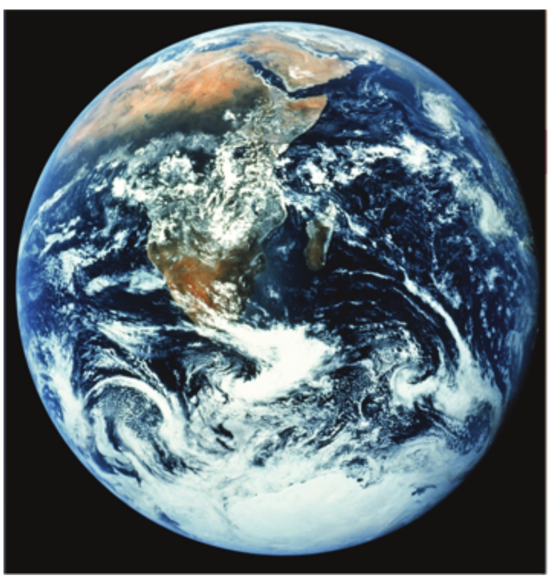
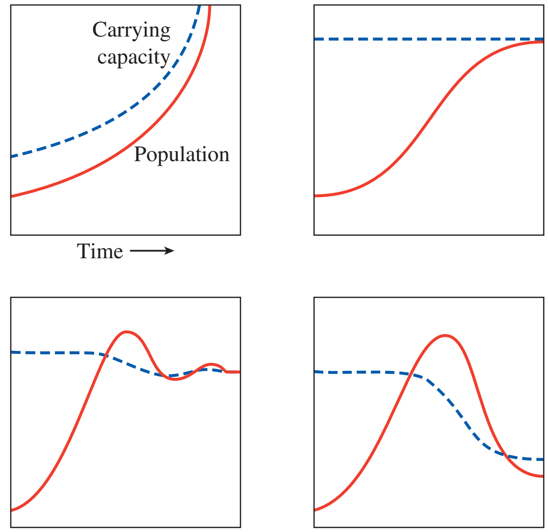
Chapter 2

Modeling with Functions



World3 is a computer model developed by a team of researchers at MIT. The model tracks population growth, use of resources, land development, industrial investment, pollution, and many other variables that describe human impact on the planet.

The figure below is taken from *Limits to Growth: The 30-Year Update*. The graphs represent four possible answers to World3's core question: How may the global population and economy interact with and adapt to Earth's limited carrying capacity (the maximum it can sustain) over the coming decades?



Source: Meadows, Randers, and Meadows, 2004

In this chapter, we examine the properties and features of some basic nonlinear functions and how they may be used as mathematical models.

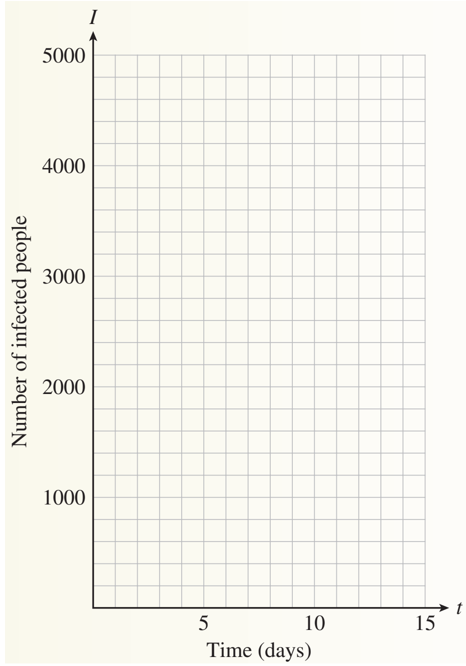
Investigation 8 Epidemics. A contagious disease whose spread is unchecked can devastate a confined population. For example, in the early sixteenth century Spanish troops introduced smallpox into the Aztec population in Central America, and the resulting epidemic contributed significantly to the fall of Montezuma's empire.

Suppose that an outbreak of cholera follows severe flooding in an isolated town of 5000 people. Initially (on Day 0), 40 people are infected. Every day after that, 25% of those still healthy fall ill.

1 At the beginning of the first day (Day 1), how many people are still healthy? _______ How many will fall ill during the first day? _______ What is the total number of people infected after the first day? _______

2 Check your results against the first two rows of the table. Subtract the total number of infected residents from 5000 to find the number of healthy residents at the beginning of the second day. Then fill in the rest of the table for 10 days. (Round off decimal results to the nearest whole number.)

Day	Number Healthy	New Patients	Total Infected
0	5000	40	40
1	4960	1240	1280
2			
3			
4			
5			
6			
7			
8			
9			
10			



3 Use the last column of the table to plot the total number of infected residents, I, against time, t. Connect your data points with a smooth curve.

4 Do the values of I approach some largest value? Draw a dotted horizontal line at that value of I. Will the values of I ever exceed that value?

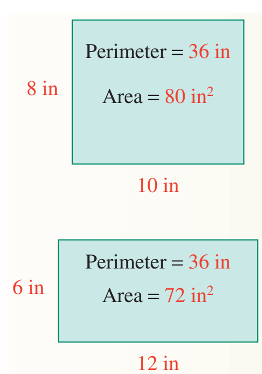
5 What is the first day on which at least 95% of the population is infected?

6 Look back at the table. What is happening to the number of new patients each day as time goes on? How is this phenomenon reflected in the graph? How would your graph look if the number of new patients every day were a constant?

7 Summarize your work: In your own words, describe how the number of residents infected with cholera changes with time. Include a description of your graph.

Investigation 9 Perimeter and Area.

Do all rectangles with the same perimeter, say 36 inches, have the same area? Two different rectangles with perimeter 36 inches are shown at right. The first rectangle has base 10 inches and height 8 inches, and its area is 80 square inches. The second rectangle has base 12 inches and height 6 inches. Its area is 72 square inches.



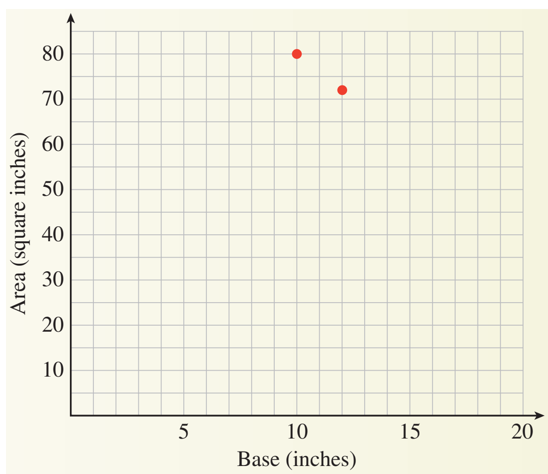
Part I

1 The table shows the bases of various rectangles, in inches. Each rectangle has a perimeter of 36 inches. Fill in the height and the area of each rectangle. (To find the height of the rectangle, reason as follows: The base plus the height makes up half of the rectangle's perimeter.)

2 What happens to the area of the rectangle when we change its base but still keep the perimeter at 36 inches? Plot the points with coordinates (Base, Area). (For this graph, we will not use the heights of the rectangles.) The first two points, $(10, 80)$ and $(12, 72)$, are shown. Connect your data points with a smooth curve.

3 What are the coordinates of the highest point on your graph?

Base	Height	Area
10	8	80
12	6	72
4		
14		
5		
17		
9		
2		
11		
4		
16		
15		
1		
6		
8		
4		
13		
7		



Part II

1 Each point on your graph represents a particular rectangle with perimeter 36 inches. The first coordinate of the point gives the base of the rectangle, and the second coordinate gives the area of the rectangle. What is the largest area you found among rectangles with perimeter 36 inches? What is the base for that rectangle? What is its height?

2 Give the dimensions of the rectangle that corresponds to the point $(13, 65)$.

3 Find two points on your graph with vertical coordinate 80.

4 If the rectangle has area 80 square inches, what is its base? Why are there two different answers? Describe the rectangle corresponding to each answer.

5 Now we will write an algebraic expression for the area of the rectangle in terms of its base. Let x represent the base of the rectangle. First, express

the height of the rectangle in terms of x. (Hint: If the perimeter of the rectangle is 36 inches, what is the sum of the base and the height?) Now write an expression for the area of the rectangle in terms of x.

6 Use your formula from part (8) to compute the area of the rectangle when the base is 5 inches. Does your answer agree with the values in your table and the point on your graph?

7 Use your formula to compute the area of the rectangle when $x = 0$ and when $x = 18$. Describe the rectangles that correspond to these data points.

8 Continue your graph to include the points corresponding to $x = 0$ and $x = 18$.

2.1 Nonlinear Models

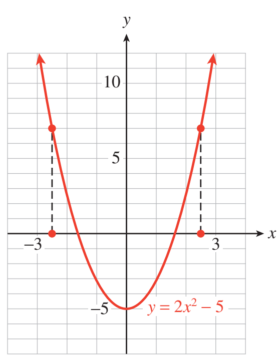
In Chapter 1, we considered models described by linear functions. In this chapter, we begin our study of nonlinear models.

2.1.1 Solving Nonlinear Equations

When studying nonlinear models, we will need to solve nonlinear equations. For example, in Investigation 9, p. 151 we used a graph to solve the quadratic equation

$$18x - x^2 = 80$$

Here is another example. The figure shows a table and a graph for the function $y = 2x^2 - 5$.



x	-3	-2	-1	0	1	2	3
y	13	3	-3	-5	-3	3	13

You can see that there are two points on the graph for each y-value greater than -5. For example, the two points with y-coordinate 7 are shown. To solve the equation

$$2x^2 - 5 = 7$$

we need only find the x-coordinates of these points. From the graph, the solutions appear to be about 2.5 and -2.5.

How can we solve this equation algebraically? The opposite operation for squaring a number is taking a square root. So we can undo the operation of squaring by extracting square roots. We first solve for x^2 to get

$$2x^2 = 12$$
$$x^2 = 6$$

and then take square roots to find

$$x = \pm\sqrt{6}$$

Caution 2.1.1 Don't forget that every positive number has two square roots. The symbol $\pm$ (read ``plus or minus'') is a shorthand notation used to indicate both square roots of 6.

The exact solutions are thus $\sqrt{6}$ and $-\sqrt{6}$. We can also find decimal approximations for the solutions using a calculator. Rounded to two decimal places, the approximate solutions are 2.45 and -2.45.

In general, we can solve equations of the form $ax^2 + c = 0$ by isolating x^2 on one side of the equation and then taking the square root of each side. This method for solving equations is called **extraction of roots**.

Extraction of Roots.

To solve the equation

$$ax^2 + c = 0$$

1. Isolate x^2.
2. Take square roots of both sides. There are two solutions.

Example 2.1.2 If a cat falls off a tree branch 20 feet above the ground, its height t seconds later is given by $h = 20 - 16t^2$.

a What is the height of the cat 0.5 second later?

b How long does the cat have to get in position to land on its feet before it reaches the ground?

Solution.

a In this question, we are given a value of t and asked to find the corresponding value of h. To do this, we evaluate the formula for $t = 0.5$. We substitute **0.5** for t into the formula and simplify.

$$\begin{aligned} h &= 20 - 16(\mathbf{0.5})^2 && \text{Compute the power.} \\ &= 20 - 16(0.25) && \text{Multiply; then subtract.} \\ &= 20 - 4 = 16 \end{aligned}$$

The cat is 16 feet above the ground after 0.5 second.

b We would like to find the value of t when the height, h, is known. We substitute $h = \mathbf{0}$ into the equation to obtain

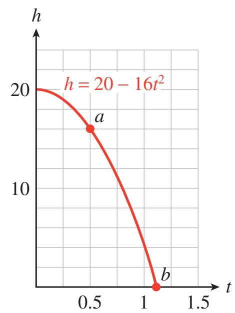
$$\mathbf{0} = 20 - 16t^2$$

To solve this equation, we use extraction of roots. First we isolate t^2 on one side of the equation.

$$\begin{aligned} 16t^2 &= 20 && \text{Divide by 16.} \\ t^2 &= \frac{20}{16} = 1.25 \end{aligned}$$

Then we take the square root of both sides of the equation to find

$$t = \pm\sqrt{1.25} \approx \pm 1.118$$



Only the positive solution makes sense here, so the cat has approximately 1.12 seconds to get into position for landing. A graph of the cat's height after t seconds is shown at left. The points corresponding to parts (a) and (b) are labeled.

□

Note 2.1.3 In part (a) of Example 2.1.2, p. 154 we **evaluated** the expression $20 - 16t^2$ to find a value for h, and in part (b) we **solved** the equation $0 = 20 - 16t^2$ to find a value for t.

Checkpoint 2.1.4

a Solve by extracting roots $\dfrac{3x^2 - 8}{5} = 10$.

First, isolate $\boldsymbol{x^2}$.

Take the square root of both sides.

b Give exact answers; then give approximations rounded to two decimal places.

Answer. $x = \pm\sqrt{\dfrac{58}{3}} \approx \pm 4.40$

2.1.2 Solving Formulas

We can use extraction of roots to solve many formulas involving the square of the variable.

Example 2.1.5 The formula $V = \dfrac{1}{3}\pi r^2 h$ gives the volume of a cone in terms of its height and radius. Solve the formula for r in terms of V and h.

Solution. Because the variable we want is squared, we use extraction of roots. First, we multiply both sides by 3 to clear the fraction.

$$\begin{aligned} \mathbf{3}V &= \mathbf{3}\left(\frac{1}{3}\pi r^2 h\right) \\ 3V &= \pi r^2 h && \text{Divide both sides by } \boldsymbol{\pi h}\text{.} \\ \frac{3V}{\pi h} &= r^2 && \text{Take square roots.} \\ \pm\sqrt{\frac{3V}{\pi h}} &= r \end{aligned}$$

Because the radius of a cone must be a positive number, we use only the positive square root: $r = \sqrt{\dfrac{3V}{\pi h}}$.

□

Checkpoint 2.1.6 Find a formula for the radius of a circle in terms of its area.

Hint. Start with the formula for the area of a circle: $A =$ ________
Solve for r in terms of A.

Answer. $r = \sqrt{A/\pi}$

2.1.3 More Extraction of Roots

Equations of the form

$$a(px+q)^2+r=0$$

can also be solved by extraction of roots after isolating the squared expression, $(px+q)^2$.

Example 2.1.7 Solve the equation $3(x-2)^2=48$

Solution. First, we isolate the perfect square, $(x-2)^2$.

$$\begin{aligned} 3(x-2)^2 &= 48 && \text{Divide both sides by } \mathbf{3}. \\ (x-2)^2 &= 16 && \text{Take the square root of each side.} \\ x-2 &= \pm\sqrt{16}=\pm 4 \end{aligned}$$

This gives us two equations for x,

$$\begin{aligned} x-2=4 \quad &\text{or} \quad x-2=-4 && \text{Solve each equation.} \\ x=6 \quad &\text{or} \quad x=-2 \end{aligned}$$

The solutions are 6 and -2. □

Here is a general strategy for solving equations by extraction of roots.

