6-4x}{-3} = \frac{6}{-3} + \frac{-4x}{-3}$$
$$y = -2 + \frac{4}{3}x$$

b We see that the slope of the line is $m = \frac{4}{3}$ and its y-intercept is $b = -2$. We begin by plotting the y-intercept, $(0, -2)$. We then use the slope to find another point on the line. We have

$$m = \frac{\Delta y}{\Delta x} = \frac{4}{3}$$

so starting at $(0, -2)$, we move 4 units in the y-direction and 3 units in the x-direction, to arrive at the point $(3, 2)$. Finally, we draw the line through these two points.



□

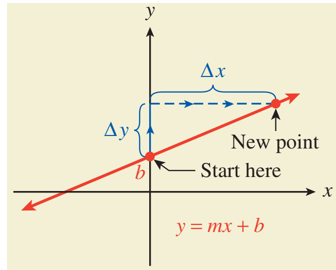
Note 1.5.5 The slope of a line is a ratio and can be written in many equivalent ways. In Example 1.5.4, p. 111, the slope is equal to $\frac{8}{6}$, $\frac{12}{9}$, and $\frac{-4}{-3}$. We can use any of these fractions to locate a third point on the line as a check. If we use $m = \frac{\Delta y}{\Delta x} = \frac{-4}{-3}$, we move down 4 units and left 3 units from the y-intercept to find the point $(-3, -6)$ on the line.

Slope-Intercept Method for Graphing a Line.

a Plot the y-intercept $(0, b)$.

b Use the definition of slope to find a second point on the line: Starting at the y-intercept, move Δy units in the y-direction and Δx units in the x-direction. Plot a second point at this location.

c Use an equivalent form of the slope to find a third point, and draw a line through the points.



Checkpoint 1.5.6

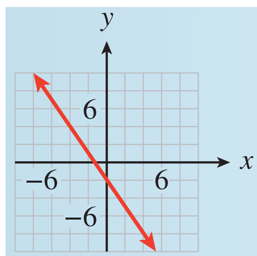
a Write the equation $2y + 3x + 4 = 0$ in slope-intercept form.

b Use the slope-intercept method to graph the line.

Answer.

a $y = -2 - \dfrac{3}{2}x$

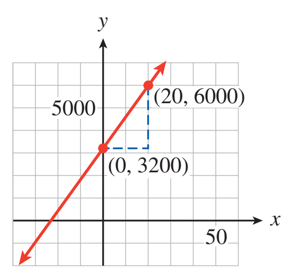
b



1.5.3 Finding a Linear Equation from a Graph

We can also use the slope-intercept form to find the equation of a line from its graph. First, we note the value of the y-intercept from the graph, and then we calculate the slope using two convenient points.

Example 1.5.7
Find an equation for the line shown at right.



Solution. The line crosses the y-axis at the point $(0, 3200)$, so the y-intercept is 3200. To calculate the slope of the line, we locate another point, say $(20, 6000)$, and compute:

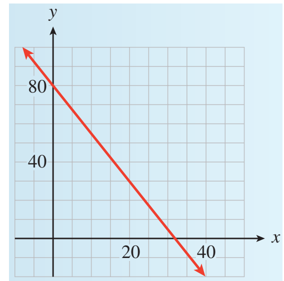
$$m = \frac{\Delta y}{\Delta x} = \frac{6000 - 3200}{20 - 0}$$
$$= \frac{2800}{20} = 140$$

The slope-intercept form of the equation, with $m = 140$ and $b = 3200$, is

$$y = 3200 + 140x$$

□

Checkpoint 1.5.8
Find an equation for the line shown at right.
$b =$
$m =$
$y =$



Answer. $b = 80,\ m = \dfrac{-5}{2},\ y = 80 - \dfrac{5}{2}x$

1.5.4 Point-Slope Form

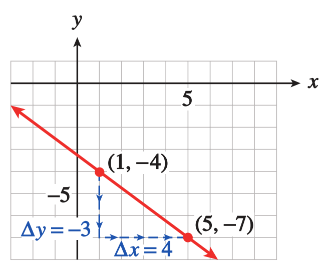
We can find the equation for a line if we know its slope and y-intercept. What if we do not know the y-intercept, but instead know some other point on the line? There is only one line that passes through a given point and has a given slope, so we should be able to find its equation.

For example, we can graph the line of slope $\dfrac{-3}{4}$ that passes through the point $(1, -4)$. We first plot the given point, $(1, -4)$, as shown in the figure below.

Then we use the slope to find another point on the line. The slope is

$$m = \frac{-3}{4} = \frac{\Delta y}{\Delta x}$$

so we move down 3 units and then 4 units to the right, starting from $(1, -4)$. This brings us to the point $(5, -7)$. We can then draw the line through these two points.



We can also find an equation for the line, as shown in Example 1.5.9, p. 113.

Example 1.5.9 Find an equation for the line that passes through $(1, -4)$ and has slope $\dfrac{-3}{4}$.

Solution. We will use the formula for slope,

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

We substitute $\dfrac{-3}{4}$ for the slope, m, and $(1, -4)$ for (x_1, y_1). For the second point, (x_2, y_2), we use the variable point (x, y). Substituting these values into the slope formula gives us

$$\frac{-3}{4} = \frac{y - (-4)}{x - 1} = \frac{y + 4}{x - 1}$$

To solve for y we first multiply both sides by $x - 1$.

$$\begin{aligned}
(x-1)\frac{-3}{4} &= \frac{y+4}{x-1}(x-1) \\
\frac{-3}{4}(x-1) &= y+4 && \text{Apply the distributive law.} \\
\frac{-3}{4}x + \frac{3}{4} &= y+4 && \text{Subtract 4 from both sides.} \\
\frac{-3}{4}x - \frac{13}{4} &= y && \frac{3}{4} - 4 = \frac{3}{4} - \frac{16}{4} = \frac{-13}{4}
\end{aligned}$$

The equation of the line is $y = \frac{-13}{4} - \frac{3}{4}x$ □

When we use the slope formula in this way to find the equation of a line, we substitute a variable point (x, y) for the second point. This version of the formula,

$$m = \frac{y - y_1}{x - x_1}$$

is called the **point-slope form** for a linear equation. It is sometimes stated in another form obtained by clearing the fraction to get

$$\begin{aligned}
(x - x_1)m &= \frac{y - y_1}{x - x_1}(x - x_1) && \text{Multiply both sides by } (x - x_1) \\
(x - x_1)m &= y - y_1 && \text{Clear fractions and solve for } y. \\
y &= y_1 + m(x - x_1)
\end{aligned}$$

Point-Slope Form.

The equation of the line that passes through the point (x_1, y_1) and has slope m is

$$y = y_1 + m(x - x_1)$$

Checkpoint 1.5.10 Use the point-slope form to find the equation of the line that passes through the point $(-3, 5)$ and has slope -1.4.

$$y = y_1 + m(x - x_1) \quad \text{Substitute } -1.4 \text{ for } m \text{ and } (-3, 5) \text{ for } (x_1, y_1).$$

Simplify: Apply the distributive law.

Answer. $y = 0.8 - 1.4x$

The point-slope form is useful for modeling linear functions when we don't know the initial value but do know some other point on the line.

Example 1.5.11 Under a proposed graduated income tax system, single taxpayers would owe \$1500 plus 20% of the amount of their income over %13,000. (For example, if your income is \$18,000, you would pay \$1500 plus 20% of \$5000.)

a Complete the table of values for the tax, T, on various incomes, I.

I	$15,000$	$20,000$	$22,000$
T			

b Write a linear equation in point-slope form for the tax, T, on an income I.

c Write the equation in slope-intercept form.

Solution.

a On an income of \$15,000, the amount of income over \$13,000 is \$15,000 - \$13,000 = \$2000, so you would pay \$1500 plus 20% of \$2000, or

$$T = 1500 + 0.20(2000) = 1900$$

You can compute the other function values in the same way.

I	$15,000$	$20,000$	$22,000$
T	1900	2900	3300

b On an income of I, the amount of income over \$13,000 is $I - \$13,000$, so you would pay 41500 plus 20% of $I - 13,000$, or

$$T = 1500 + 0.20(I - 13,000)$$

c Simplify the right side of the equation to get

$$T = 1500 + 0.20I - 2600$$
$$T = -1100 + 0.20I$$

□

Checkpoint 1.5.12 A healthy weight for a young woman of average height, 64 inches, is 120 pounds. To calculate a healthy weight for a woman taller than 64 inches, add 5 pounds for each inch of height over 64.

a Write a linear equation in point-slope form for the healthy weight, W, for a woman of height, H, in inches.

b Write the equation in slope-intercept form.