Extraction of Roots.

To solve the equation

$$a(px+q)^2+r=0$$

1. Isolate the squared expression, $(px+q)^2$.
2. Take the square root of each side of the equation. Remember that a positive number has two square roots.
3. Solve each equation. There are two solutions.

Checkpoint 2.1.8 Solve $2(5x+3)^2=38$ by extracting roots.

a Give your answers as exact values.

b Find approximations for the solutions to two decimal places.

Answer.

a $x=\dfrac{-3\pm\sqrt{19}}{5}$

b $x\approx -1.47$ or $x\approx 0.27$

2.1.4 Compound Interest and Inflation

Many savings institutions offer accounts on which the interest is *compounded annually*. At the end of each year, the interest earned is added to the principal, and the interest for the next year is computed on this larger sum of money.

Compound Interest.

If interest is compounded annually for n years, the amount, A, of money in an account is given by

$$A=P(1+r)^n$$

where P is the principal and r is the interest rate, expressed as a decimal fraction.

Example 2.1.9 Carmella invests \$3000 in an account that pays an interest rate, r, compounded annually.

a Write an expression for the amount of money in Carmella's account after two years.

b What interest rate would be necessary for Carmella's account to grow to \$3500 in two years?

Solution.

a We use the formula above with $P = 3000$ and $n = 2$. Carmella's account balance will be

$$A = 3000(1 + r)^2$$

b We substitute 3500 for A in the equation.

$$\mathbf{3500} = 3000(1 + r)^2$$

We can solve this equation in r by extraction of roots. First, we isolate the perfect square.

$$\begin{aligned} 3500 &= 3000(1 + r)^2 && \text{Divide both sides by 3000.} \\ 1.1\overline{6} &= (1 + r)^2 && \text{Take the square root of both sides.} \\ \pm 1.0801 &\approx 1 + r && \text{Subtract 1 from both sides.} \\ r &\approx 0.0801 \text{ or } r \approx -2.0801 \end{aligned}$$

Because the interest rate must be a positive number, we discard the negative solution. Carmella needs an account with interest rate $r \approx 0.0801$, or just over 8%, to achieve an account balance of \$3500 in two years.

□

The formula for compound interest also applies to the effects of inflation. For instance, if there is a steady inflation rate of 4% per year, in two years an item that now costs \$100 will cost

$$\begin{aligned} A &= P(1 + r)^2 \\ &= 100(1 + 0.04)2 = \$108.16 \end{aligned}$$

Checkpoint 2.1.10 Two years ago, the average cost of dinner and a movie was \$24. This year the average cost is \$25.44. What was the rate of inflation over the past two years?

Answer. $r \approx 2.96\%$

2.1.5 Other Nonlinear Equations

Because squaring and taking square roots are opposite operations, we can solve the equation

$$\sqrt{x} = 8.2$$

by squaring both sides to get

$$\begin{aligned} (\sqrt{x})^2 &= 8.2^2 \\ x &= 67.24 \end{aligned}$$

Similarly, we can solve

$$x^3 = 258$$

by taking the cube root of both sides, because cubing and taking cube roots are opposite operations. Rounding to three places, we find

$$\begin{aligned} \sqrt[3]{x^3} &= 258 \\ x &\approx 6.366 \end{aligned}$$

The notion of undoing operations can help us solve a variety of simple nonlinear equations. The operation of taking a reciprocal is its own opposite, so we solve the equation

$$\frac{1}{x} = 50$$

by taking the reciprocal of both sides to get

$$x = \frac{1}{50} = 0.02$$

Example 2.1.11 Solve $\dfrac{3}{x-2} = 4$

Solution. We begin by taking the reciprocal of both sides of the equation to get

$$\frac{x-2}{3} = \frac{1}{4}$$

We continue to undo the operations in reverse order. First, we multiply both sides by 3.

$$\begin{aligned} x - 2 &= \frac{3}{4} && \text{Add 2 to both sides.} \\ x &= 2 + \frac{3}{4} = \frac{11}{4} && \mathbf{\tfrac{2}{1} = \tfrac{8}{4}}, \text{ so } \mathbf{\tfrac{2}{1} + \tfrac{3}{4} = \tfrac{8}{4} + \tfrac{3}{4} = \tfrac{11}{4}} \end{aligned}$$

The solution is $\dfrac{11}{4}$, or 2.75. □

Checkpoint 2.1.12 Solve $2\sqrt{x+4} = 6$

Answer. $x = 5$

Checkpoint 2.1.13 Use the intersect feature to solve the equation $2x^2 - 5 = 7$. Round your answers to three decimal places.

Answer. $x = \pm 2.449$

2.1.6 Section Summary

2.1.6.1 Vocabulary

Look up the definitions of new terms in the Glossary.

- Quadratic
- Compound interest
- Exact solution
- Perfect square
- Extraction of roots
- Inflation
- Area
- Cube root
- Isolate
- Height
- Perimeter
- Reciprocal

2.1.6.2 CONCEPTS

1 Extraction of Roots.

To solve the equation

$$a(px+q)^2 + r = 0$$

1 Isolate the squared expression, $(px+q)^2$.

2 Take the square root of each side of the equation. Remember that a positive number has two square roots.

3 Solve each equation. There are two solutions.

2 Compound Interest.

If interest is compounded annually for n years, the amount, A, of money in an account is given by

$$A = P(1+r)^n$$

where P is the principal and r is the interest rate, expressed as a decimal fraction.

3 We can give exact answers to a simple nonlinear equation, or we can give decimal approximations.

4 Simple nonlinear equations can be solved by undoing the operations on the variable.

2.1.6.3 STUDY QUESTIONS

1 How many square roots does a positive number have?

2 What is the first step in solving the equation $a(px+q)^2 = r$ by extraction of roots?

3 Give the exact solutions of the equation $x^2 = 10$, and then give decimal approximations rounded to hundredths.

4 State a formula for the amount in an account on which 5% interest is compounded annually.

5 Give an example of two rectangles with the same perimeter but different areas.

6 The perimeter of a rectangle is 50 meters. Write an expression for the length of the rectangle in terms of its width.

7 What is the opposite operation for taking a reciprocal?

8 What is the reciprocal of $\dfrac{1}{\sqrt{x}}$?

2.1.6.4 SKILLS

Practice each skill in the Homework 2.1.7, p. 160 problems listed.

1 Solve equations by extraction of roots: #1–12, 31–42

2 Solve formulas: #13–16, 63–68

3 Use the Pythagorean theorem: #19–24

4 Solve equations graphically: #25–30

5 Solve simple nonlinear equations: #43–54

6 Solve problems: #55–62

2.1.7 Nonlinear Models (Homework 2.1)

For Problems 1-6, solve by extracting roots. Give exact values for your answers.

1. $9x^2 = 25$

Answer. $\pm\dfrac{5}{3}$

2. $4x^2 = 9$

3. $4x^2 - 24 = 0$

Answer. $\pm\sqrt{6}$

4. $3x^2 - 9 = 0$

5. $\dfrac{2x^2}{3} = 4$

Answer. $\pm\sqrt{6}$

6. $\dfrac{3x^2}{5} = 6$

For Problems 7-12, solve by extracting roots. Round your answers to two decimal places.

7. $2x^2 = 14$

Answer. ± 2.65

8. $3x^2 = 15$

9. $1.5x^2 = 0.7x^2 + 26.2$

Answer. ± 5.72

10. $0.4x^2 = 2x^2 - 8.6$

11. $5x^2 - 97 = 3.2x^2 - 38$

Answer. ± 5.73

12. $17 - \dfrac{x^2}{4} = 43 - x^2$

>For Problems 13-16, solve the formulas for the specified variable.

13. $F = \dfrac{mv^2}{r}$, for v

Answer. $\pm\sqrt{\dfrac{Fr}{m}}$

14. $A = \dfrac{\sqrt{3}}{4}s^2$, for s

15. $s = \dfrac{1}{2}gt^2$, for t

Answer. $\pm\sqrt{\dfrac{2s}{g}}$

16. $S = 4\pi r^2$, for r

For Problems 17 and 18, refer to the geometric formulas in Appendix E, p. 1027.

17. A conical coffee filter is 8.4 centimeters tall.

a Write a formula for the filter's volume in terms of its widest radius (at the top of the filter).

b Complete the table of values for the volume equation. If you double the radius of the filter, by what factor does the volume increase?

r	1	2	3	4	5	6	7	8
V								

c If the volume of the filter is 302.4 cubic centimeters, what is its radius?

d Use your calculator to graph the volume equation. Locate the point on the graph that corresponds to the filter in part (c).

Answer.

a $V = 2.8\pi r^2 \approx 8.8r^2$

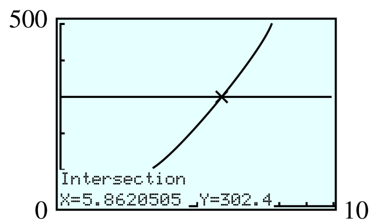
b

r	1	2	3	4	5	6	7	8
V	8.8	35.2	79.2	140.7	219.9	316.7	431.0	563.0

The volume increases by a factor of 4.

c 5.86 cm

d



18. A large bottle of shampoo is 20 centimeters tall and cylindrical in shape.

a Write a formula for the volume of the bottle in terms of its radius.

b Complete the table of values for the volume equation. If you halve the radius of the bottle, by what factor does the volume decrease?

r	1	2	3	4	5	6	7	8
V								

c What radius should the bottle have if it must hold 240 milliliters of shampoo? (One milliliter is equal to 1 cubic centimeter.)

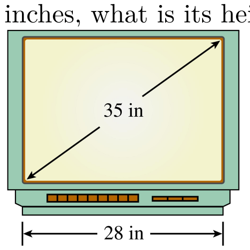
d Use your calculator to graph the volume equation. Locate the point on the graph that corresponds to the bottle in part (c).

For Problems 19–24,

a Make a sketch of the situation described, and label a right triangle.

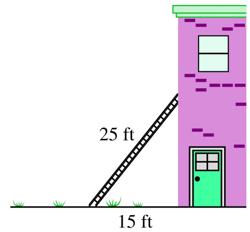
b Use the Pythagorean theorem to solve each problem. (See Algebra Skills Refresher Section A.11, p. 944 to review the Pythagorean theorem.)

19. The size of a TV screen is the length of its diagonal. If the width of a 35-inch TV screen is 28 inches, what is its height?

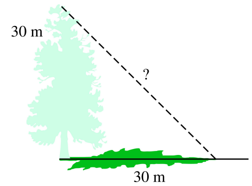


Answer. 21 in.

20. How high on a building will a 25-foot ladder reach if its foot is 15 feet away from the base of the wall?

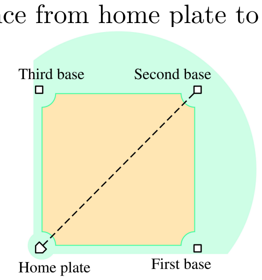


21. If a 30-meter pine tree casts a shadow of 30 meters, how far is the tip of the shadow from the top of the tree?

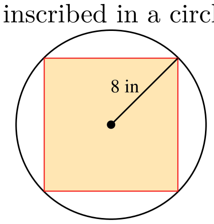


Answer. $\sqrt{1800} \approx 42.4$ m

22. A baseball diamond is a square whose sides are 90 feet in length. Find the straight-line distance from home plate to second base.

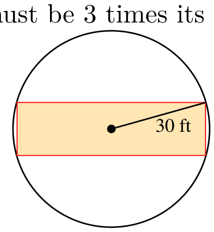


23. What size square can be inscribed in a circle of radius 8 inches?



Answer. $\sqrt{128}$ in. by $\sqrt{128}$ in. ≈ 11.3 in. $\times$ 11.3 in.

24. What size rectangle can be inscribed in a circle of radius 30 feet if the length of the rectangle must be 3 times its width?



For Problems 25–30,

a Use a calculator or computer to graph the function in the suggested window.

b Use your graph to find two solutions for the given equation. (See Section 1.3, p. 57 to review graphical solution of equations.)

c Check your solutions algebraically, using mental arithmetic.

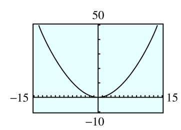
25.

a $y = \dfrac{1}{4}x^2$

Xmin $= -15$ Xmax $= 15$
Ymin $= -10$ Ymax $= 40$

b $\dfrac{1}{4}x^2 = 36$

Answer.

a 

b $x = \pm 12$

26.

a $y = 8x^2$

Xmin $= -15$ Xmax $= 15$
Ymin $= -50$ Ymax $= 450$

b $8x^2 = 392$

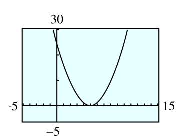
27.

a $y = (x-5)^2$

Xmin $= -5$ Xmax $= 15$
Ymin $= -5$ Ymax $= 25$

b $(x-5)^2 = 16$

Answer.

a 

b $x = 1$ or $x = 9$

28.

a $y = (x+2)^2$

Xmin $= -10$ Xmax $= 10$
Ymin $= -2$ Ymax $= 12$

b $(x+2)^2 = 9$

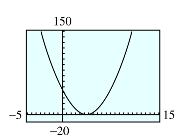
29.

a $y = 3(x-4)^2$

Xmin $= -5$ Xmax $= 15$
Ymin $= -20$ Ymax $= 130$

b $3(x-4)^2 = 108$

Answer.

a 

b $x = 10$ or $x = -2$

30.

a $y = \dfrac{1}{2}(x+3)^2$

Xmin $= -15$ Xmax $= 5$
Ymin $= -5$ Ymax $= 15$

b $\dfrac{1}{2}(x+3)^2 = 8$

For Problems 31-42, solve by extraction of roots.

31. $(x-2)^2 = 9$

Answer. $5, -1$

32. $(x+3)^2 = 4$

33. $(2x-1)^2 = 16$

Answer.
$\dfrac{5}{2}, \dfrac{-3}{2}$

34. $(3x+1)^2 = 25$

35. $4(x+2)^2 = 12$

Answer.
$-2 \pm \sqrt{3}$

36. $6(x-5)^2 = 42$

37. $\left(x - \frac{1}{2}\right)^2 = \frac{3}{4}$

Answer.
$\frac{1}{2} \pm \frac{\sqrt{3}}{2}$

38. $\left(x - \frac{2}{3}\right)^2 = \frac{5}{9}$

39. $81\left(x + \frac{1}{3}\right)^2 = 1$

Answer.
$\frac{-2}{9}, \frac{-4}{9}$

40. $16\left(x + \frac{1}{2}\right)^2 = 1$

41. $3(8x-7)^2 = 24$

Answer.
$\frac{7}{8} \pm \frac{\sqrt{8}}{8}$

42. $-2(5x-12)^2 = 48$

For Problems 43–54,

a Solve algebraically.

b Use the **intersect** feature on a graphing calculator to solve.

43. $4x^3 - 12 = 852$

Answer. 6

44. $\frac{8x^3+6}{3} = 74$

45. $5\sqrt{x} - 9 = 31$

Answer. 64

46. $25 - 2\sqrt{x} = 1$

47. $\frac{1}{2x-3} = \frac{3}{4}$

Answer. $\frac{13}{6}$

48. $\frac{15}{x+16} = 3$

49. $8 - 6\sqrt[3]{x} = -4$

Answer. 8

50. $\frac{4\sqrt[3]{x}}{5} + 3 = 7$

51. $\sqrt{3x-2} + 3 = 8$

Answer. 9

52. $6\sqrt{1-2x} = 30$

53. $\frac{2}{\sqrt{4x-2}} = 8$

Answer. $\frac{33}{64}$

54. $\frac{1}{\sqrt{x+2}} = \frac{3}{4}$

55. Cyril plans to invest $5000 in a money market account that pays interest compounded annually.

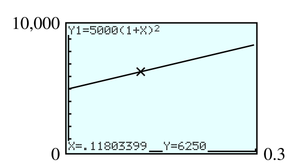
a Write a formula for the balance, B, in Cyril's account after two years as a function of the interest rate, r.

b If Cyril would like to have $6250 in two years, what interest rate must the account pay?

c Use your calculator to graph the formula for Cyril's account balance. Locate the point on the graph that corresponds to the amount in part (b).

Answer.

a $B = 5000(1+r)^2$

c 

b 11.8%

56. You plan to deposit your savings of $1600 in an account that compounds interest annually.

a Write a formula for the amount in your savings account after two years as a function of the interest rate, r.

b To the nearest tenth of a percent, what interest rate will you require if you want your \$1600 to grow to \$2000 in two years?

c Use your calculator to graph the formula for the account balance. Locate the point on the graph that corresponds to the amount in part (b).

57. Carol's living expenses two years ago were \$1200 per month. This year, the same items cost Carol \$1400 per month. What was the annual inflation rate for the past two years?

Answer. 8%

58. Two years ago, the average price of a house in the suburbs was $\$188,600$. This year, the average price is $\$203,700$. What was the annual percent increase in the cost of a house?

59. A machinist wants to make a metal section of pipe that is 80 millimeters long and has an interior volume of 9000 cubic millimeters. If the pipe is 2 millimeters thick, its interior volume is given by the formula

$$V = \pi(r-2)^2 h$$

where h is the length of the pipe and r is its radius. What should the radius of the pipe be?

Answer. 7.98 mm

60. A storage box for sweaters is constructed from a square sheet of corrugated cardboard measuring x inches on a side. The volume of the box, in cubic inches, is

$$V = 10(x-20)^2$$

If the box should have a volume of 1960 cubic inches, what size cardboard square is needed?

61. The area of an equilateral triangle is given by the formula $A = \dfrac{\sqrt{3}}{4}s^2$, where s is the length of the side.

a Find the areas of equilateral triangles with sides of length 2 centimeters, 4 centimeters, and 10 centimeters. First give exact values, then approximations to hundredths.

b Graph the area equation in the window

Xmin = 0	Xmax = 14.1
Ymin = 0	Ymax = 60

Use the `TRACE` or **value** feature to verify your answers to part (a).

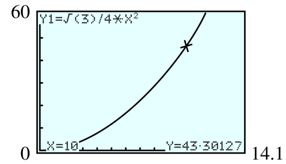
c What does the coordinate $(5.1, 11.26266)$ represent?

d Use your graph to estimate the side of an equilateral triangle whose area is 20 square centimeters.

e Write and solve an equation to answer part (d).

f If the area of an equilateral triangle is $100\sqrt{3}$ square centimeters, what is the length of its side?

Answer.

a $\sqrt{3} \approx 1.73$ sq cm, $4\sqrt{3} \approx 6.93$ sq cm, $25\sqrt{3} \approx 43.3$ sq cm b 

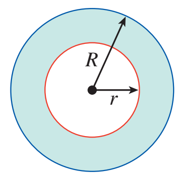
c An equilateral triangle with side 5.1 cm has area 11.263 cm^2.

d side $\approx$ 6.8 cm

e $\dfrac{\sqrt{3}}{4}s^2 = 20$; $s \approx 6.8$

f $\approx$ 20 cm

62. The area of the ring in the figure is given by the formula $A = \pi R^2 - \pi r^2$, where R is the radius of the outer circle and r is the radius of the inner circle.



a Suppose the inner radius of the ring is kept fixed at $r = 4$ centimeters, but the radius of the outer circle, R, is allowed to vary. Find the area of the ring when the outer radius is 6 centimeters, 8 centimeters, and 12 centimeters. First give exact values, then approximations to hundredths.

b Graph the area equation, with r = 4, in the window

Xmin = 0	Xmax = 14.1
Ymin = 0	Ymax = 400

Use the `TRACE` feature to verify your answers to part (a).

c **Trace** along the curve to the point $(9.75, 248.38217)$. What do the coordinates of this point represent?

d Use your graph to estimate the outer radius of the ring when its area is 100 square centimeters.

e Write and solve an equation to answer part (d).

f If the area of the ring is 9π square centimeters, what is the radius of the outer circle?

For Problems 63–68, solve for x in terms of a, b, and c.

63. $\dfrac{ax^2}{b} = c$
Answer. $\pm\sqrt{\dfrac{bc}{a}}$

64. $\dfrac{bx^2}{c} - a = 0$

65. $(x-a)^2 = 16$
Answer. $a \pm 4$

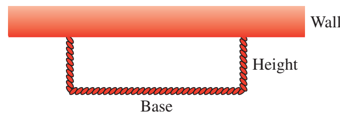
66. $(x+a)^2 = 36$

67. $(ax+b)^2 = 9$
Answer. $\dfrac{-b \pm 3}{a}$

68. $(ax-b)^2 = 25$

69. You have 36 feet of rope and you want to enclose a rectangular display area against one wall of an exhibit hall. The area enclosed depends on the dimensions of the rectangle you make. Because the wall makes one side of the rectangle, the length of the rope accounts for only three sides. Thus

$$\text{Base} + 2(\text{Height}) = 36$$



a Complete the table showing the base and the area of the rectangle for the given heights.

Height	Base	Area
1	34	34
2	32	64
3		
4		
5		
6		
7		
8		
9		

Height	Base	Area
10		
11		
12		
13		
14		
15		
16		
17		
18		

b Make a graph with *Height* on the horizontal axis and *Area* on the vertical axis. Draw a smooth curve through your data points.

c What is the area of the largest rectangle you can enclose in this way? What are its dimensions? On your graph, label the point that corresponds to this rectangle with the letter M.

d Let x stand for the height of a rectangle and write algebraic expressions for the base and the area of the rectangle.

e Enter your algebraic expression for the area in your calculator, then use the **Table** feature to verify the entries in your table in part (a).

f Graph your formula for area on your graphing calculator. Use your table of values and your handdrawn graph to help you choose appropriate `WINDOW` settings.

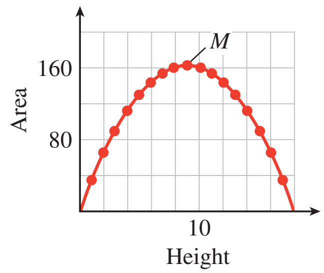
g Use the **intersect** command to find the height of the rectangle whose area is 149.5 square feet.

Answer.

a

Height	Base	Area
1	34	34
2	32	64
3	30	90
4	28	112
5	26	130
6	24	144
7	22	154
8	20	160
9	18	162

Height	Base	Area
10	16	160
11	14	154
12	12	144
13	10	130
14	8	112
15	6	90
16	4	64
17	2	34
18	0	0

b 

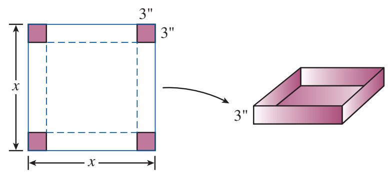
c 162 sq ft, with base 18 ft, height 9 ft

d Base: $36 - 2x$; area: $x(36 - 2x)$

e See (a)

f 6.5 ft or 11.5 ft

70. We are going to make an open box from a square piece of cardboard by cutting 3-inch squares from each corner and then turning up the edges as shown in the figure.



a Complete the table showing the side of the original sheet of cardboard, the dimensions of the box created from it, and the volume of the box.

Side	Length of box	Width of box	Height of box	Volume of box
7	1	1	3	3
8	2	2	3	12
9				
10				
11				
12				
13				
14				
15				

Explain why the side of the cardboard square cannot be smaller than 6 inches. What happens if the cardboard is exactly 6 inches on a side?

b Make a graph with *Side* on the horizontal axis and *Volume* on the vertical axis. Draw a smooth curve through your data points. (Use your table to help you decide on appropriate scales for the axes.)

c Let x represent the side of the original sheet of cardboard. Write algebraic expressions for the dimensions of the box and for its volume.

d Enter your expression for the volume of the box in your calculator; then use the **Table** feature to verify the values in your table in part (a).

e Graph your formula for volume on your graphing calculator. Use your table of values and your handdrawn graph to help you choose appropriate `WINDOW` settings.

f Use the **intersect** command to find out how large a square of cardboard you need to make a box with volume 126.75 cubic inches.

g Does your graph have a highest point? What happens to the volume of the box as you increase x?

71. The jump height, J, in meters, achieved by a pole vaulter is given approximately by $J = v^2/(2g)$, where v is the vaulter's speed in meters per second at the end of his run, and $g = 9.8$ is the gravitational acceleration. (Source: Alexander, 1992)

a Fill in the table of values for jump heights achieved with values of v from 0 to 11 meters per second.

v	0	1	2	3	4	5	6	7	8	9	10	11
J												

b Graph the jump height versus final speed. (Use the table values to help you choose a window for the graph.)

c The jump height should be added to the height of the vaulter's center of gravity (at about hip level) to give the maximum height, H, he can clear. For a typical pole vaulter, his center of gravity at the end of the run is 0.9 meters from the ground. Complete the table of values for maximum heights, H, and graph H on your graph of J.

v	0	1	2	3	4	5	6	7	8	9	10	11
H												

d A good pole vaulter can reach a final speed of 9.5 meters per second. What height will he clear?

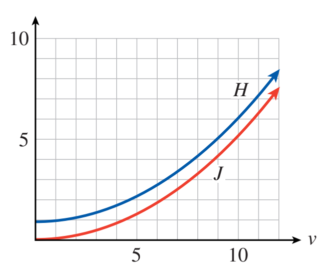
e In 2016, the world record in pole vaulting, established by Renaud Lavillenie in 2014, was 6.16 meters. What was the vaulter's speed at the end of his run?

Answer.

a

v	0	1	2	3	4	5	6	7	8	9	10	11
J	0	0.05	0.2	0.46	0.82	1.28	1.84	2.5	3.27	4.13	5.1	6.17

b



c

v	0	1	2	3	4	5	6	7	8	9	10	11
H	0.9	0.95	1.1	1.36	1.72	2.18	2.74	3.4	4.17	5.03	6.0	7.07

d 5.5 meters

e 10.15 meters per second

72. To be launched into space, a satellite must travel fast enough to escape Earth's gravity. This escape velocity, v, satisfies the equation

$$\frac{1}{2}mv^2 = \frac{GMm}{R}$$

where m is the mass of the satellite, M is the mass of the Earth, R is the radius of the Earth, and G is the universal gravitational constant.

a Solve the equation for v in terms of the other variables.

b The equation

$$mg = \frac{GMm}{R^2}$$

gives the force of gravity at the Earth's surface. We can use this equation to simplify the expression for v: First, multiply both sides of the equation by $\frac{R}{m}$. You now have an expression for $\frac{GM}{R}$. Substitute this new expression into your formula for v.

c The radius of the Earth is about 6400 km, and $g = 0.0098\frac{\text{km}}{\text{s}^2}$. Calculate the escape velocity from Earth in kilometers per second. Convert your answer to miles per hour. (One kilometer is 0.621 miles.)

d The radius of the moon is 1740 km, and the value of g at the moon's surface is 0.0016. Calculate the escape velocity from the moon in kilometers per second and convert to miles per hour.

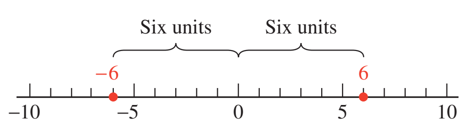
2.2 Some Basic Functions

In this section, we study the graphs of some important basic functions. Many functions fall into families or classes of similar functions, and recognizing the appropriate family for a given situation is an important part of modeling.

We begin by reviewing the absolute value.

2.2.1 Absolute Value

The absolute value is used to discuss problems involving distance. For example, consider the number line shown below. Starting at the origin, we travel in opposite directions to reach the two numbers 6 and −6, but the distance we travel in each case is the same.



The distance from a number c to the origin is called the **absolute value** of c, denoted by $|c|$. Because distance is never negative, the absolute value of a number is always positive (or zero). Thus, $|6| = 6$ and $|-6| = 6$. In general, we define the absolute value of a number x as follows.

Absolute Value.

The absolute value of x is defined by

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

Note 2.2.1 This definition says that the absolute value of a positive number (or zero) is the same as the number. To find the absolute value of a negative number, we take the opposite of the number, which results in a positive number. For instance,

$$|-6| = -(-6) = 6$$

Absolute value bars act like grouping devices in the order of operations: You should complete any operations that appear inside absolute value bars before you compute the absolute value.

Example 2.2.2 Simplify each expression.

a $|3-8|$ b $|3|-|8|$

Solution.

a We simplify the expression inside the absolute value bars first.

$$|3-8| = |-5| = 5$$

b We simplify each absolute value; then subtract.

$$|3|-|8| = 3-8 = -5$$

□

Checkpoint 2.2.3 Simplify each expression.

a $12-3\,|-6|$ b $-7-3\,|2-9|$

Answer.

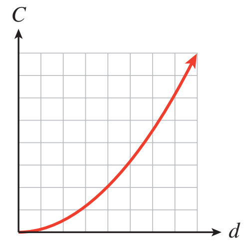
a -6 b -28

2.2.2 Examples of Models

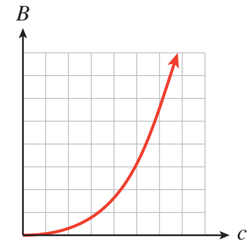
Many situations can be modeled by a handful of simple functions. The following examples represent applications of eight useful functions.

The contractor for a new hotel is estimating the cost of the marble tile for a circular lobby. The cost is a function of the *square* of the diameter of the lobby.

The number of board-feet that can be cut from a Ponderosa pine is a function of the *cube* of the circumference of the tree at a standard height.