Answer.

a $W = 120 + 5(H - 64)$

b $W = -200 + 5H$

1.5.5 Section Summary

1.5.5.1 Vocabulary

Look up the definitions of new terms in the Glossary.

- Slope-intercept form
- Point-slope form
- Parameter

1.5.5.2 CONCEPTS

1 Linear functions form a two-parameter family, $f(x) = b + mx$.

2 The initial value of a linear function and the y-intercept of its graph are given by b. The rate of change of the function and the slope of its graph are given by m.

3 The slope-intercept form, $y = b + mx$, is useful when we know the initial value and the rate of change.

4 The point-slope form, $y = y_1 + m(x - x_1)$, is useful when we know the rate of change and one point on the line.

1.5.5.3 STUDY QUESTIONS

1 How can you put a linear equation into slope-intercept form?

2 What do the coefficients in the slope-intercept form tell you about the line?

3 Explain how to graph a line using the slope-intercept method.

4 Explain how to find an equation for a line from its graph.

5 Explain how to use the point-slope form for a linear equation.

6 Francine says that the slope of the line $y = 4x - 6$ is $4x$. Is she correct? Explain your answer

7 Delbert says that the slope of the line $3x - 4y = 8$ is 3. Is he correct? Explain your answer.

1.5.5.4 SKILLS

Practice each skill in the Homework 1.5.6, p. 116 problems listed.

1 Write a linear equation in slope-intercept form: #1–14

2 Identify the slope and y-intercept: #1–10

3 Graph a line by the slope-intercept method: #11–14

4 Find a linear equation from its graph: #21–26, 29–32, 53–56

5 Interpret the slope and y-intercept: #21–28, 63 and 64

6 Find a linear equation from one point and the slope: #33–50

1.5.6 Linear Functions (Homework 1.5)

In Problems 1–10,

a Write each equation in slope-intercept form.

b State the slope and y-intercept of the line.

1. $3x + 2y = 1$

Answer.

a $y = \frac{1}{2} - \frac{3}{2}x$

b Slope $\frac{-3}{2}$, y-intercept $\frac{1}{2}$

2. $5x - 4y = 0$

3. $\frac{1}{4}x + \frac{3}{2}y = \frac{1}{6}$

Answer.

a $y = \frac{1}{9} - \frac{1}{6}x$

b Slope $\frac{-1}{6}$, y-intercept $\frac{1}{9}$

4. $\frac{7}{6}x - \frac{2}{9}y = 3$

5. $4.2x - 0.3y = 6.6$

Answer.

a $y = -22 + 14x$

b Slope 14, y-intercept -22

6. $0.8x + 0.004y = 0.24$

7. $y + 29 = 0$

Answer.

a $y = -29$

b Slope 0, y-intercept -29

8. $0.7x - 12 = 0$

9. $250x + 150y = 2450$

Answer.

a $y = \dfrac{49}{3} - \dfrac{5}{3}x$

b Slope $\dfrac{-5}{3}$, y-intercept $\dfrac{49}{3}$

10. $80x - 360y = 6120$

In Problems 11–14,

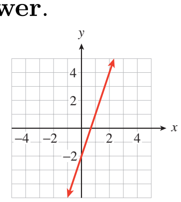
a Sketch by hand the graph of the line with the given slope and y-intercept.

b Write an equation for the line.

c Find the x-intercept of the line.

11. $m = 3$ and $b = -2$

Answer.

a 

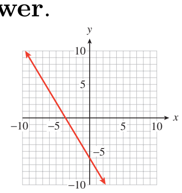
b $y = -2 + 3x$

c $\dfrac{2}{3}$

12. $m = -4$ and $b = 1$

13. $m = -\dfrac{5}{3}$ and $b = -6$

Answer.

a 

b $y = -6 + \dfrac{5}{3}x$

c $\dfrac{-18}{5}$

14. $m = \dfrac{3}{4}$ and $b = -2$

15. The point $(2, -1)$ lies on the graph of $f(x) = -3x + b$. Find b.

Answer. 5

16. The point $(-3, -8)$ lies on the graph of $f(x) = \frac{2}{3}x + b$. Find b.

17. The point $(8, -5)$ lies on the graph of $f(x) = mx - 3$. Find m.

Answer. $\frac{-1}{4}$

18. The point $(-5, -6)$ lies on the graph of $f(x) = mx + 2$. Find m.

19. Find the slope and intercepts of the line $Ax + By = C$

Answer. $m = \frac{-A}{B}$, x-intercept $\left(\frac{C}{A}, 0\right)$, y-intercept $\left(0, \frac{C}{B}\right)$

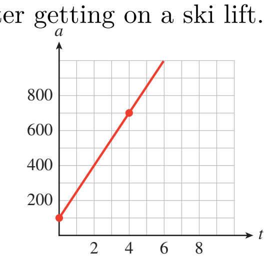
20. Find the slope and intercepts of the line $\frac{x}{a} + \frac{y}{b} = 1$

In Problems 21–26,

a Find a formula for the function whose graph is shown.

b Say what the slope and the vertical intercept tell us about the problem.

21. The graph shows the altitude, a (in feet), of a skier t minutes after getting on a ski lift.

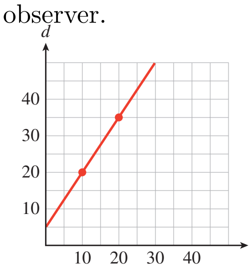


Answer.

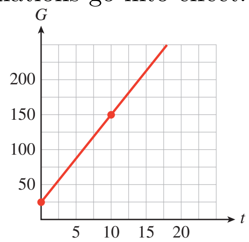
a $a = 100 + 150t$

b The slope tells us that the skier's altitude is increasing at a rate of 150 feet per minute, the vertical intercept that the skier began at an altitude of 200 feet.

22. The graph shows the distance, d (in meters), traveled by a train t seconds after it passes an observer.



23. The graph shows the amount of garbage, G (in tons), that has been deposited at a dump site t years after new regulations go into effect.

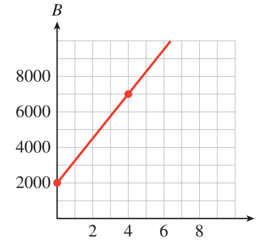


Answer.

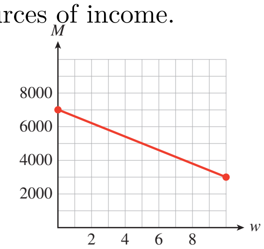
a $G = 25 + 12.5t$

b The slope tells us that the garbage is increasing at a rate of 12.5 tons per year, the vertical intercept that the dump already had 25 tons (when the new regulations went into effect).

24. The graph shows the number of barrels of oil, B, that has been pumped at a drill site t days after a new drill is installed.



25. The graph shows the amount of money, M (in dollars), in Tammy's bank account w weeks after she loses all sources of income.

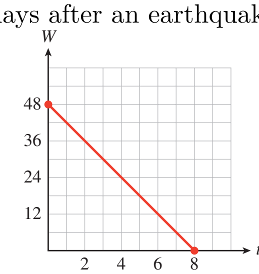


Answer.

a $M = 7000 - 400w$

b The slope tells us that Tammy's bank account is diminishing at a rate of \$400 per week, the vertical intercept that she had \$7000 (when she lost all sources of income).

26. The graph shows the amount of emergency water, W (in liters), remaining in a southern California household t days after an earthquake.



27. The formula $F = \dfrac{9}{5}C + 32$ defines a function that converts the temperature in degrees Celsius to degrees Fahrenheit.

a What is the Fahrenheit temperature when it is $10°$ Celsius?

b What is the Celsius temperature when it is $-4°$ Fahrenheit?

c Choose appropriate `WINDOW` settings and graph the equation $y = \frac{9}{5}x + 32$.

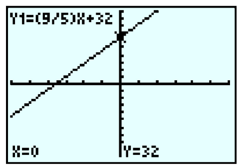
d Find the slope and explain its meaning for this problem.

e Find the intercepts and explain their meanings for this problem.

Answer.

a 50°F

b -20°C

c 

d The slope, $\frac{9}{5} = 1.8$, tells us that Fahrenheit temperatures increase by 1.8° for each increase of 1° Celsius.

e C-intercept $\left(-17\frac{7}{9}, 0\right)$: $-17\frac{7}{9}^{\circ}$ C is the same as 0°F; F-intercept $(0, 32)$: 0°C is the same as 32°F.

28. If the temperature on the ground is 70° Fahrenheit, the formula $T = 70 - \dfrac{3}{820}h$ defines a function that gives the temperature at an altitude of h feet.

a What is the temperature at an altitude of 4100 feet?

b At what altitude is the temperature 34° Fahrenheit?

c Choose appropriate `WINDOW` settings and graph the equation $y = 70 - \dfrac{3}{820}x$.

d Find the slope and explain its meaning for this problem.

e Find the intercepts and explain their meanings for this problem.