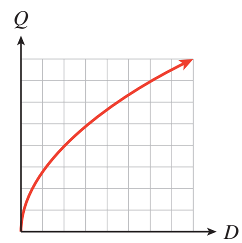


$C = kd^2$



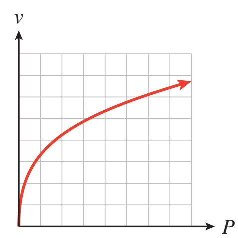
$B = kc^3$

The manager of an appliance store must decide how many coffeemakers to order every quarter. The optimal order size is a function of the *square root* of the annual demand for coffeemakers.



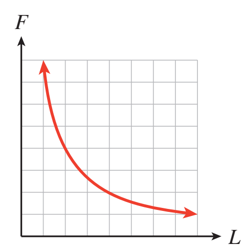
$Q = k\sqrt{D}$

Investors are deciding whether to support a windmill farm. The wind speed needed to generate a given amount of power is a function of the *cube root* of the power.



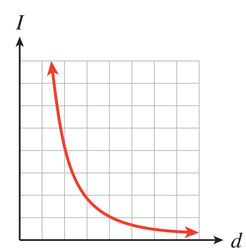
$v = k\sqrt[3]{P}$

The frequency of the note produced by a violin string is a function of the *reciprocal* of the length of the string.



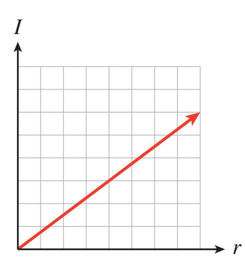
$F = \dfrac{k}{L}$

The loudness, or intensity, of the music at a concert is a function of the *reciprocal of the square* of your distance from the speakers.



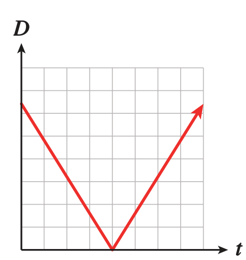
$I = \dfrac{k}{d^2}$

The annual return on an investment is a linear function of the interest rate.



$I = kr$

You are flying from Los Angeles to New York. Your distance from the Mississippi River is an *absolute value* function of time.



$D = k|t - h|$

We will consider each of these functions and their applications in more detail in later sections. For now, you should become familiar with the properties of each graph and be able to sketch them easily from memory.

Investigation 10 Eight Basic Functions. Part I Some Powers

1. Complete the table of values for the squaring function, $f(x) = x^2$, and the cubing function, $g(x) = x^3$. Then sketch each function on graph paper, using the table values to help you scale the axes.
2. Verify both graphs with your graphing calculator.
3. State the intervals on which each graph is increasing.
4. Write a few sentences comparing the two graphs. The graph of $y = x^2$ is called a **parabola**, and the graph of $y = x^3$ is called a **cubic**.

x	$f(x) = x^2$	$g(x) = x^3$
-3		
-2		
-1		
$-\frac{1}{2}$		
0		
$\frac{1}{2}$		
1		
2		
3		

Part II Some Roots

1. Complete the tables for the square root function, $f(x) = \sqrt{x}$, and the cube root function, $g(x) = \sqrt[3]{x}$. (Round your answers to two decimal places.) Then sketch each function on graph paper, using the table values to help you scale the axes.
2. Verify both graphs with your graphing calculator.
3. State the intervals on which each graph is increasing.
4. Write a few sentences comparing the two graphs.

x	$f(x) = \sqrt{x}$	x	$g(x) = \sqrt[3]{x}$
0		-8	
$\frac{1}{2}$		-4	
1		-1	
2		$-\frac{1}{2}$	
3		0	
4		$\frac{1}{2}$	
5		1	
7		4	
9		8	

Part III Asymptotes

1. Complete the table for the functions
$$f(x) = \frac{1}{x} \quad \text{and} \quad g(x) = \frac{1}{x^2}$$
What is true about $f(0)$ and $g(0)$?
2. Prepare a grid on graph paper, scaling both axes from -5 to 5. Plot the points from the table and connect them with smooth curves.
3. As x increases through larger and larger values, what happens to the values of $f(x)$? Extend your graph to reflect your answer.
4. What happens to $f(x)$ as x decreases through larger and larger negative values (that is, for $x = -5, -6, -7, \ldots$)? Extend your graph for these x-values.

x	$f(x) = \frac{1}{x}$	$g(x) = \frac{1}{x^2}$
-4		
-3		
-2		
-1		
$-\frac{1}{2}$		
0		
$\frac{1}{2}$		
1		
2		
3		
4		

As the values of x get larger in absolute value, the graph approaches the x-axis. However, because $\frac{1}{x}$ never *equals* zero for any x-value, the graph never actually touches the x-axis. We say that the x-axis is a **horizontal asymptote** for the graph.

Repeat step (3) for the graph of $g(x)$.

Next we'll examine the graphs of f and g near $x = 0$.

1. Use your calculator to evaluate f for several x-values close to zero and record the results in the tables below.

x	$f(x) = \frac{1}{x}$	$g(x) = \frac{1}{x^2}$
-2		
-1		
-0.1		
-0.01		
-0.001		

x	$f(x) = \frac{1}{x}$	$g(x) = \frac{1}{x^2}$
2		
1		
0.1		
0.01		
0.001		

   What happens to the values of $f(x)$ as x approaches zero? Extend your graph of f to reflect your answer.

   As x approaches zero from the left (through negative values), the function values decrease toward $-\infty$. As x approaches zero from the right (through positive values), the function values increase toward ∞. The graph approaches but never touches the vertical line $x = 0$ (the y-axis.)We say that the graph of f has a **vertical asymptote** at $x = 0$.

2. Repeat step (1) for the graph of $g(x)$.

3. The functions $f(x) = \frac{1}{x}$ and $g(x) = \frac{1}{x^2}$ are examples of **rational functions**, so called because they are fractions, or ratios. Verify both graphs with your graphing calculator. Use the window

$$\begin{array}{ll} \text{Xmin} = -4 & \text{Xmax} = 4 \\ \text{Ymin} = -4 & \text{Ymax} = 4 \end{array}$$

4. State the intervals on which each graph is increasing.

5. Write a few sentences comparing the two graphs.

Part IV Absolute Value

1. Complete the table for the two functions $f(x) = x$ and $g(x) = |x|$. Then sketch each function on graph paper, using the table values to help you scale the axes.

2. Verify both graphs with your graphing calculator. Your calculator uses the notation $abs(x)$ instead of $|x|$ for the absolute value of x. First, position the cursor after $Y_1 =$ in the graphing window. Now access the absolute value function by pressing `2nd` 0 for *CATALOG*; then `ENTER` for $abs($. Don't forget to press X if you want to graph $y = |x|$.

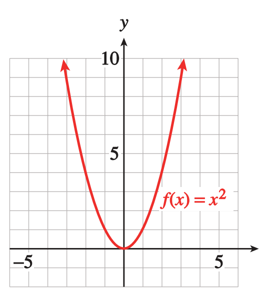
3. State the intervals on which each graph is increasing.

4. Write a few sentences comparing the two graphs.

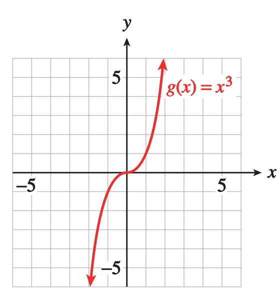
x	$f(x) = x$	$g(x) = \|x\|$
-4		
-3		
-2		
-1		
$-\frac{1}{2}$		
0		
$\frac{1}{2}$		
1		
2		
3		
4		

2.2.3 Graphs of Eight Basic Functions

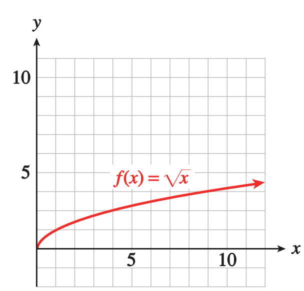
The graphs of the eight basic functions considered in Investigation 10, p. 173 are shown below. Once you know the shape of each graph, you can sketch an accurate picture by plotting a few guidepoints and drawing the curve through those points. Usually, points (or vertical asymptotes!) at $x = -1$, 0, and 1 make good guidepoints.



(a)



(b)



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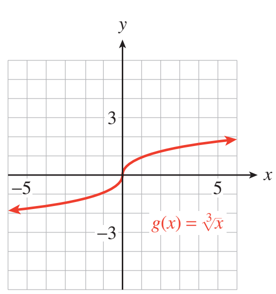

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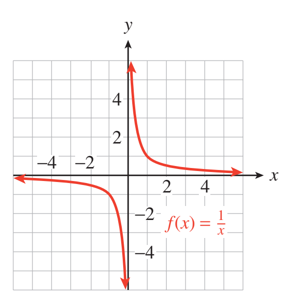

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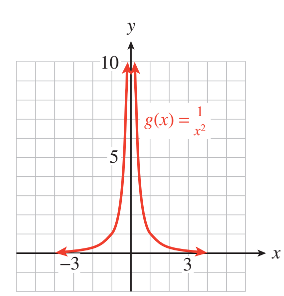
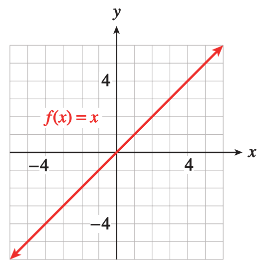
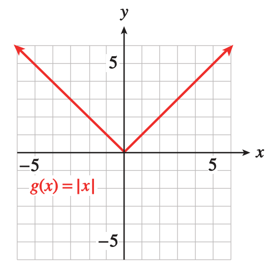


(f)



(g)



(h)

2.2.4 Properties of the Basic Functions

In Section 1.2, p. 27, we saw that for most functions, $f(a+b)$ is not equal to $f(a)+f(b)$. We may be able to find *some* values of a and b for which $f(a+b)=f(a)+f(b)$ is true, but if it is not true for *all* values of a and b, we cannot claim that $f(a+b)=f(a)+f(b)$ for that function.

For example, for the function $f(x)=x^2$, if we choose $a=3$ and $b=4$, then

$$\begin{aligned} & f(3+4)=f(7)=7^2=49 \\ \text{but} \qquad & f(3)+f(4)=3^2+4^2=9+16=25 \end{aligned}$$

so we have proved that $f(a+b)\neq f(a)+f(b)$ for the squaring function. (In fact, we already knew this because $(a+b)^2\neq a^2+b^2$ as long as neither a nor b is 0.)

What about multiplication? Which of the basic functions have the property that $f(ab)=f(a)f(b)$ for all a and b? You will consider this question in the homework problems, but in particular you will need to recall the following properties of absolute value.

Properties of Absolute Value.

$\lvert a+b\rvert \leq \lvert a\rvert + \lvert b\rvert$	Triangle inequality
$\lvert ab\rvert = \lvert a\rvert\,\lvert b\rvert$	Multiplicative property

Example 2.2.4 Verify the triangle inequality for three cases: a and b are both positive, a and b are both negative, and a and b have opposite signs.

Solution.

- We choose positive values for a and b, say $a=3$ and $b=5$. Then

  $$|3+5|=|8|=8 \quad\text{and}\quad |3|+|5|=3+5=8$$

  so $|3+5|=|3|+|5|$.

- For the second case, we choose $a=-3$ and $b=-5$. Then

  $$\big|-3+(-5)\big|=|-8|=8 \quad\text{and}\quad |-3|+|-5|=3+5=8$$

  so $\big|-3+(-5)\big|=|-3|+|-5|$.

- For the third case, we choose $a=3$ and $b=-5$. Then

  $$\big|3+(-5)\big|=|-2|=2 \quad\text{and}\quad |3|+|-5|=3+5=8$$

  so $\big|3+(-5)\big|<|3|+|-5|$. In each case, $|a+b|\leq|a|+|b|$.

□

Note 2.2.5 Note that *verifying* a statement for one or two values of the variables does not *prove* the statement is true for *all* values of the variables. However, working with examples can help us understand the meaning and significance of mathematical properties.

Checkpoint 2.2.6 Verify the multiplicative property of absolute value for the three cases in Example 2.2.4, p. 176.

Answer.

- $|3|\,|5| = 15 = |3 \cdot 5|$
- $|-3|\,|-5| = 15 = \left|(-3)\cdot(-5)\right|$
- $|3|\,|-5| = 15 = \left|3(-5)\right|$

2.2.5 Functions Defined Piecewise

A function may be defined by different formulas on different portions of the x-axis. Such a function is said to be defined **piecewise**. To graph a function defined piecewise, we consider each piece of the x-axis separately.

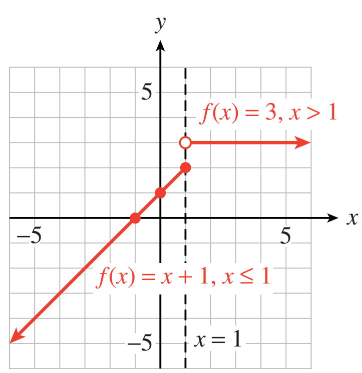
Example 2.2.7 Graph the function defined by

$$f(x) = \begin{cases} x+1 & \text{if } x \le 1 \\ 3 & \text{if } x > 1 \end{cases}$$

Solution. Think of the plane as divided into two regions by the vertical line $x = 1$, as shown below. In the left-hand region ($x \le 1$), we graph the line $y = x + 1$. (The fastest way to graph the line is to plot its intercepts, $(-1, 0)$ and $(0, 1)$.)

Notice that the value $x = 1$ is included in the first region, so $f(1) = 1+1 = 2$, and the point $(1, 2)$ is included on the graph. We indicate this with a solid dot at the point $(1, 2)$.

In the right-hand region ($x > 1$), we graph the horizontal line $y = 3$. The value $x = 1$ is not included in the second region, so the point $(1, 3)$ is not part of the graph. We indicate this with an open circle at the point $(1, 3)$.

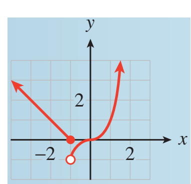


□

Checkpoint 2.2.8 Graph the piecewise defined function

$$g(x) = \begin{cases} -1-x & \text{if } x \le -1 \\ x^3 & \text{if } x > -1 \end{cases}$$

Answer.



The absolute value function $f(x) = |x|$ is an example of a function that is defined piecewise.

$$f(x) = |x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

To sketch the absolute value function, we graph the line $y = x$ in the first quadrant and the line $y = -x$ in the second quadrant.

Example 2.2.9

a Write a piecewise definition for $g(x) = |x - 3|$.

b Sketch a graph of $g(x) = |x - 3|$.

Solution.

a In the definition for $|x|$, we replace x by $x - 3$ to get

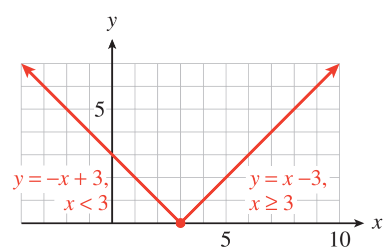
$$g(x) = |x - 3| = \begin{cases} x - 3 & \text{if } \; x - 3 \geq 0 \\ -(x - 3) & \text{if } \; x - 3 < 0 \end{cases}$$

We can simplify this expression to

$$g(x) = |x - 3| = \begin{cases} x - 3 & \text{if } \; x \geq 3 \\ -x + 3 & \text{if } \; x < 3 \end{cases}$$

b In the first region, $x \geq 3$, we graph the line $y = x - 3$. Because $x = 3$ is included in this region, the endpoint of this portion of the graph, $(3, 0)$, is included, too.

In the second region, $x < 3$, we graph the line $y = -x + 3$. Note that the two pieces of the graph meet at the point $(0, 3)$, as shown below.



□

Checkpoint 2.2.10

a Use your calculator to graph $g(x) = |x - 3|$ and $h(x) = |x| + |-3|$. Are the graphs the same?

b Explain why the functions $f(x) = |x + k|$ and $g(x) = |x| + |k|$ are not the same if $k \neq 0$.

Answer.

a No

b Because $|x + k| \neq |x| + |k|$ when x and k have opposite signs.

2.2.6 Section Summary

2.2.6.1 Vocabulary

Look up the definitions of new terms in the Glossary.

- Absolute value
- Verify
- Triangle inequality
- Vertical asymptote
- Rational function
- Parabola
- Piecewise defined function
- Multiplicative property
- Guidepoints
- Horizontal asymptote
- Cubic

2.2.6.2 CONCEPTS

1 The **absolute value** of x is defined by

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

2 The absolute value has the following properties:

$$\begin{array}{ll} |a+b| \leq |a| + |b| & \text{Triangle inequality} \\ |ab| = |a|\,|b| & \text{Multiplicative property} \end{array}$$

3 Many useful functions fall into families or classes of variations on basic functions.

4 We can make sketches of the eight basic functions using guidepoints.

5 Functions can be defined piecewise, with different formulas on different intervals

2.2.6.3 STUDY QUESTIONS

1 Is it true that $-x$ must be a negative number? Why or why not?

2 Are there any numbers for which $x = -x$?

3 If $0 < x < 1$, which is larger, x^2 or x^3?

4 If $0 < x < 1$, $\sqrt{x}$ or $\sqrt[3]{x}$?

5 List the eight basic functions considered in this section.

6 Which of the eight basic functions have a horizontal asymptote? A vertical asymptote?

7 What does an open circle on a graph mean?

8 For what value(s) of x does $|x+6| = 0$?

2.2.6.4 SKILLS

Practice each skill in the Homework 2.2.7, p. 180 problems listed.

1 Simplify expressions containing absolute values: #1–10

2 Sketch graphs of the basic functions by hand: #15–18

3 Identify the graph of a basic function: #19–26

4 Solve equations and inequalities graphically: #11–14, 27–34

5 Graph functions defined piecewise: #41–58

2.2.7 Some Basic Functions (Homework 2.2)

For problems 1-10, simplify the expression according to the order of operations.

1.

a $-\lvert -9\rvert$

b $-(-9)$

Answer.

a -9 b 9

2.

a $2-(-6)$

b $2-\lvert -6\rvert$

3.

a $\lvert -8\rvert - \lvert 12\rvert$

b $\lvert -8-12\rvert$

Answer.

a -4 b 20

4.

a $\lvert -3\rvert + \lvert -5\rvert$

b $\bigl\lvert -3+(-5)\bigr\rvert$

5. $4-9\,\lvert 2-8\rvert$

Answer. -50

6. $2-5\,\lvert -6-3\rvert$

7. $\lvert -4-5\rvert\,\bigl\lvert 1-3(-5)\bigr\rvert$

Answer. 144

8. $\lvert -3+7\rvert\,\bigl\lvert -2(6-10)\bigr\rvert$

9. $\bigl\lvert \lvert -5\rvert - \lvert -6\rvert \bigr\rvert$

Answer. 1

10. $\bigl\lvert \lvert 4\rvert - \lvert -6\rvert \bigr\rvert$

In Problems 11–14, show how to use the graphs to find the values. Estimate your answers to one decimal point. Compare your estimates to values obtained with a calculator.

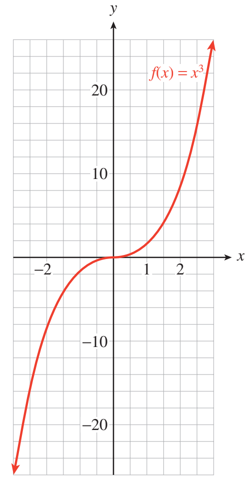
11. Refer to the graph of $f(x) = x^3$.

a Estimate the value of $(1.4)^3$.

b Find all numbers whose cubes are -20.

c Find all solutions of the equation $x^3 = 6$.

d Estimate the value of $\sqrt[3]{24}$.



Answer.

a 2.7   b -2.7 c 1.8   d 2.9

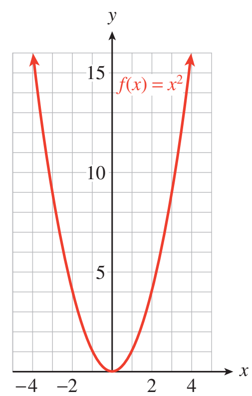
12. Refer to the graph of $f(x) = x^2$.

a Estimate the value of $(-2.5)^2$.

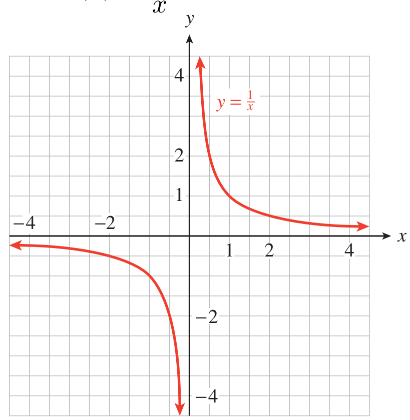
b Find all numbers whose squares are 12.

c Find all solutions of the equation $x^2 = 15$.

d Estimate the value of $\sqrt{10.5}$.



13. Refer to the graph of $f(x) = \dfrac{1}{x}$.



a Estimate the value of $\dfrac{1}{3.4}$.

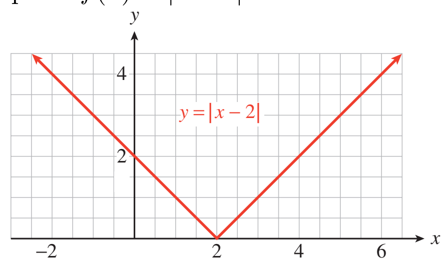
b Find all numbers whose reciprocals are -2.5.

c Find all solutions of the equation $\dfrac{1}{x} = 4.8$.

Answer.

a 0.3 b −0.4 c 0.2

14. Refer to the graph of $f(x) = |x - 2|$.



a Estimate the value of $|1.6 - 2|$.

b Find all values of x for which $|x - 2| = 3$.

c Find all solutions of the equation $|x - 2| = 0.4$.

For Problems 15–18,

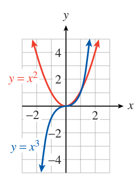
a Sketch both functions on the same grid, paying attention to the shape of the graph. Plot at least three guidepoints for each graph to ensure accuracy.

b Use the graph to find all solutions of the equation $f(x) = g(x)$.

c On what intervals is $f(x) > g(x)$?

15. $f(x) = x^2,\ g(x) = x^3$

Answer.

a 

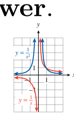
b $x = 0,\ x = 1$

c $(-\infty, 0)$ and $(0, 1)$

16. $f(x) = \sqrt{x},\ g(x) = \sqrt[3]{x}$

17. $f(x) = \dfrac{1}{x},\ g(x) = \dfrac{1}{x^2}$

Answer.

a 

b $x = 1$

c $(1, +\infty)$

18. $f(x) = x,\ g(x) = |x|$

For Problems 19-24, graph each set of functions together in the **ZDecimal** window. Describe how graphs (b) and (c) are different from the basic graph.

19.

a $f(x) = x^3$

b $g(x) = x^3 - 2$

c $h(x) = x^3 + 1$

Answer. Graph (b) is the basic graph shifted 2 units down; graph (c) is the basic graph shifted 1 unit up.

20.

a $f(x) = |x|$

b $g(x) = |x - 2|$

c $h(x) = |x + 1|$

21.

a $f(x) = \dfrac{1}{x}$

b $g(x) = \dfrac{1}{x + 1.5}$

c $h(x) = \dfrac{1}{x - 1}$

Answer. Graph (b) is the basic graph shifted 1.5 units left; graph (c) is the basic graph shifted 1 unit right.

22.

a $f(x) = \dfrac{1}{x^2}$

b $g(x) = \dfrac{1}{x^2} + 2$

c $h(x) = \dfrac{1}{x^2} - 1$

23.

a $f(x) = \sqrt{x}$

b $g(x) = -\sqrt{x}$

c $h(x) = \sqrt{-x}$

Answer. Graph (b) is the basic graph reflected about the x-axis; graph (c) is the basic graph reflected about the y-axis.

24.

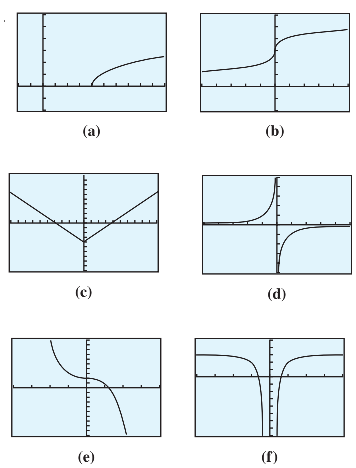
a $f(x) = \sqrt[3]{x}$

b $g(x) = -\sqrt[3]{x}$

c $h(x) = \sqrt[3]{-x}$

Each graph in Problems 25-26 is a variation of one of the eight basic graphs of Investigation 10, p. 173. Identify the basic graph for each problem.

25.



Answer.

a $\sqrt{x}$

b $\sqrt[3]{x}$

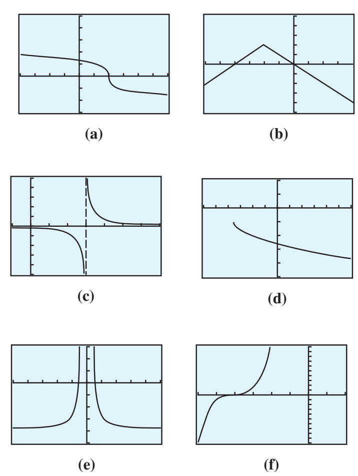
c $|x|$

d $\dfrac{1}{x}$

e x^3

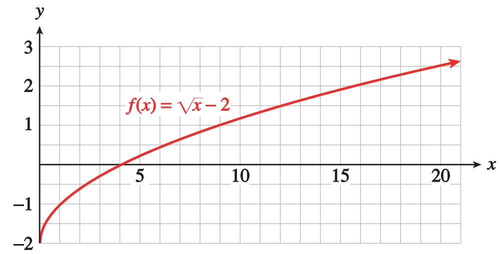
f $\dfrac{1}{x^2}$

26.



In Problems 27–30, use the graph to estimate the solution to the equation or inequality. Show the solution or solutions on the graph.

27. The figure shows a graph of $f(x) = \sqrt{x} - 2$, for $x > 0$. Solve the following:



a $\sqrt{x} - 2 = 1.5$

b $\sqrt{x} - 2 = 2.25$

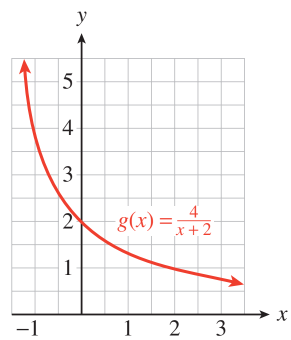
c $\sqrt{x} - 2 < 1$

d $\sqrt{x} - 2 > -0.25$

Answer.

a $x \approx 12$ b $x \approx 18$ c $x < 9$ d $x > 3$

28. The figure shows a graph of $g(x) = \dfrac{4}{x+2}$, for $x > -2$. Solve the following:



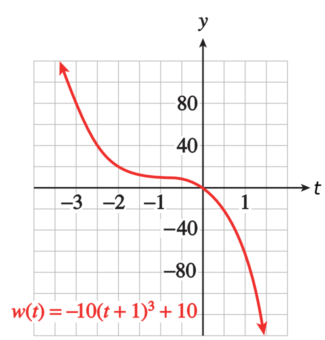
a $\dfrac{4}{x+2} = 4$

b $\dfrac{4}{x+2} = 0.8$

c $\dfrac{4}{x+2} > 1$

d $\dfrac{4}{x+2} < 3$

29. The figure shows a graph of $w(t) = -10(t+1)^3 + 10$. Solve the following:



a $-10(t+1)^3 + 10 = 100$

b $-10(t+1)^3 + 10 = -140$

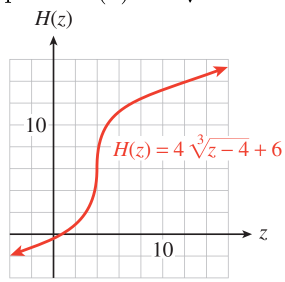
c $-10(t+1)^3 + 10 > -50$

d $-20 < -10(t+1)^3 + 10 < 40$

Answer.

a $t \approx -3.1$ b $t \approx 1.5$ c $t < 0.8$ d $-2.4 < t < 0.4$

30. The figure shows a graph of $H(z) = 4\sqrt[3]{z-4}+6$. Solve the following:



a $4\sqrt[3]{z-4}+6=2$

b $4\sqrt[3]{z-4}+6=12$

c $4\sqrt[3]{z-4}+6>14$

d $4\sqrt[3]{z-4}+6<6$

For Problems 31-34, gaph the function with the **ZInteger** setting. Use the graph to solve each equation or inequality. Check your solutionsalgebraically.

31. Graph $F(x) = 4\sqrt{x-25}$.

a Solve $4\sqrt{x-25} = 16$

b Solve $8 < 4\sqrt{x-25} \le 24$

Answer.

a $x = 41$ b $29 < x < 61$

32. Graph $G(x) = 15 - 0.01(x-2)^3$.

a Solve $15 - 0.01(x-2)^3 = -18.75$

b Solve $15 - 0.01(x-2)^3 \le 25$

33. Graph $H(x) = 24 - 0.25(x-6)^2$.

a Solve $24 - 0.25(x-6)^2 = -6.25$

b Solve $24 - 0.25(x-6)^2 > 11.75$

Answer.

a $x = -5$ or $x = 17$ b $-1 < x < 13$

34. Graph $R(x) = 0.1(x+12)^2 - 18$.

a Solve $0.1(x+12)^2 - 18 = 14.4$

b Solve $0.1(x+12)^2 - 18 < 4.5$

For Problems 35–40,

a Graph the equation by completing the table and plotting points.

b Does the equation define y as a function of x? Why or why not?