29. In England, oven cooking temperatures are often given as Gas Marks rather than degrees Fahrenheit. The table shows the equivalent oven temperatures for various Gas Marks.

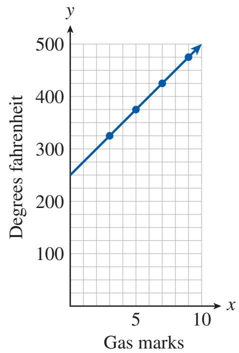
Gas Mark	3	5	7	9
Degrees (F)	325	375	425	475

a Plot the data and draw a line through the data points.

b Calculate the slope of your line. Estimate the y-intercept from the graph.

c Find an equation that gives the temperature in degrees Fahrenheit in terms of the Gas Mark.

Answer.

a 

b $m = 25,\ b = 250$

c $y = 250 + 25x$

30. European shoe sizes are scaled differently than American shoe sizes. The table shows the European equivalents for various American shoe sizes.

American shoe size	5.5	6.5	7.5	8.5
European shoe size	37	38	39	40

a Plot the data and draw a line through the data points.

b Calculate the slope of your line. Estimate the y-intercept from the graph.

c Find an equation that gives the European shoe size in terms of American shoe size.

31. A spring is suspended from the ceiling. The table shows the length of the spring in centimeters as it is stretched by hanging various weights from it.

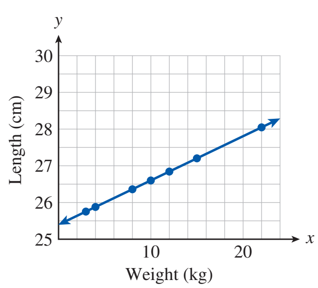
Weight, kg	3	4	8	10	12	15	22
Length, cm	25.76	25.88	26.36	26.6	26.84	27.2	28.04

a Plot the data on graph paper and draw a straight line through the points. Estimate the y-intercept of your graph.

b Find an equation for the line.

c If the spring is stretched to 27.56 cm, how heavy is the attached weight?

Answer.

a 

b $y = 0.12x + 25.4$

c 18 kg

32. The table shows the amount of ammonium chloride salt, in grams, that can be dissolved in 100 grams of water at different temperatures.

Temperature, °C	10	12	15	21	25	40	52
Grams of salt	33	34	35.5	38.5	40.5	48	54

a Plot the data on graph paper and draw a straight line through the points. Estimate the y-intercept of your graph.

b Find an equation for the line.

c At what temperature will 46 grams of salt dissolve?

In Problems 33–36,

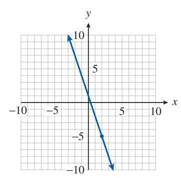
a Sketch by hand the graph of the line that passes through the given point and has the given slope.

b Write an equation for the line in point-slope form.

c Put your equation from part (b) into slope-intercept form.

33. $(2, -5)$; $m = -3$

Answer.

a 

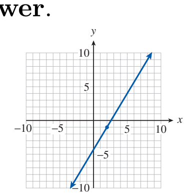
b $y + 5 = -3(x - 2)$

c $y = 1 - 3x$

34. $(-6, -1)$; $m = 4$

35. $(2, -1)$; $m = \dfrac{5}{3}$

Answer.

a 

b $y + 1 = \dfrac{5}{3}(x - 2)$

c $y = \dfrac{-13}{3} + \dfrac{5}{3}x$

36. $(-1, 2)$; $m = -\dfrac{3}{2}$

For Problems 37–40,

a Write an equation in point-slope form for the line that passes through the given point and has the given slope.

b Put your equation from part (a) into slope-intercept form.

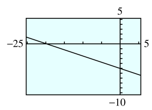
c Use your graphing calculator to graph the line.

37. $(-6.4, -3.5)$, $m = -0.25$

Answer.

a $y + 3.5 = -0.25(x + 6.4)$

b $y = -5.1 - 0.25x$

c 

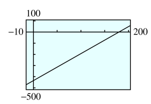
38. $(7.2, -5.6)$, $m = 1.6$

39. $(80, -250)$, $m = 2.4$

Answer.

a $y + 250 = 2.4(x - 80)$

b $y = -442 + 2.4x$

c 

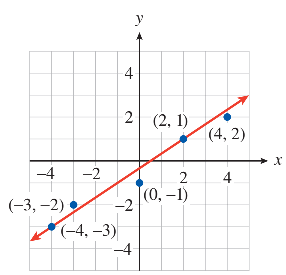
40. $(-150, 1800)$, $m = -24$

For Problems 41 and 42,

a Find the slope of the line. (Note that not all the labeled points lie on the line.)

b Find an equation for the line.

41.

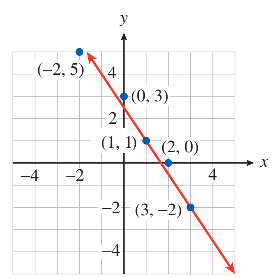


Answer.

a $m = \dfrac{2}{3}$

b $y = \dfrac{-1}{3} + \dfrac{2}{3}x$

42.



For Problems 43 and 44, the equation of line l_1 is $y = q + px$, and the equation of line l_2 is $y = v + tx$.

a Decide whether the coordinates of each labeled point are

I a solution of $y = q + px$,

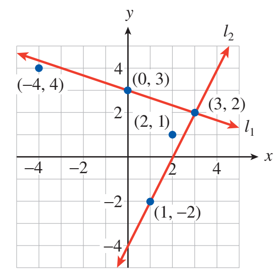
II a solution of $y = v + tx$,

III a solution of both equations, or

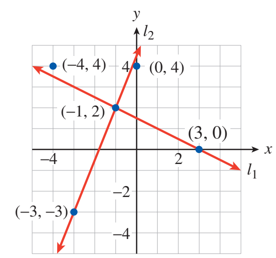
IV a solution of neither equation.

b Find p, q, t, and v.

43.



44.



Answer.

a $(-4, 4)$: neither; $(0, 3)$: $y = px + q$; $(3, 2)$: both; $(2, 1)$: neither; $(1, -2)$: $y = tx + v$

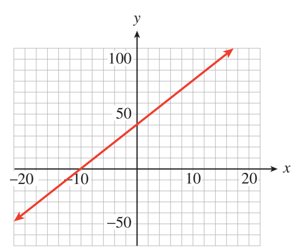
b $p = \dfrac{-1}{3}$, $q = 3$, $t = 2$, $v = -4$

For Problems 45–50,

a Estimate the slope and vertical intercept of each line. (Hint: To calculate the slope, find two points on the graph that lie on the intersection of grid lines.)

b Using your estimates from (a), write an equation for the line.

45.

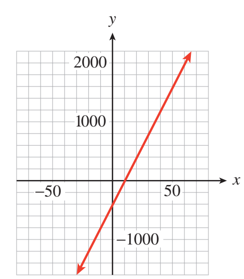


Answer.

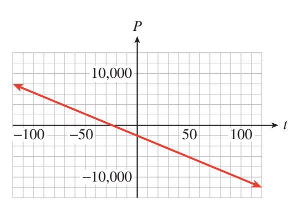
a $m = 4,\ b = 40$

b $y = 40 + 4x$

46.



47.

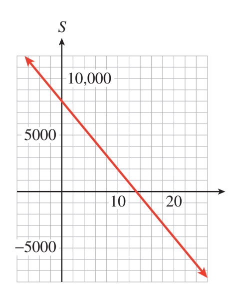


Answer.

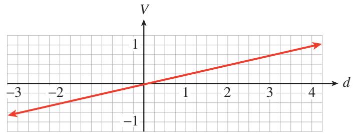
a $m = -80,\ b = -2000$

b $P = -2000 - 80t$

48.



49.

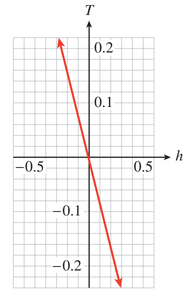


Answer.

a $m = \frac{1}{4}$, $b = 0$

b $V = \frac{1}{4}d$

50.



51.

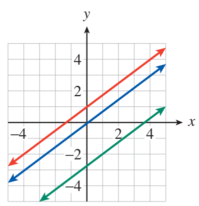
a Write equations for three lines with slope $m = \frac{3}{4}$. (Many answers are possible.)

b Graph all three lines on the same axis. What do you notice about the lines?

Answer.

a $y = \frac{3}{4}x$, $y = 1 + \frac{3}{4}x$, $y = -2.7 + \frac{3}{4}x$

b



The lines are parallel.

52.

a Write equations for three lines with slope $m = 0$. (Many answers are possible.)

b Graph all three lines in the same window. What do you notice about the lines?

In Problems 53–56, choose the correct graph for each equation. The scales on both axes are the same.

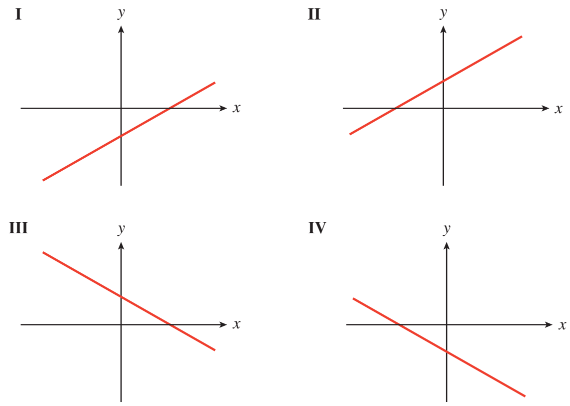
53.

a $y = \frac{3}{4}x + 2$

b $y = \frac{-3}{4}x + 2$

c $y = \frac{3}{4}x - 2$

d $y = \frac{-3}{4}x - 2$



Answer.

a II　　b III　　c I　　d IV

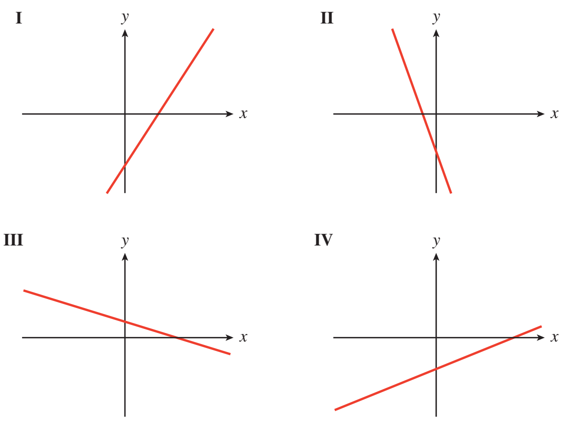
54.

a $m < 0$, $b > 0$

c $0 < m < 1$, $b < 0$

b $m > 1$, $b < 0$

d $m < -1$, $b < 0$



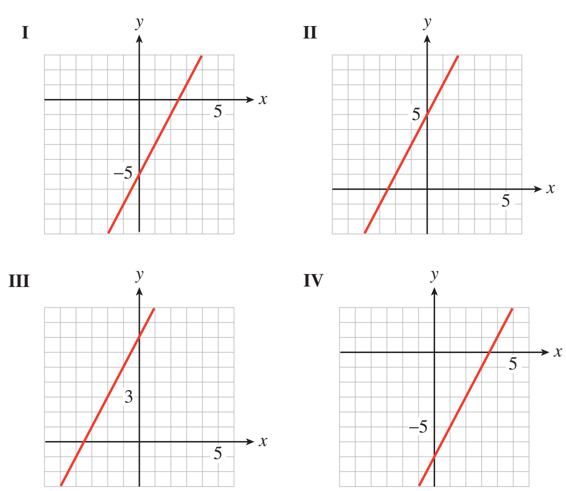
55.

a $y = 1 + 2(x + 3)$

c $y = -1 + 2(x + 3)$

b $y = -1 + 2(x - 3)$

d $y = 1 + 2(x - 3)$



Answer.

a III b IV c II d I

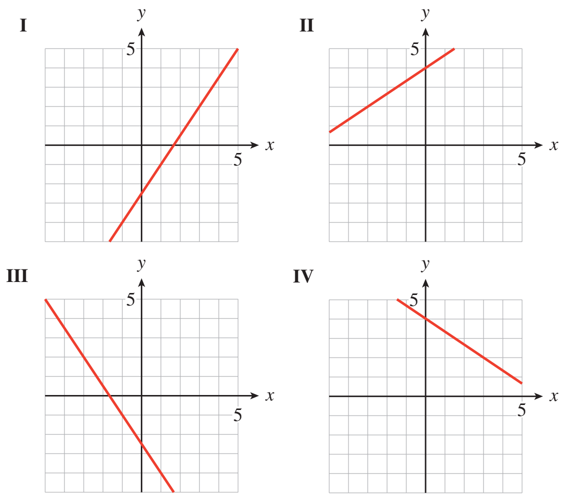
56.

a $y = 2 - \frac{2}{3}(x-3)$

b $y = 2 - \frac{3}{2}(x+3)$

c $y = 2 + \frac{3}{2}(x-3)$

d $y = 2 + \frac{2}{3}(x+3)$



In Problems 57–60, find the slope of each line and the coordinates of one point on the line. (No calculation is necessary!)

57. $y + 1 = 2(x - 6)$

Answer. $m = 2$; $(6, -1)$

58. $2(y - 8) = 5(x + 2)$

59. $y = 3 - \frac{4}{3}(x + 5)$

Answer. $m = \frac{-4}{3}$; $(-5, 3)$

60. $7x = -3y$

61.

a Draw a set of coordinate axes with a square grid (i.e., with units the same size in both directions). Sketch four lines through the point (0, 4) with the following slopes:

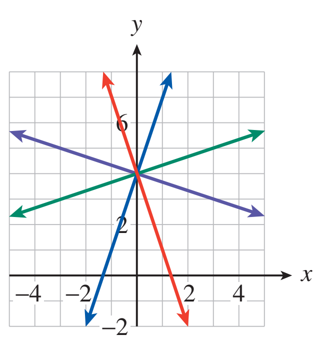
$$m = 3, \quad m = -3, \quad m = \frac{1}{3}, \quad m = \frac{-1}{3}$$

b What do you notice about these lines?

Hint. Look for perpendicular lines.

Answer.

a



b The lines with slope 3 and $\frac{-1}{3}$ are perpendicular to each other, and the lines with slope -3 and $\frac{1}{3}$ are perpendicular to each other.

62.

a Draw a set of coordinate axes with a square grid (see Problem 61). Sketch four lines through the point (0, -3) with the following slopes:

$$m = \frac{2}{5}, \quad m = \frac{-2}{5}, \quad m = \frac{5}{2}, \quad m = \frac{-5}{2}$$

b What do you notice about these lines?

63. The boiling point of water changes with altitude and is approximated by the formula

$$B = f(h) = 212 - 0.0018h$$

where B is in degrees Fahrenheit and h is in feet. State the slope and vertical intercept of the graph, including units, and explain their meaning in this context.

Answer. $m = -0.0018$ degree/foot, so the boiling point drops with altitude at a rate of 0.0018 degree per foot. $b = 212$, so the boiling point is 212° at sea level (where the elevation $h = 0$).

64. The height of a woman in centimeters is related to the length of her femur (in centimeters) by the formula

$$H = f(x) = 2.47x + 54.10$$

State the slope and the vertical intercept of the graph, including units, and explain their meaning in this context.

1.6 Chapter Summary and Review

1.6.1 Key Concepts

1 We can describe a relationship between variables with a table of values, a graph, or an equation.

2 Linear models have equations of the following form:

$$y = (\text{starting value}) + (\text{rate of change}) \cdot x$$

3 The general form for a linear equation is $Ax + By = C$.

4 We can use the **intercepts** to graph a line. The intercepts are also useful for interpreting a model.

5 A **function** is a rule that assigns to each value of the input variable a unique value of the output variable.

6 Function notation: $y = f(x)$, where x is the input and y is the output.

7 The point (a, b) lies on the graph of the function f if and only if $f(a) = b$

8 Each point on the graph of the function f has coordinates $(x, f(x))$ for some value of x.

9 The **vertical line test** tells us whether a graph represents a function.

10 Lines have constant slope.

11 The slope of a line gives us the **rate of change** of one variable with respect to another

12 **Formulas for Linear Functions.**

$$\text{Slope:} \quad m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

$$\text{Slope-intercept form:} \quad y = b + mx$$

$$\text{Point-slope form:} \quad y = y_1 + m(x - x_1)$$

13 The **slope-intercept form** is useful when we know the initial value and the rate of change.

14 The **point-slope form** is useful when we know the rate of change and one point on the line.

15 Linear functions form a **two-parameter family**, $f(x) = b + mx$.

16 We can approximate a linear pattern by a **regression line**.

17 We can use **interpolation** or **extrapolation** to make estimates and predictions.

18 If we extrapolate too far beyond the known data, we may get unreasonable results.

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1.6.2 Chapter 1 Review Problems

Write and graph a linear equation for each situation. Then answer the questions.

1. Last year, Pinwheel Industries introduced a new model calculator. It cost \$2000 to develop the calculator and \$20 to manufacture each one.

a Complete the table of values showing the total cost, C, of producing n calculators.

n	100	500	800	1200	1500
C					

b Write an equation that expresses C in terms of n.

c Graph the equation by hand.

d What is the cost of producing 1000 calculators? Illustrate this as a point on your graph.

e How many calculators can be produced for \$10,000? Illustrate this as a point on your graph.