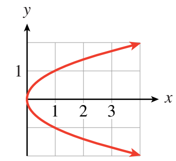
35. $x = y^2$

x							
y	-2	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	2

Answer.

a

x	4	1	$\frac{1}{4}$	0	$\frac{1}{4}$	1	4
y	-2	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	2



b no

36. $x = y^3$

x							
y	-2	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	2

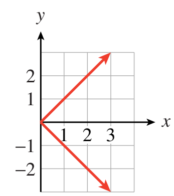
37. $x = |y|$

x							
y	-2	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	2

Answer.

a

x	2	1	$\frac{1}{2}$	0	$\frac{1}{2}$	1	2
y	-2	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	2



b no

38. $|x| = |y|$

x							
y	-2	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	2

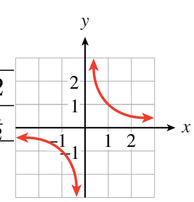
39. $x = \dfrac{1}{y}$

x							
y	-2	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	2

Answer.

a

x	$-\frac{1}{2}$	-1	-2	undefined	2
y	-2	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$



b yes

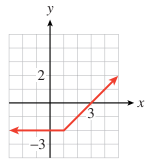
40. $x = \dfrac{1}{y^2}$

x							
y	-2	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	2

For Problems 41–52, graph the following piecewise defined functions. Indicate whether the endpoints of each piece are included on the graph.

41. $f(x) = \begin{cases} -2 & \text{if } x \leq 1 \\ x - 3 & \text{if } x > 1 \end{cases}$

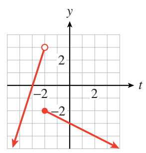
Answer.



42. $h(x) = \begin{cases} -x + 2 & \text{if } x \leq -1 \\ 3 & \text{if } x > -1 \end{cases}$

43. $G(t) = \begin{cases} 3t + 9 & \text{if } t < -2 \\ -3 - \dfrac{1}{2}t & \text{if } t \geq -2 \end{cases}$

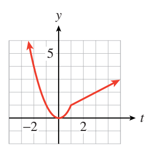
Answer.



44. $F(s) = \begin{cases} \dfrac{1}{3}s + 3 & \text{if } s < 3 \\ 2s - 3 & \text{if } s \geq 3 \end{cases}$

45. $H(t) = \begin{cases} t^2 & \text{if } t \leq 1 \\ \dfrac{1}{2}t + \dfrac{1}{2} & \text{if } t > 1 \end{cases}$

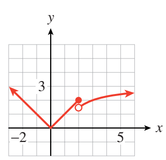
Answer.



46. $g(t) = \begin{cases} \dfrac{3}{2}t + 7 & \text{if } t \leq -2 \\ t^2 & \text{if } t > -2 \end{cases}$

47. $k(x) = \begin{cases} |x| & \text{if } x \leq 2 \\ \sqrt{x} & \text{if } x > 2 \end{cases}$

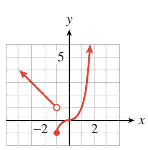
Answer.



48. $S(x) = \begin{cases} \dfrac{1}{x} & \text{if } x < 1 \\ |x| & \text{if } x \geq 1 \end{cases}$

49. $D(x) = \begin{cases} |x| & \text{if } x < -1 \\ x^3 & \text{if } x \geq -1 \end{cases}$

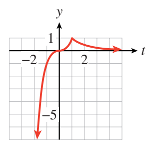
Answer.



50. $m(x) = \begin{cases} x^2 & \text{if } x \leq \dfrac{1}{2} \\ |x| & \text{if } x > \dfrac{1}{2} \end{cases}$

51. $P(t) = \begin{cases} t^3 & \text{if } t \leq 1 \\ \dfrac{1}{t^2} & \text{if } t > 1 \end{cases}$

Answer.



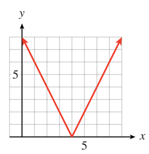
52. $Q(t) = \begin{cases} t^2 & \text{if } t \leq -1 \\ \sqrt[3]{t} & \text{if } t > -1 \end{cases}$

For Problems 53-58, write a piecewise definition for the function and sketch its graph.

53. $f(x) = |2x - 8|$

Answer.

$$f(x) = \begin{cases} 8 - 2x & x < 4 \\ 2x - 8 & x \geq 4 \end{cases}$$

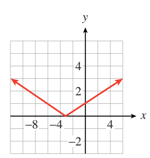


54. $g(x) = |3x + 6|$

55. $g(t) = \left|1 + \dfrac{t}{3}\right|$

Answer.

$$g(t) = \begin{cases} -1 - \dfrac{t}{3} & t < -3 \\ 1 + \dfrac{t}{3} & t \geq -3 \end{cases}$$

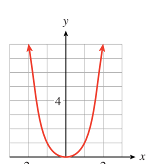


56. $f(t) = \left|\dfrac{1}{2}t - 3\right|$

57. $F(x) = \left|x^3\right|$

Answer.

$$F(x) = \begin{cases} -x^3 & x < 0 \\ x^3 & x \geq 0 \end{cases}$$



58. $G(x) = \left|\dfrac{1}{x}\right|$

In Problems 59–64, decide whether each statement is true for all values of a and b. If the statement is true, give an algebraic justification. If it is false, find values of a and b to disprove it.

a $f(a + b) = f(a) + f(b)$

b $f(ab) = f(a)f(b)$

59. $f(x) = x^2$

Answer.

a Not always true:
$f(1+2) \neq f(1) + f(2)$
because $9 \neq 5$.

b True: $(ab)^2 = a^2b^2$

60. $f(x) = x^3$

61. $f(x) = \dfrac{1}{x}$

Answer.

a Not always true:
$f(1+2) \neq f(1) + f(2)$
because $\frac{1}{3} \neq \frac{3}{2}$.

b True: $\dfrac{1}{ab} = \dfrac{1}{a} \cdot \dfrac{1}{b}$

62. $f(x) = \sqrt{x}$

63. $f(x) = mx + b$

Answer.

a Not always true (unless $b = 0$):
$f(1+2) \neq f(1) + f(2)$
because
$3m + b \neq 3m + 2b$.

b Not always true:
$f(1 \cdot 2) \neq f(1) \cdot f(2)$
because
$2m + b \neq 2m^2 + 3mb + b^2$.

64. $f(x) = kx$

65. Verify that $|a - b|$ gives the distance between a and b on a number line.

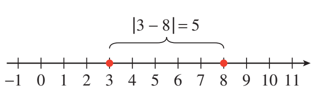
a $a = 3,\ b = 8$

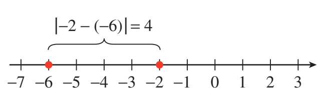
b $a = -2,\ b = -6$

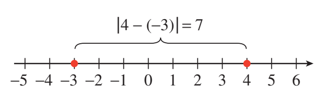
c $a = 4,\ b = -3$

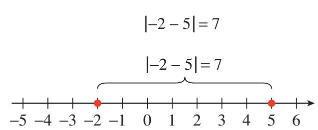
d $a = -2,\ b = 5$

Answer.

a


b


c


d


66. Which of the following statements is true for all values of a and b?

1 $|a - b| = |a| + |b|$

2 $|a - b| \leq |a| + |b|$

3 $|a - b| \geq |a| + |b|$

67. Explain how the distributive law, $a(b + c) = ab + ac$, is different from the equation $f(a + b) = f(a) + f(b)$.

Answer. The distributive law shows a relationship between multiplication and addition that always holds. The equation $f(a + b) = f(a) + f(b)$

is not about multiplication and may or may not be true.

68. For each function, decide whether $f(kx) = kf(x)$ for all $x \neq 0$, where $k \neq 0$ is a constant.

a $f(x) = x^2$

b $f(x) = \dfrac{1}{x}$

c $f(x) = \sqrt{x}$

d $f(x) = |x|$

For Problems 69-70,find the indicated value.

69. Use the function
$$F(s) = \begin{cases} \frac{1}{3}s + 3 & \text{if } s < 3 \\ 2s - 3 & \text{if } s \geq 3 \end{cases}$$
(from Problem 44) and add the indicated values

a $F(0)$

b $F(3)$

c $F(-9)$

70. Use the function
$$k(x) = \begin{cases} |x| & \text{if } x \leq 2 \\ \sqrt{x} & \text{if } x > 2 \end{cases}$$
(from Problem 47) and add the indicated values

a $k(2)$

b $k(1)$

c $k(-3)$

d $k(4.5)$

2.3 Transformations of Graphs

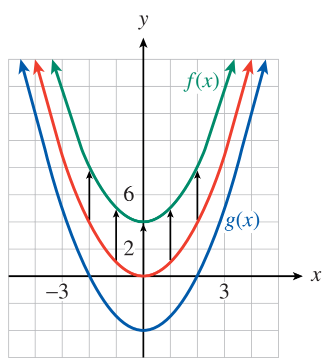
Models for real situations are often variations of the basic functions introduced in Section 2.2, p. 170. In this section, we explore how certain changes in the formula for a function affect its graph. In particular, we will compare the graph of $y = f(x)$ with the graphs of

$$y = f(x) + k, \qquad y = f(x + h), \qquad \text{and} \qquad y = af(x)$$

for different values of the constants k, h, and a. Such variations are called **transformations** of the graph.

2.3.1 Vertical Translations

The figure below shows the graphs of $f(x) = x^2 + 4$, $g(x) = x^2 - 4$, and the basic parabola, $y = x^2$. By comparing tables of values, we can see exactly how the graphs of f and g are related to the basic parabola.



x	-2	-1	0	1	2
$y = x^2$	4	1	0	1	4
$f(x) = x^2 + 4$	8	5	4	5	8

x	-2	-1	0	1	2
$y = x^2$	4	1	0	1	4
$g(x) = x^2 - 4$	0	-3	-4	-3	0

Each y-value in the table for $f(x)$ is four units greater than the corresponding y-value for the basic parabola. Consequently, each point on the graph of $f(x)$ is four units higher than the corresponding point on the basic parabola, as shown by the arrows. Similarly, each point on the graph of $g(x)$ is four units lower than the corresponding point on the basic parabola.

The graphs of $y = f(x)$ and $y = g(x)$ are said to be **translations** of the graph of $y = x^2$. They are shifted to a different location in the plane but retain the same size and shape as the original graph. In general, we have the following principles.

Vertical Translations.

Compared with the graph of $y = f(x)$,

1. The graph of $y = f(x) + k$, $(k > 0)$ is shifted *upward* k units.
2. The graph of $y = f(x) - k$, $(k > 0)$ is shifted *downward* k units.

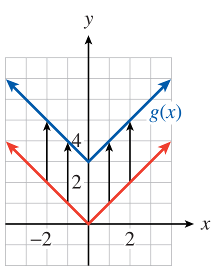
Example 2.3.1 Graph the following functions.

a $g(x) = |x| + 3$

b $h(x) = \dfrac{1}{x} - 2$

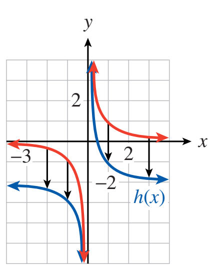
Solution.

a The table shows that the y-values for $g(x)$ are each three units greater than the corresponding y-values for the absolute value function. The graph of $g(x) = |x| + 3$ is a translation of the basic graph of $y = |x|$, shifted upward three units, as shown below.



x	-2	-1	0	1	2
$y = \|x\|$	2	1	0	1	2
$g(x) = \|x\| + 3$	5	4	3	4	5

b The table shows that the y-values for $h(x)$ are each two units smaller than the corresponding y-values for $y = \dfrac{1}{x}$. The graph of $h(x) = \dfrac{1}{x} - 2$ is a translation of the basic graph of $y = \dfrac{1}{x}$, shifted downward two units, as shown below.



x	-2	-1	$\frac{1}{2}$	1	2
$y = \frac{1}{x}$	$\frac{-1}{2}$	-1	2	1	$\frac{1}{2}$
$h(x) = \frac{1}{x} - 2$	$\frac{-5}{2}$	-3	0	-1	$\frac{-3}{2}$

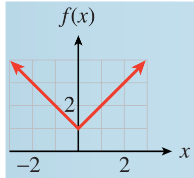
□

Checkpoint 2.3.2

a Graph the function $f(x) = |x| + 1$.

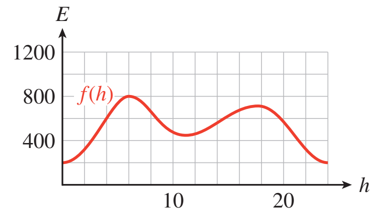
b How is the graph of f different from the graph of $y = |x|$?

Answer.

a 

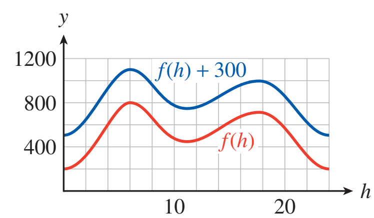
b Translate $y = |x|$ one unit up.

Example 2.3.3
The function $E = f(h)$ graphed at right gives the amount of electrical power, in megawatts, drawn by a community from its local power plant as a function of time during a 24-hour period in 2002. Sketch a graph of $y = f(h) + 300$ and interpret its meaning.



Solution. The graph of $y = f(h) + 300$ is a vertical translation of the graph of f,

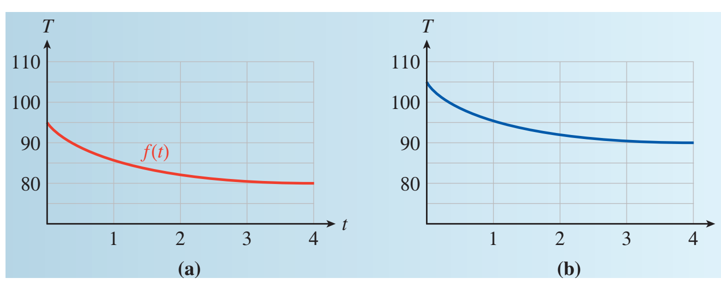
as shown at right. At each hour of the day, or for each value of h, the y-coordinate is 300 greater than on the graph of f. So at each hour, the community is drawing 300 megawatts more power than in 2002.



□

Checkpoint 2.3.4 An evaporative cooler, or swamp cooler, is an energy-efficient type of air conditioner used in dry climates. A typical swamp cooler can reduce the temperature inside a house by 15 degrees.

Figure (a) shows the graph of $T = f(t)$, the temperature inside Kate's house t hours after she turns on the swamp cooler. Write a formula in terms of f for the function g shown in figure (b), and give a possible explanation of its meaning.



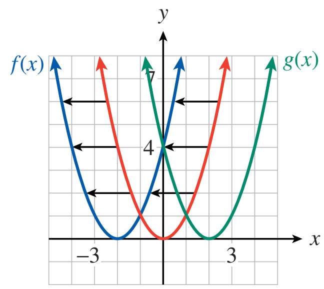
Answer. $g(t) = f(t) + 10$. The outside temperature was 10° hotter.