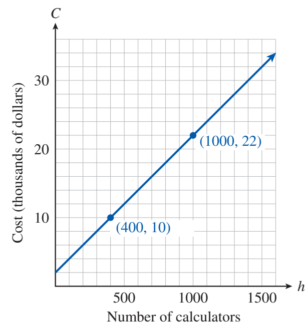
Answer.

a

n	100	500	800	1200	1500
C	4000	12,000	18,000	26,000	32,000

b $C = 20n + 2000$

c



d $\$22,000$

e 400

2. Megan weighed 5 pounds at birth and gained 18 ounces per month during her first year.

a Complete the table of values for Megan's weight, w, in terms of her age, m, in months.

m	2	4	6	9	12
w					

b Write an equation that expresses 2 in terms of m.

c Graph the equation by hand.

d How much did Megan weigh at 9 months? Illustrate this as a point on your graph.

e When did Megan weigh 9 pounds? Illustrate this as a point on your graph.

3. The total amount of oil remaining in 2005 is estimated at 2.1 trillion barrels, and total annual consumption is about 28 billion barrels.

a Assuming that oil consumption continues at the same level, write an equation for the remaining oil, R, as a function of time, t (in years since 2005).

b Find the intercepts and graph the equation by hand.

c What is the significance of the intercepts to the world's oil supply?

Answer.

a $R = 2100 - 28t$

b $(75, 0)$, $(0, 2100)$

c t-intercept: The oil reserves will be gone in 2080; R-intercept: There were 2100 billion barrels of oil reserves in 2005.

4. The world's copper reserves were 950 million tons in 2004; total annual consumption was 16.8 million tons.

a Assuming that copper consumption continues at the same level, write an equation for the remaining copper reserves, R, as a function of time, t (in years since 2004).

b Find the intercepts and graph the equation by hand.

c What is the significance of the intercepts to the world's copper supply?

5. The owner of a movie theater needs to bring in \$1000 at each screening in order to stay in business. He sells adult tickets at \$5 apiece and children's tickets at \$2 each.

a Write an equation that relates the number of adult tickets, A, he must sell and the number of children's tickets, C.

b Find the intercepts and graph the equation by hand.

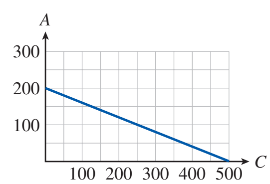
c If the owner sells 120 adult tickets, how many children's tickets must he sell?

d What is the significance of the intercepts to the sale of tickets?

Answer.

a $2C + 5A = 1000$

b $(500, 0)$, $(0, 200)$



c C-intercept: If no adult tickets are sold, he must sell 500 children's tickets; A-intercept: If no children's tickets are sold, he must sell 200 adult tickets.

6. Alida plans to spend part of her vacation in Atlantic City and part in Saint-Tropez. She estimates that after airfare her vacation will cost \$60 per day in Atlantic City and \$100 per day in Saint-Tropez. She has \$1200 to spend after airfare.

a Write an equation that relates the number of days, C, Alida can spend in Atlantic City and the number of days, T, in Saint-Tropez.

b Find the intercepts and graph the equation by hand.

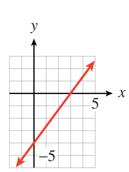
c If Alida spends 10 days in Atlantic City, how long can she spend in Saint-Tropez?

d What is the significance of the intercepts to Alida's vacation?

Graph each equation on graph paper. Use the most convenient method for each problem.

7. $4x - 3y = 12$

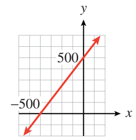
Answer.



8. $\dfrac{x}{6} - \dfrac{y}{12} = 1$

9. $50x = 40y - 20,000$

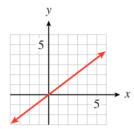
Answer.



10. $1.4x + 2.1y = 8.4$

11. $3x - 4y = 0$

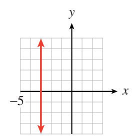
Answer.



12. $x = -4y$

13. $4x = -12$

Answer.



14. $2y - x = 0$

Which of the following tables describe functions? Explain.

15.

x	-2	-1	0	1	2	3
y	6	0	1	2	6	8

Answer. A function: Each x has exactly one associated y-value.

16.

p	3	-3	2	-2	-2	0
q	2	-1	4	-4	3	0

17.

Student	Score on IQ test	Score on SAT test
(A)	118	649
(B)	98	450
(C)	110	590
(D)	105	520
(E)	98	490
(F)	122	680

Answer. Not a function: The IQ of 98 has two possible SAT scores.

18.

Student	Correct answers on math quiz	Quiz grade
(A)	13	85
(B)	15	89
(C)	10	79
(D)	12	82
(E)	16	91
(F)	18	95

19. The total number of barrels of oil pumped by the AQ oil company is given by the formula

$$N(t) = 2000 + 500t$$

where N is the number of barrels of oil t days after a new well is opened. Evaluate $N(10)$ and explain what it means.

Answer. $N(10) = 7000$: Ten days after the new well is opened, the company has pumped a total of 7000 barrels of oil.

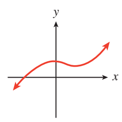
20. The number of hours required for a boat to travel upstream between two cities is given by the formula

$$H(v) = \frac{24}{v-8}$$

where v represents the boat's top speed in miles per hour. Evaluate $H(16)$ and explain what it means.

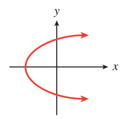
Which of the following graphs represent functions?

21.

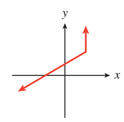


Answer. Function

22.

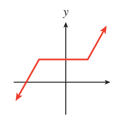


23.



Answer. Not a function

24.



Evaluate each function for the given values.

25. $F(t) = \sqrt{1+4t^2}, \quad F(0) \quad \text{and} \quad F(-3)$

Answer. $F(0) = 1, \quad F(-3) = \sqrt{37}$

26. $G(x) = \sqrt[3]{x-8}, \quad G(0) \quad \text{and} \quad G(20)$

27. $h(v) = 6 - |4-2v|, \quad h(8) \quad \text{and} \quad h(-8)$

Answer. $h(8) = -6, \quad h(-8) = -14$

28. $m(p) = \dfrac{120}{p+15}, \quad m(5) \quad \text{and} \quad m(-40)$

Refer to the graphs shown for Problems 29 and 30.

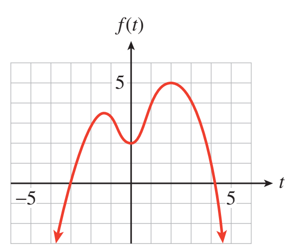
29.

a Find $f(-2)$ and $f(2)$.

b For what value(s) of t is $f(t) = 4$?

c Find the t- and $f(t)$-intercepts of the graph.

d What is the maximum value of f? For what value(s) of t does f take on its maximum value?



Answer.

a $f(-2) = 3, \quad f(2) = 5$

b $t = 1, \quad t = 3$

c t-intercepts $(-3, 0), (4, 0)$; $f(t)$-intercept: $(0, 2)$

d Maximum value of 5 occurs at $t = 2$

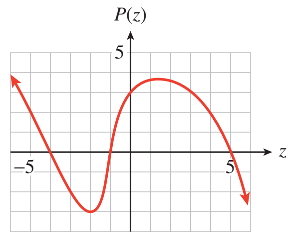
30.

a Find $P(-3)$ and $P(3)$.

b For what value(s) of z is $P(z) = 2$?

c Find the z- and $P(z)$-intercepts of the graph.

d What is the minimum value of P? For what value(s) of z does P take on its minimum value?



Graph the given function on a graphing calculator. Then use the graph to solve the equations and inequalities. Round your answers to one decimal place if necessary.

31. $y = \sqrt[3]{x}$

a Solve $\sqrt[3]{x} = 0.8$

b Solve $\sqrt[3]{x} = 1.5$

c Solve $\sqrt[3]{x} > 1.7$

d Solve $\sqrt[3]{x} \leq 1.26$

Answer.

a $x = \dfrac{1}{2} = 0.5$

b $x = \dfrac{27}{8} \approx 3.4$

c $x > 4.9$

d $x \leq 2.0$

32. $y = \dfrac{1}{x}$

a Solve $\dfrac{1}{x} = 2.5$

b Solve $\dfrac{1}{x} = 0.3125$

c Solve $\dfrac{1}{x} \geq 0.\overline{2}$

d Solve $\dfrac{1}{x} < 5$

33. $y = \dfrac{1}{x^2}$

a Solve $\dfrac{1}{x^2} = 0.03$

b Solve $\dfrac{1}{x^2} = 6.25$

c Solve $\dfrac{1}{x^2} > 0.16$

d Solve $\dfrac{1}{x^2} \leq 4$

Answer.

a $x \approx \pm 5.8$

b $x = \pm 0.4$

c $-2.5 < x < 0$ or $0 < x < 2.5$

d $x \leq -0.5$ or $x \geq 0.5$

34. $y = \sqrt{x}$

a Solve $\sqrt{x} = 0.707$

b Solve $\sqrt{x} = 1.7$

c Solve $\sqrt{x} < 1.5$

d Solve $\sqrt{x} \geq 1.3$

Evaluate each function.

35. $H(t) = t^2 + 2t, \quad H(2a) \quad \text{and} \quad H(a+1)$

Answer. $H(2a) = 4a^2 + 4a, \quad H(a+1) = a^2 + 4a + 3$

36. $F(x) = 2 - 3x, \quad F(2) + F(3) \quad \text{and} \quad F(2+3)$

37. $f(x) = 2x^2 - 4, \quad f(a) + f(b) \quad \text{and} \quad f(a+b)$

Answer. $f(a)+f(b) = 2a^2+2b^2-8, \quad f(a+b) = 2a^2+4ab+2b^2-4$

38. $G(t) = 1 - t^2, \quad G(3w) \quad \text{and} \quad G(s+1)$

39. A spiked volleyball travels 6 feet in 0.04 seconds. A pitched baseball travels 66 feet in 0.48 seconds. Which ball travels faster?

Answer. The volleyball

40. Kendra needs $4\frac{1}{2}$ gallons of Luke's Brand primer to cover 1710 square feet of wall. She uses $5\frac{1}{3}$ gallons of Slattery's Brand primer for 2040 square feet of wall. Which brand covered more wall per gallon?

41. Which is steeper, Stone Canyon Drive, which rises 840 feet over a horizontal distance of 1500 feet, or Highway 33, which rises 1150 feet over a horizontal distance of 2000 feet?

Answer. Highway 33

42. The top of Romeo's ladder is on Juliet's window sill that is 11 feet above the ground, and the bottom of the ladder is 5 feet from the base of the wall. Is the incline of this ladder as steep as a firefighter's ladder that rises a height of 35 feet over a horizontal distance of 16 feet?

43. The table shows the amount of oil, B (in thousands of barrels), left in a tanker t minutes after it hits an iceberg and springs a leak.

t	0	10	20	30
B	800	750	700	650

a Write a linear function for B in terms of t.

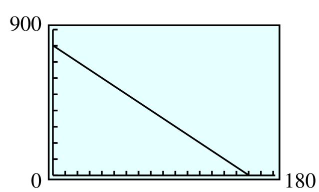
b Choose appropriate window settings on your calculator and graph your function.

c Give the slope of the graph, including units, and explain the meaning of the slope in terms of the oil leak.

Answer.

a $B = 800 - 5t$

b



c $m = -5$ thousand barrels/minute: The amount of oil in the tanker is decreasing by 5000 barrels per minute.

44. A traditional first experiment for chemistry students is to make 98 observations about a burning candle. Delbert records the height, h, of the candle in inches at various times t minutes after he lit it.

t	0	10	30	45
h	12	11.5	10.5	9.75

a Write a linear function for h in terms of t.

b Choose appropriate window settings on your calculator and graph your function.

c Give the slope of the graph, including units, and explain the meaning of the slope in terms of the candle.

45. An interior decorator bases her fee on the cost of a remodeling job. The accompanying table shows her fee, F, for jobs of various costs, C, both given in dollars.

C	5000	10,000	20,000	50,000
F	1000	1500	2500	5500

a Write a linear function for F in terms of C.

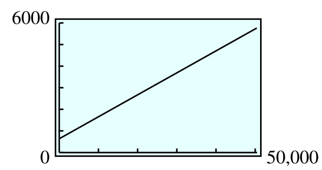
b Choose appropriate window settings on your calculator and graph your function.

c Give the slope of the graph, including units, and explain the meaning of the slope in terms of the the decorator's fee.

Answer.

a $F = 500 + 0.10C$

b



c $m = 0.10$: The fee increases by \$0.10 for each dollar increase in the remodeling job.

46. Auto registration fees in Connie's home state depend on the value of the automobile. The table below shows the registration fee, R, for a car whose value is V, both given in dollars.

V	5000	10,000	15,000	20,000
R	135	235	335	435

a Write a linear function for R in terms of V.

b Choose appropriate window settings on your calculator and graph your function.

c Give the slope of the graph, including units, and explain the meaning of the slope in terms of the registration fee.

Find the slope of the line segment joining each pair of points.

47. $(-1, 4),\ (3, -2)$

Answer. $\dfrac{-3}{2}$

48. $(5, 0),\ (2, -6)$

49. $(6.2, 1.4),\ (-2.1, 4.8)$

Answer. $\dfrac{-34}{83} \approx -0.4$

50. $(0, -6.4),\ (-5.6, 3.2)$

51. The planners at AquaWorld want the small water slide to have a slope of 25%. If the slide is 20 feet tall, how far should the end of the slide be from the base of the ladder?

Answer. 80 ft

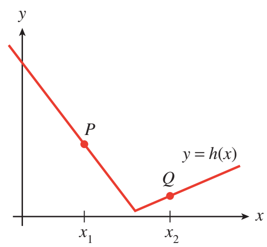
52. In areas with heavy snowfall, the pitch (or slope) of the roof of an A-frame house should be at least 1.2. If a small ski chalet is 40 feet wide at its base, how tall is the center of the roof?

Find the coordinates of the indicated points, and then write an algebraic expression using function notation for the indicated quantities.

53.

a Δy as x increases from x_1 to x_2

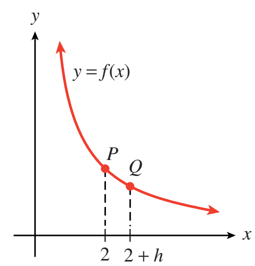
b The slope of the line segment joining P to Q



54.

a Δy as x increases from 2 to $2+h$

b The slope of the line segment joining P to Q



Answer.

a $h(x_2) - h(x_1)$

b $\dfrac{h(x_2) - h(x_1)}{x_2 - x_1}$

Which of the following tables could represent linear functions?

55.

a

r	E
1	5
2	$\frac{5}{2}$
3	$\frac{5}{3}$
4	$\frac{5}{4}$
5	1

b

s	t
10	6.2
20	9.7
30	12.6
40	15.8
50	19.0

Answer. Neither

56.

a

w	A
2	−13
4	−23
6	−33
8	−43
10	−53

b

x	C
0	0
2	5
4	10
8	20
16	40

Each table gives values for a linear function. Fill in the missing values.

57.

d	V
-5	-4.8
-2	-3
	-1.2
6	1.8
10	

Answer.

d	V
-5	-4.8
-2	-3
	-1.2
6	1.8
10	4.2

58.

q	S
-8	-8
-4	56
3	
	200
9	264

Find the slope and y-intercept of each line.

59. $2x - 4y = 5$

Answer. $m = \frac{1}{2},\ b = \frac{-5}{4}$

60. $\frac{1}{2}x + \frac{2}{3}y = \frac{5}{6}$

61. $8.4x + 2.1y = 6.3$

Answer. $m = -4,\ b = 3$

62. $y - 3 = 0$

For Problems 63 and 64,

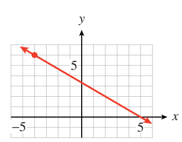
a Graph by hand the line that passes through the given point with the given slope.

b Find an equation for the line.

63. $(-4, 6)$; $m = \frac{-2}{3}$

Answer.

a



b $y = \frac{10}{3} - \frac{2}{3}x$

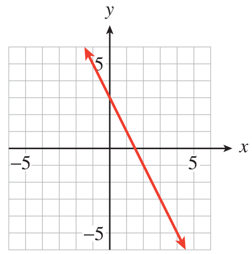
64. $(2, -5)$; $m = \frac{3}{2}$

For Problems 65 and 66,

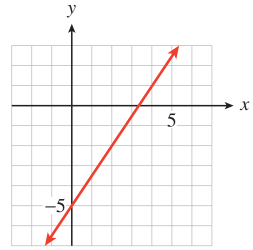
a Find the slope and y-intercept of each line.

b Write an equation for the line.

65.



66.



Answer.

a $m = -2,\ b = 3$

b $y = 3 - 2x$

67. What is the slope of the line whose intercepts are $(-5, 0)$ and $(0, 3)$?

Answer. $\frac{3}{5}$

68.

a Find the x- and y-intercepts of the line $\frac{x}{4} - \frac{y}{6} = 1$.

b What is the slope of the line in part (a)?