2.3.2 Horizontal Translations

Now consider the graphs of

$$f(x) = (x+2)^2 \quad \text{and} \quad g(x) = (x-2)^2$$

shown below. Compared with the graph of the basic function $y = x^2$, the graph of $f(x) = (x+2)^2$ is shifted two units to the *left*, as shown by the arrows.



You can see why this happens by studying the function values in the table.

Locate a particular y-value for $y = x^2$, say, $y = 4$. You must move two units to the left in the table to find the same y-value for $f(x)$, as shown by the arrow. In fact, each y-value for $f(x)$ occurs two units to the left when compared to the same y-value for $y = x^2$.

x	−3	−2	−1	0	1	2	3
$y = x^2$	9	4	1	0	1	4	9
$f(x) = (x + 2)^2$	1	0	1	4	9	16	25

x	−3	−2	−1	0	1	2	3
$y = x^2$	9	4	1	0	1	4	9
$g(x) = (x - 2)^2$	25	16	9	4	1	0	1

Similarly, the graph of $g(x) = (x - 2)^2$ is shifted two units to the *right* compared to the graph of $y = x^2$. In the table for g, each y-value for $g(x)$ occurs two units to the right of the same y-value for $y = x^2$. In general, we have the following principle.

Horizontal Translations.

Compared with the graph of $y = f(x)$,

1. The graph of $y = f(x + h)$, $(h > 0)$ is shifted h units to the *left*.
2. The graph of $y = f(x - h)$, $(h > 0)$ is shifted h units to the *right*.

Note 2.3.5 At first, the direction of a horizontal translation may seem counterintuitive. Look again at the tables above to help you see how the shift occurs.

Example 2.3.6 Graph the following functions.

a $g(x) = \sqrt{x + 1}$

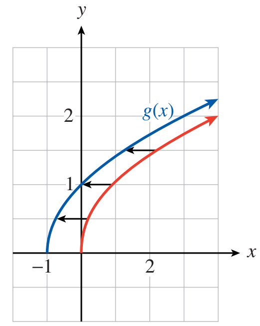
b $h(x) = \dfrac{1}{(x - 3)^2}$

Solution.

a Consider the table of values for the function.

x	-1	0	1	2	3
$y = \sqrt{x}$	undefined	0	1	1.414	1.732
$y = \sqrt{x + 1}$	0	1	1.414	1.732	2

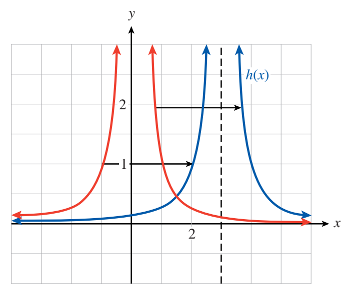
The table shows that each y-value for $g(x)$ occurs one unit to the left of the same y-value for the graph of $y = \sqrt{x}$. Consequently, each point on the graph of $y = g(x)$ is shifted one unit to the left of $y = \sqrt{x}$, as shown at right.



b Consider the table of values for the function.

x	-1	0	1	2	3	4
$y = \frac{1}{x}$	1	undefined	1	$\frac{1}{4}$	$\frac{1}{9}$	$\frac{1}{16}$
$y = \frac{1}{(x-3)^2}$	$\frac{1}{16}$	$\frac{1}{9}$	$\frac{1}{4}$	1	undefined	1

The table shows that each y-value for $h(x)$ occurs three units to the right of the same y-value for the graph of $y = \frac{1}{x^2}$. Consequently, each point on the graph of $y = h(x)$ is shifted three units to the right of $y = \frac{1}{x^2}$, as shown at right.



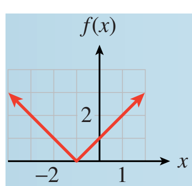
□

Checkpoint 2.3.7

a Graph the function $f(x) = |x + 1|$.

b How is the graph of f different from the graph of $y = |x|$?

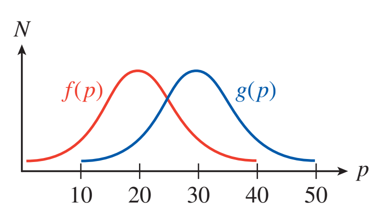
Answer.

a 

b Translate $y = |x|$ one unit left.

Example 2.3.8

The function $N = f(p)$ graphed at right gives the number of people who have a given eye pressure level p from a sample of 100 people with healthy eyes, and the function g gives the number of people with pressure level p in a sample of 100 glaucoma patients.



a Write a formula for g as a transformation of f.

b For what pressure readings could a doctor be fairly certain that a patient has glaucoma?

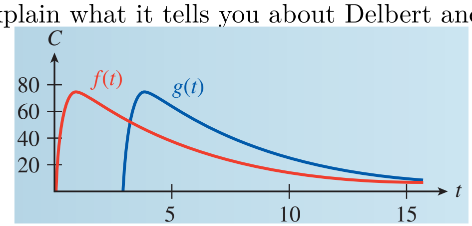
Solution.

a The graph of g is translated 10 units to the right of f, so $g(p) = f(p - 10)$.

b Pressure readings above 40 are a strong indication of glaucoma. Readings between 10 and 40 cannot conclusively distinguish healthy eyes from those with glaucoma.

□

Checkpoint 2.3.9 The function $C = f(t)$ shown below gives the caffeine level in Delbert's bloodstream at time t hours after he drinks a cup of coffee, and $g(t)$ gives the caffeine level in Francine's bloodstream. Write a formula for g in terms of f, and explain what it tells you about Delbert and Francine.

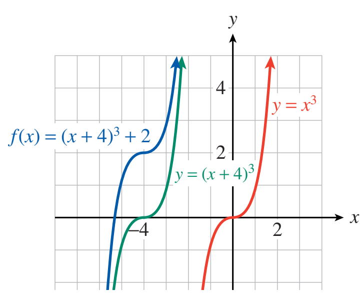


Answer. $g(t) = f(t - 3)$. Francine drank her coffee 3 hours after Delbert drank his.

Example 2.3.10 Graph $f(x) = (x + 4)^3 + 2$

Solution. We identify the basic graph from the structure of the formula for $f(x)$. In this case, the basic graph is $y = x^3$, so we begin by locating a few points on that graph, say, $(0, 0)$, $(1, 1)$, and $(-1, -1)$.

We'll perform the translations separately, following the order of operations. First, we sketch a graph of $y = (x + 4)^3$ by shifting each point on the basic graph four units to the left. We then move each point up two units to obtain the graph of $f(x) = (x + 4)^3 + 2$. All three graphs are shown below.



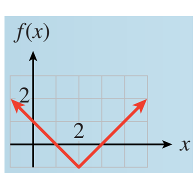
□

Checkpoint 2.3.11

a Graph the function $f(x) = |x - 2| - 1$.

b How is the graph of f different from the graph of $y = |x|$?

Answer.

a 

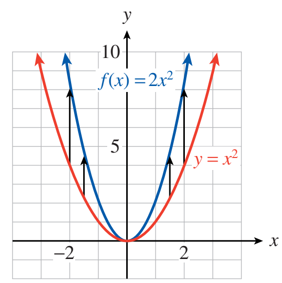
b Translate $y = |x|$ one unit down and two units right.

2.3.3 Scale Factors

We have seen that *adding* a constant to the expression defining a function results in a translation of its graph. What happens if we *multiply* the expression by a constant? Consider the graphs of the functions

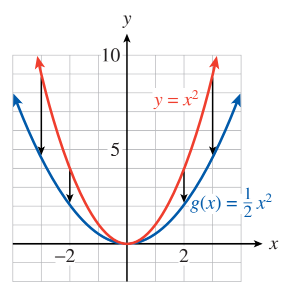
$$f(x) = 2x^2, \quad g(x) = \frac{1}{2}x^2, \quad \text{and} \quad h(x) = -x^2$$

shown below, and compare each to the graph of $y = x^2$.



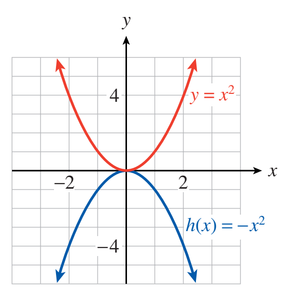
x	$y = x^2$	$f(x) = 2x^2$
-2	4	8
-1	1	2
0	0	0
1	1	2
2	4	8

Compared to the graph of $y = x^2$, the graph of $f(x) = 2x^2$ is expanded, or stretched, vertically by a factor of 2. The y-coordinate of each point on the graph has been doubled, as you can see in the table of values, so each point on the graph of f is twice as far from the x-axis as its counterpart on the basic graph $y = x^2$.



x	$y = x^2$	$g(x) = \frac{1}{2}x^2$
-2	4	2
-1	1	$\frac{1}{2}$
0	0	0
1	1	$\frac{1}{2}$
2	4	2

The graph of $g(x) = \dfrac{1}{2}x^2$ is compressed vertically by a factor of $\dfrac{1}{2}$; each point is half as far from the x-axis as its counterpart on the graph of $y = x^2$.



x	$y = x^2$	$h(x) = -x^2$
-2	4	-4
-1	1	-1
0	0	0
1	1	-1
2	4	-4

The graph of $h(x) = -x^2$ is flipped, or reflected, about the x-axis; the y-coordinate of each point on the graph of $y = x^2$ is replaced by its opposite.

In general, we have the following principles.

Scale Factors and Reflections.

Compared with the graph of $y = f(x)$, the graph of $y = af(x)$, where $a \neq 0$, is

1. stretched vertically by a factor of $|a|$ if $|a| > 1$,

2. compressed vertically by a factor of $|a|$ if $0 < |a| < 1$, and
3. reflected about the x-axis if $a < 0$.

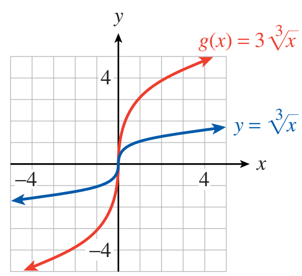
Example 2.3.12 Graph the following functions.

a $g(x) = 3\sqrt[3]{x}$

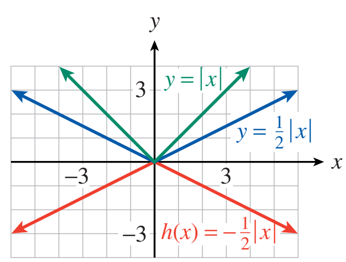
b $h(x) = \dfrac{-1}{2}|x|$

Solution.

a The graph of $g(x) = 3\sqrt[3]{x}$ is a vertical expansion of the basic graph $y = \sqrt[3]{x}$ by a factor of 3, as shown below . Each point on the basic graph has its y-coordinate tripled.



b The graph of $h(x) = \dfrac{-1}{2}|x|$ is a vertical compression of the basic graph $y = |x|$ by a factor of $\dfrac{1}{2}$, combined with a reflection about the x-axis. You may find it helpful to graph the function in two steps, as shown below.



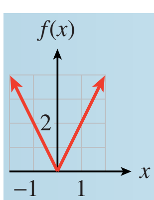
□

Checkpoint 2.3.13

a Graph the function $f(x) = 2|x|$.

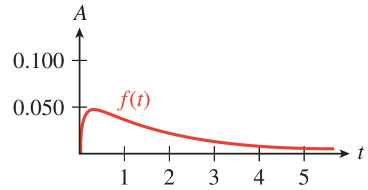
b How is the graph of f different from the graph of $y = |x|$?

Answer.

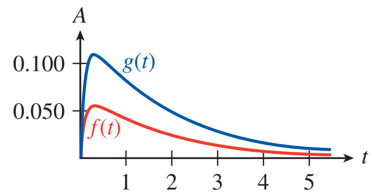
a 

b Stretch $y = |x|$ vertically by a factor of 2.

Example 2.3.14 The function $A = f(t)$ graphed below gives a person's blood alcohol level t hours after drinking a martini. Sketch a graph of $g(t) = 2f(t)$ and explain what it tells you.



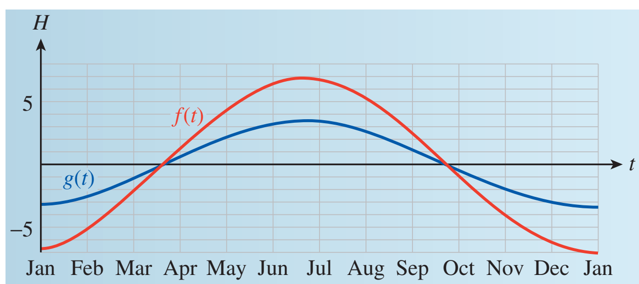
Solution. To sketch a graph of g, we stretch the graph of f vertically by a factor of 2, as shown below. At each time t, the person's blood alcohol level is twice the value given by f. The function g could represent a person's blood alcohol level t hours after drinking two martinis.



□

Checkpoint 2.3.15 If the Earth were not tilted on its axis, there would be 12 daylight hours every day all over the planet. But in fact, the length of a day in a particular location depends on the latitude and the time of year.

The graph below shows $H = f(t)$, the length of a day in Helsinki, Finland, t days after January 1, and $R = g(t)$, the length of a day in Rome. Each is expressed as the number of hours greater or less than 12. Write a formula for f in terms of g. What does this formula tell you?



Answer. $f(t) \approx 2g(t)$. On any given day, the number of daylight hours varies from 12 hours about twice as much in Helsinki as it does in Rome.

2.3.4 Section Summary

2.3.4.1 Vocabulary

Look up the definitions of new terms in the Glossary.

- Transformation
- Scale factor
- Vertical stretch
- Vertical compression
- Horizontal translation

2.3.4.2 CONCEPTS

1 Vertical Translations.

Compared with the graph of $y = f(x)$,

1. The graph of $y = f(x) + k \quad (k > 0)$ is shifted *upward* k units.
2. The graph of $y = f(x) - k \quad (k > 0)$ is shifted *downward* k units.

2 Horizontal Translations.

Compared with the graph of $y = f(x)$,

1. The graph of $y = f(x + h) \quad (h > 0)$ is shifted h units to the *left*.
2. The graph of $y = f(x - h) \quad (h > 0)$ is shifted h units to the *right*.

3 Scale Factors and Reflections.

Compared with the graph of $y = f(x)$, the graph of $y = af(x)$, where $a \neq 0$, is

1. stretched vertically by a factor of $|a|$ if $|a| > 1$,
2. compressed vertically by a factor of $|a|$ if $0 < |a| < 1$, and
3. reflected about the x-axis if $a < 0$.

2.3.4.3 STUDY QUESTIONS

1. How does a vertical translation affect the formula for a function? Give an example.
2. How does a horizontal translation affect the formula for a function? Give an example.
3. How does a scale factor affect the formula for a function? Give an example.
4. How is the graph of $y = -f(x)$ different from the graph of $y = f(x)$?