69.

a What is the slope of the line $y = 2 + \frac{3}{2}(x - 4)$?

b Find the point on the line whose x-coordinate is 4. Can there be more than one such point?

c Use your answers from parts (a) and (b) to find another point on the line.

Answer.

a $\frac{3}{2}$ b $(4, 2)$, no   c $(6, 5)$

70. A line passes through the point $(-5, 3)$ and has slope $\frac{2}{3}$. Find the coordinates of two more points on the line.

71. A line passes through the point $(-2, -6)$ and has slope $-\frac{8}{5}$. Find the coordinates of two more points on the line.

Answer. $(3, -14),\ (-7, 2)$

72. Find an equation in point-slope form for the line of slope $\frac{6}{5}$ that passes through $(-3, -4)$.

73. The rate at which air temperature decreases with altitude is called the lapse rate. In the troposphere, the layer of atmosphere that extends from the Earth's surface to a height of about 7 miles, the lapse rate is about 3.6°F for every 1000 feet. (Source: Ahrens, 1998)

a If the temperature on the ground is 62°F, write an equation for the temperature, T, at an altitude of h feet.

b What is the temperature outside an aircraft flying at an altitude of 30,000 feet? How much colder is that than the ground temperature?

c What is the temperature at the top of the troposphere?

Answer.

a $T = 62 - 0.0036h$ b -46°F; 108°F c -71°F

74. In his television program *Notes from a Small Island*, aired in February 1999, Bill Bryson discussed the future of the British aristocracy. Because not all families produce an heir, 4 or 5 noble lines die out each year. At this rate, Mr. Bryson says, if no more peers are created, there will be no titled families left by the year 2175.

a Assuming that on average 4.5 titled families die out each year, write an equation for the number, N, of noble houses left in year t, where $t = 0$ in the year 1999.

b Graph your equation.

c According to your graph, how many noble families existed in 1999? Which point on the graph corresponds to this information?

Find an equation for the line passing through the two given points.

75. $(3, -5),\ (-2, 4)$

Answer. $y = \dfrac{2}{5} - \dfrac{9}{5}x$

76. $(0, 8),\ (4, -2)$

For Problems 77 and 78,

a Make a table of values showing two data points.

b Find a linear equation relating the variables.

c State the slope of the line, including units, and explain its meaning in the context of the problem.

77. The population of Maple Rapids was 4800 in 1990 and had grown to 6780 by 2005. Assume that the population increases at a constant rate. Express the population, P, of Maple Rapids in terms of the number of years, t, since 1990.

Answer.

a

t	0	15
P	4800	6780

b $P = 4800 + 132t$

c $m = 132$ people/year: the population grew at a rate of 132 people per year.

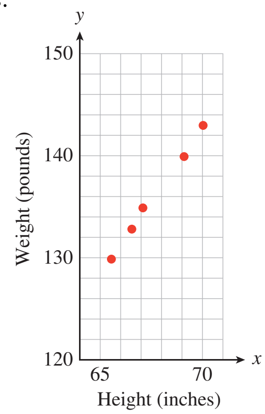
78. Cicely's odometer read 112 miles when she filled up her 14-gallon gas tank and 308 when the gas gauge read half full. Express her odometer reading, m, in terms of the amount of gas, g, she used.

79. In 1986, the space shuttle Challenger exploded because of O-ring failure on a morning when the temperature was about 30°F. Previously, there had been one incident of O-ring failure when the temperature was 70°F and three incidents when the temperature was 54°F. Use linear extrapolation to estimate the number of incidents of O-ring failure you would expect when the temperature is 30°F.

Answer. 6

80. Thelma typed a 19-page technical report in 40 minutes. She required only 18 minutes for an 8-page technical report. Use linear interpolation to estimate how long Thelma would require to type a 12-page technical report.

81. The scatterplot shows weights (in pounds) and heights (in inches) for a team of distance runners.



a Use a straightedge to draw a line that fits the data.

b Use your line to predict the weight of a 65-inch-tall runner and the weight of a 71-inch-tall runner.

c Use your answers from part (b) to approximate the equation of a regression line.

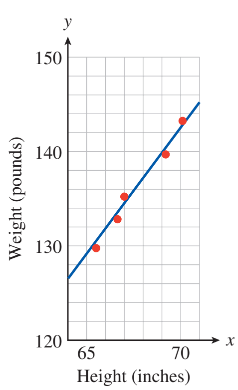
d Use your answer to part (c) to predict the weight of a runner who is 68 inches tall.

e The points on the scatterplot are $(65.5, 130)$, $(66.5, 133)$, $(67, 135)$, $(69, 140)$, and $(70, 143)$. Use your calculator to find the least squares regression line.

f Use the regression line to predict the weight of a runner who is 68 inches tall.

Answer.

a



b 129 lb, 145 lb

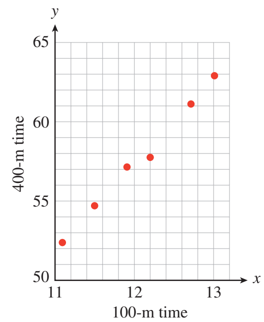
c $y = 2.\overline{6}x - 44.\overline{3}$

d 137 lb

e $y = 2.84x - 55.74$

f 137.33 lb

82. The scatterplot shows best times for various women running 400 meters and 100 meters.



a Use a straightedge to draw a line that fits the data.

b Use your line to predict the 400-meter time of a woman who runs the 100-meter dash in 11.2 seconds and the 400-meter time of a woman who runs the 100-meter dash in 13.2 seconds.

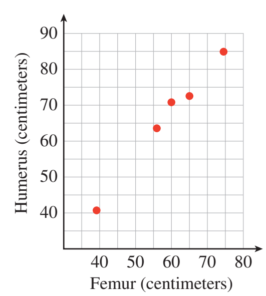
c Use your answers from part (b) to approximate the equation of a regression line.

d Use your answer to part (c) to predict the 400-meter time of a woman who runs the 100-meter dash in 12.1 seconds.

e The points on the scatterplot are $(11.1, 52.4)$, $(11.5, 54.7)$, $(11.9, 57.4)$, $(12.2, 57.9)$, $(12.7, 61.3)$, and $(13.0, 63.0)$. Use your calculator to find the least squares regression line.

f Use the regression line to predict the 400-meter time of a woman who runs the 100-meter dash in 12.1 seconds.

83. **Archaeopteryx** is an extinct creature with characteristics of both birds and reptiles. Only six fossil specimens are known, and only five of those include both a femur (leg bone) and a humerus (forearm bone) The scatterplot shows the lengths of femur and humerus for the five Archaeopteryx specimens.



a Use a straightedge to draw a line that fits the data.

b Predict the humerus length of an Archaeopteryx whose femur is 40 centimeters

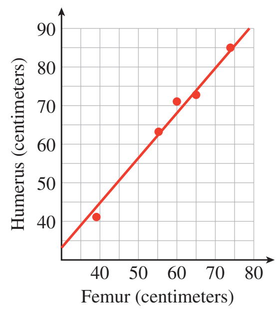
c Predict the humerus length of an Archaeopteryx whose femur is 75 centimeters

d Use your answers from parts (b) and (c) to approximate the equation of a regression line.

e Use your answer to part (d) to predict the humerus length of an Archaeopteryx whose femur is 60 centimeters.

f Use your calculator and the given points on the scatterplot to find the least squares regression line. Compare the score this equation gives for part (d) with what you predicted earlier. The ordered pairs defining the data are $(38, 41)$, $(56, 63)$, $(59, 70)$, $(64, 72)$, $(74, 84)$.

Answer.

a 

b 45 cm

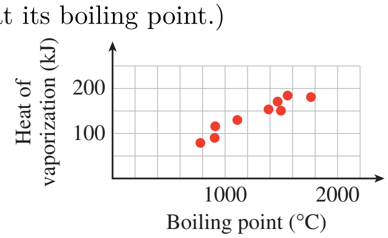
c 87 cm

d $y = 1.2x - 3$

e 69 cm

f $y = 1.197x - 3.660$; 68.16 cm

84. The scatterplot shows the boiling temperature of various substances on the horizontal axis and their heats of vaporization on the vertical axis. (The heat of vaporization is the energy needed to change the substance from liquid to gas at its boiling point.)



a Use a straightedge to estimate a line of best fit for the scatterplot.

b Use your line to predict the heat of vaporization of silver, whose boiling temperature is 2160°C.

c Find the equation of the regression line.

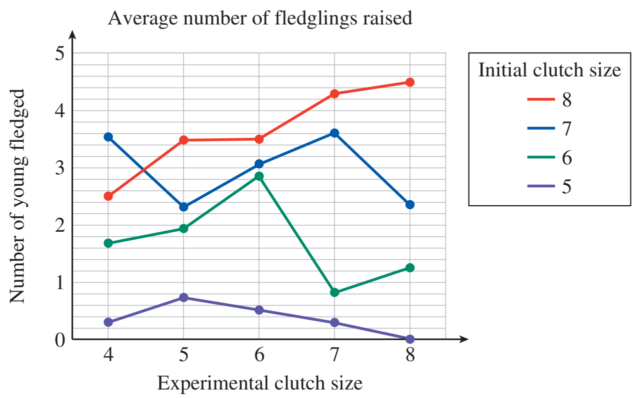
d Use the regression line to predict the heat of vaporization of potassium bromide, whose boiling temperature is 1435°C.

1.7 Projects for Chapter 1

Project 2 Optimal clutch size. The number of eggs (clutch size) that a bird lays varies greatly. Is there an optimal clutch size for birds of a given species, or does it depend on the individual bird?

In 1980, biologists in Sweden conducted an experiment on magpies as follows: They reduced or enlarged the natural clutch size by adding or removing eggs from the nests. They then computed the average number of fledglings successfully raised by the parent birds in each case.

The graph shows the results for magpies that initially laid 5, 6, 7, or 8 eggs. (Source: Högstedt, 1980, via Krebs as developed in Davies, 1993)



a Use the graph to fill in the table of values for the number of fledglings raised in each situation.

Initial clutch size laid	Experimental clutch size				
	4	5	6	7	8
5					
6					
7					
8					

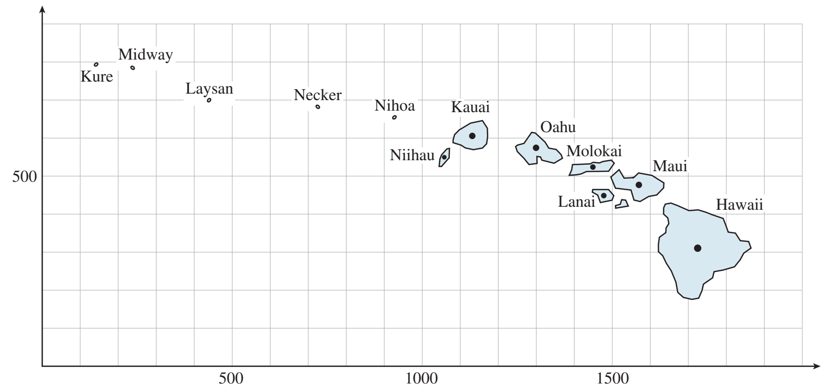
b For each initial clutch size, which experimental clutch size produced the most fledglings? Record your answers in the table.

Initial clutch size	5	6	7	8
Optimum clutch size				

c What conclusions can you draw in response to the question in the problem?

Project 3 Drift of Pacific tectonic plate. The Big Island of Hawaii is the last island in a chain of islands and submarine mountain peaks that stretch almost 6000 kilometers across the Pacific Ocean. All are extinct volcanoes except for the Big Island itself, which is still active.

The ages of the extinct peaks are roughly proportional to their distance from the Big Island. Geologists believe that the volcanic islands were formed as the tectonic plate drifted across a hot spot in the Earth's mantle. The figure shows a map of the islands, scaled in kilometers. (Source: Open University, 1998)



a The tables give the ages of the islands, in millions of years. Estimate the distance from each island to the Big Island, along a straight-line path through their centers. Fill in the third row of the tables.

Island	Hawaii	Maui	Lanai	Molokai	Oahu	Kauai
Age	0.5	0.8	1.3	1.8	3.8	5.1
Distance						

Island	Nihau	Nihoa	Necker	Laysan	Midway
Age	4.9	7.5	10	20	27
Distance					

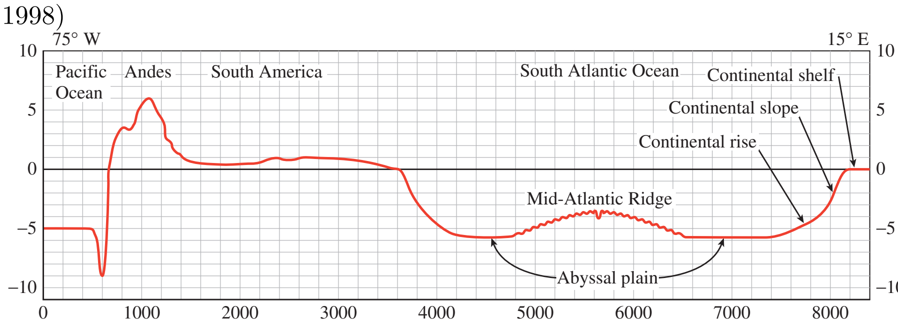
b Make a scatterplot showing the age of each island along the horizontal axis and its distance from Hawaii on the vertical axis.

c Draw a line of best fit through the data.

d Calculate the slope of the line of best fit, including units.

e Explain why the slope provides an estimate for the speed of the Pacific plate.

Project 4 Cross section of earth's surface. The graph shows a cross section of Earth's surface along an east-west line from the coast of Africa through the Atlantic Ocean to South America. Both axes are scaled in kilometers. Use the figure to estimate the distances in this problem. (Source: Open University, 1998)



a What is the highest land elevation shown in the figure? What is the lowest ocean depth shown? Give the horizontal coordinates of these two points, in kilometers west of the 75°W longitude line.

b How deep is the Atlantic Ocean directly above the crest of the Mid-Atlantic Ridge? How deep is the ocean above the abyssal plain on either side of the ridge?

c What is the height of the Mid-Atlantic Ridge above the abyssal plain? What is the width of the Mid-Atlantic Ridge?

d Using your answers to part (c), calculate the slope from the abyssal plain to the crest of the Mid-Atlantic Ridge, rounded to five decimal places

e Estimate the slopes of the continental shelf, the continental slope, and the continental rise. Use the coordinates of the points indicated on the figure

f Why do these slopes look much steeper in the accompanying figure than their numerical values suggest?

Project 5 Mid-Atlantic Range. The Mid-Atlantic Ridge is a mountain range on the sea floor beneath the Atlantic Ocean. It was discovered in the late nineteenth century during the laying of transatlantic telephone cables. The ridge is volcanic, and the ocean floor is moving away from the ridge on either side.

Geologists have estimated the speed of this sea-floor spreading by recording the age of the rocks on the sea floor and their distance from the ridge. (The age of the rocks is calculated by measuring their magnetic polarity. At known intervals over the last four million years, the Earth reversed its polarity, and this information is encoded in the rocks.) (Source: Open University, 1998)

a According to the table, rocks that are 0.78 million years old have moved 17 kilometers from the ridge. What was the speed of spreading over the past 0.78 million years? (This is the rate of spreading closest to the ridge.)

b Plot the data in the table, with age on the horizontal axis and separation distance on the vertical axis. Draw a line of best fit through the data.

c Calculate the slope of the regression line. What are the units of the slope?

d The slope you calculated in part (c) represents the average spreading rate over the past 3.58 million years. Is the average rate greater or smaller than the rate of spreading closest to the ridge?

e Convert the average spreading rate to millimeters per year

Age (millions of years)	0.78	0.99	1.07	1.79	1.95	2.60	3.04	3.11	3.22	3.33	3.58
Distance (km)	17	18	21	32	39	48	58	59	62	65	66

Project 6 Naismith's rule. Naismith's rule is used by runners and walkers to estimate journey times in hilly terrain. In 1892, Naismith wrote in the *Scottish Mountaineering Club Journal* that a person ``in fair condition should allow for easy expeditions an hour for every three miles on the map, with an additional hour for every 2000 feet of ascent.'' (Source: Scarf, 1998)

a According to Naismith, one unit of ascent requires the same time as how many units of horizontal travel? (Convert miles to feet.) This is called **Naismith's number**. Round your answer to one decimal place

b A walk in the Brecon Beacons in Wales covers 3.75 kilometers horizontally and climbs 582 meters. What is the equivalent flat distance?

c If you can walk at a pace of 15 minutes per kilometer over flat ground, how long will the walk in the Brecon Beacons take?

Project 7 Improved Naismith's number. Empirical investigations have improved Naismith's number (see Problem 5) to 8.0 for men and 9.5 for women. Part of the Karrimor International Mountain Marathon in the Arrochar Alps in Scotland has a choice of two routes. Route A is 1.75 kilometers long with a 240-meter climb, and route B is 3.25 kilometers long with a 90-meter climb. (Source: Scarf, 1998)

a Which route is faster for women?

b Which route is faster for men?

c At a pace of 6 minutes per flat kilometer, how much faster is the preferred route for women?

d At a pace of 6 minutes per flat kilometer, how much faster is the preferred route for men?