2.3.4.4 SKILLS

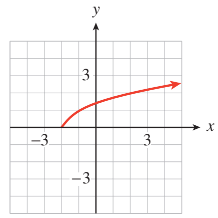
Practice each skill in the Homework 2.3.5, p. 200 problems listed.

1. Write formulas for transformations of functions: #1–6, 19–22, 35–38
2. Recognize and sketch translations of the basic graphs: #7–18
3. Recognize and sketch expansions, compression, and reflections of the basic graphs: #23–34, 43–50
4. Identify transformations from tables of values: #39–42
5. Sketch graphs obtained by two or more transformations of a basic graph: #51–62
6. Write a formula for a transformation of a graph: #63–76
7. Interpret transformations of graphs in context: #71–76

2.3.5 Transformations of Graphs (Homework 2.3)

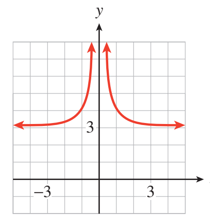
In Problems 1–6, identify the graph as a translation of a basic function, and write a formula for the graph.

1.

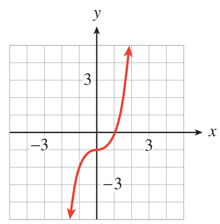


Answer. $y = \sqrt{x+2}$

2.

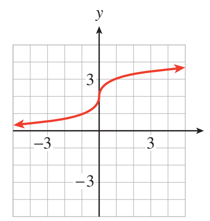


3.

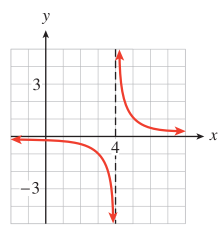


Answer. $y = x^3 - 1$

4.

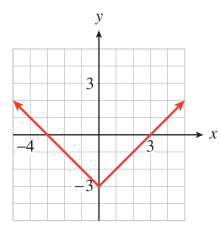


5.



Answer. $y = \dfrac{1}{x-4}$

6.



For Problems 7–18,

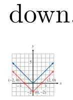
a Describe how to transform one of the basic graphs to obtain the graph of the given function.

b Using guidepoints, sketch the basic graph and the graph of the given function on the same axes. Label the coordinates of three points on the graph of the given function.

7. $f(x) = |x| - 2$

Answer.

a Translate $y = |x|$ by 2 units down.

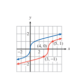
b 

8. $g(x) = (x+1)^3$

9. $g(s) = \sqrt[3]{s-4}$

Answer.

a Translate $y = \sqrt[3]{s}$ by 4 units right.

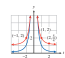
b 

10. $f(s) = s^2 + 3$

11. $F(t) = \dfrac{1}{t^2} + 1$

Answer.

a Translate $y = \dfrac{1}{t^2}$ by 1 unit up.

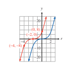
b 

12. $G(t) = \sqrt{t-2}$

13. $G(r) = (r+2)^3$

Answer.

a Translate $y = r^3$ by 2 units left.

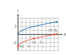
b 

14. $F(r) = \dfrac{1}{r-4}$

15. $H(d) = \sqrt{d} - 3$

Answer.

a Translate $y = \sqrt{d}$ by 3 units down.

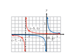
b 

16. $h(d) = \sqrt[3]{d} + 5$

17. $h(v) = \dfrac{1}{v+6}$

Answer.

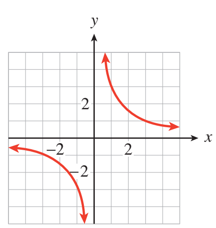
a Translate $y = \dfrac{1}{v}$ by 6 units left.

b 

18. $H(v) = \dfrac{1}{v^2} - 2$

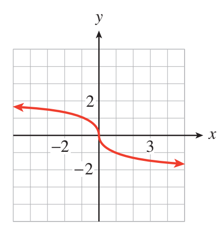
For Problems 19-22, identify the graph as a stretch, compression, or reflection of a basic function, and write a formula for the graph.

19.

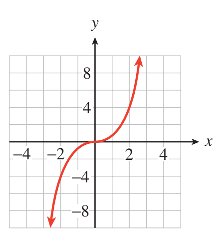


Answer. A vertical stretch by a factor of 3: $y = \dfrac{3}{x}$

20.

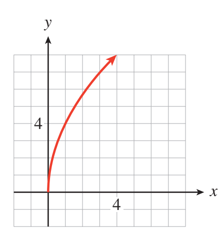


21.



Answer. A vertical compression, the scale factor is $\dfrac{1}{2}$: $y = \dfrac{1}{2}x^3$

22.



For Problems 23–32,

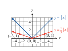
a Identify the scale factor for each function and describe how it affects the graph of the corresponding basic function.

b Using guidepoints, sketch the basic graph and the graph of the given function on the same axes. Label the coordinates of three points on the graph of the given function.

23. $f(x) = \dfrac{1}{3}\,|x|$

Answer.

a Scale factor $\frac{1}{3}$; $y = |x|$ is compressed vertically by the scale factor.

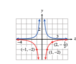
b 

24. $H(x) = -3\,|x|$

25. $h(z) = \dfrac{-2}{z^2}$

Answer.

a Scale factor -2; $y = \frac{1}{z^2}$ is reflected over the z-axis and stretched vertically by a factor of 2.

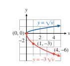
b 

26. $g(z) = \dfrac{2}{z}$

27. $G(v) = -3\sqrt{v}$

Answer.

a Scale factor -3; $y = \sqrt{v}$ is reflected over the v-axis and stretched vertically by a factor of 3.

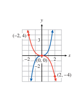
b 

28. $F(v) = -4\sqrt[3]{v}$

29. $g(s) = \dfrac{-1}{2}s^3$

Answer.

a Scale factor $\frac{-1}{2}$; $y = s^3$ is reflected over the s-axis and compressed vertically by a factor of $\frac{1}{2}$.

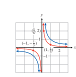
b 

30. $f(s) = \dfrac{1}{8}s^3$

31. $H(x) = \dfrac{1}{3x}$

Answer.

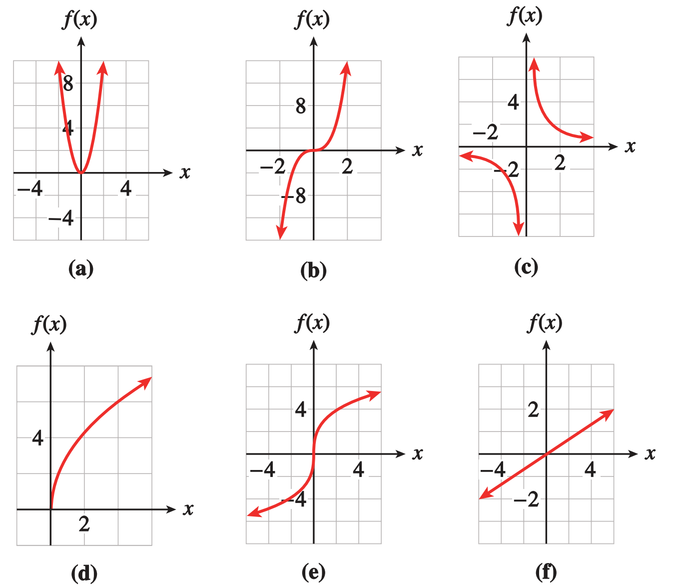
a Scale factor $\frac{1}{3}$; $y = \frac{1}{x}$ is compressed vertically by the scale factor.

b 

32. $h(x) = \dfrac{-1}{4x^2}$

In Problems 33 and 34, match each graph with its equation.

33.



i $f(x) = 3\sqrt{x}$

ii $f(x) = 2x^3$

iii $f(x) = \dfrac{x}{3}$

iv $f(x) = \dfrac{3}{x}$

v $f(x) = 2\sqrt[3]{x}$

vi $f(x) = 3x^2$

Answer.

a vi

b ii

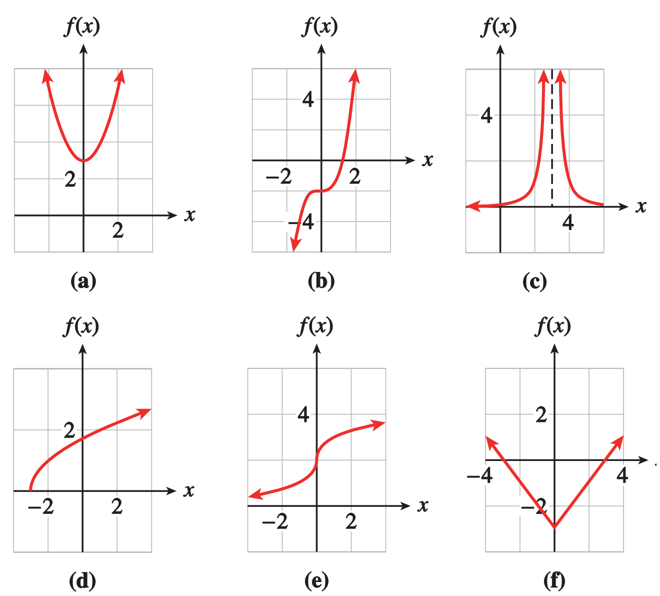
c iv

d i

e v

f iii

34.



i $f(x) = x^3 - 2$

ii $f(x) = \sqrt[3]{x} + 2$

iii $f(x) = \dfrac{1}{(x-3)^2}$

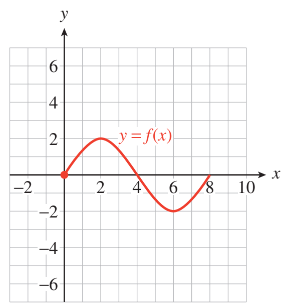
iv $f(x) = |x| - 3$

v $f(x) = x^2 + 3$

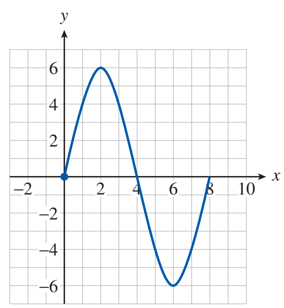
vi $f(x) = \sqrt{x-3}$

In Problems 35–38, the graph of a function is shown. Describe each transformation of the graph; then give a formula for each in terms of the original function.

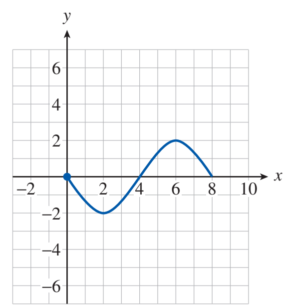
35.



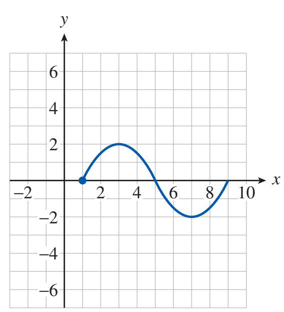
(a)



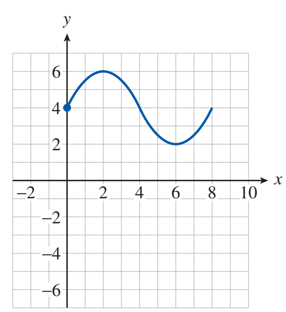
(b)



(c)



(d)



Answer.

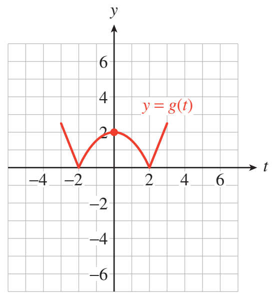
a Vertical stretch by a factor of 3: $y = 3f(x)$

b Reflection about the x-axis: $y = -f(x)$

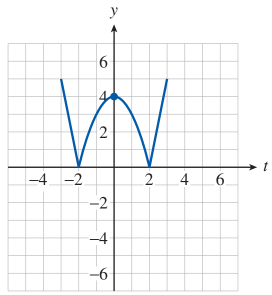
c Translation 1 unit right: $y = f(x - 1)$

d Translation 4 units up: $y = f(x) + 4$

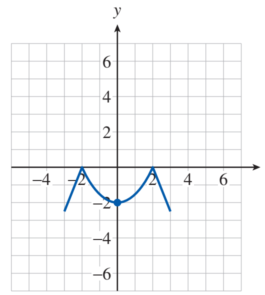
36.


y
6
4
y = g(t)
2
t
−4 −2 2 4 6
−2
−4
−6

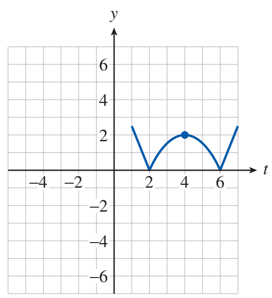
(a)


y
6
4
2
t
−4 −2 2 4 6
−2
−4
−6

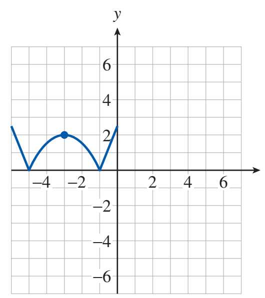
(b)


y
6
4
2
t
−4 −2 2 4 6
−2
−4
−6

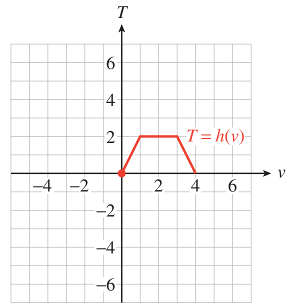
(c)


y
6
4
2
t
−4 −2 2 4 6
−2
−4
−6

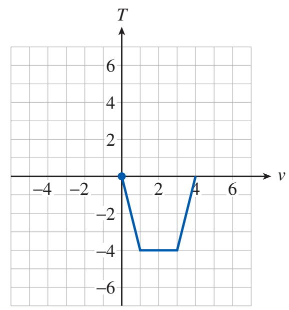
(d)


y
6
4
2
t
−4 −2 2 4 6
−2
−4
−6

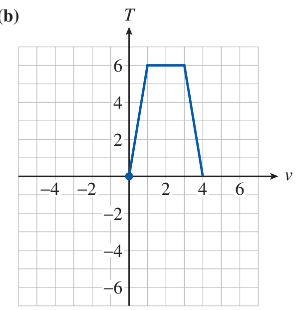
37.



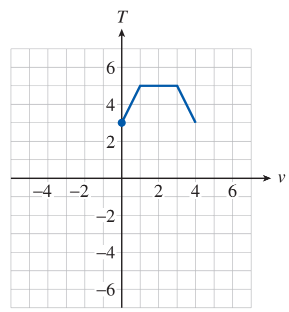
(a)



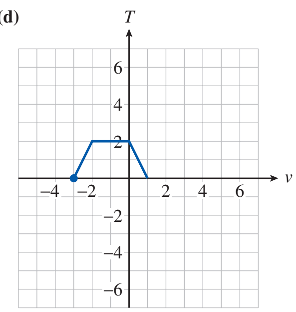
(b)



(c)



(d)



Answer.

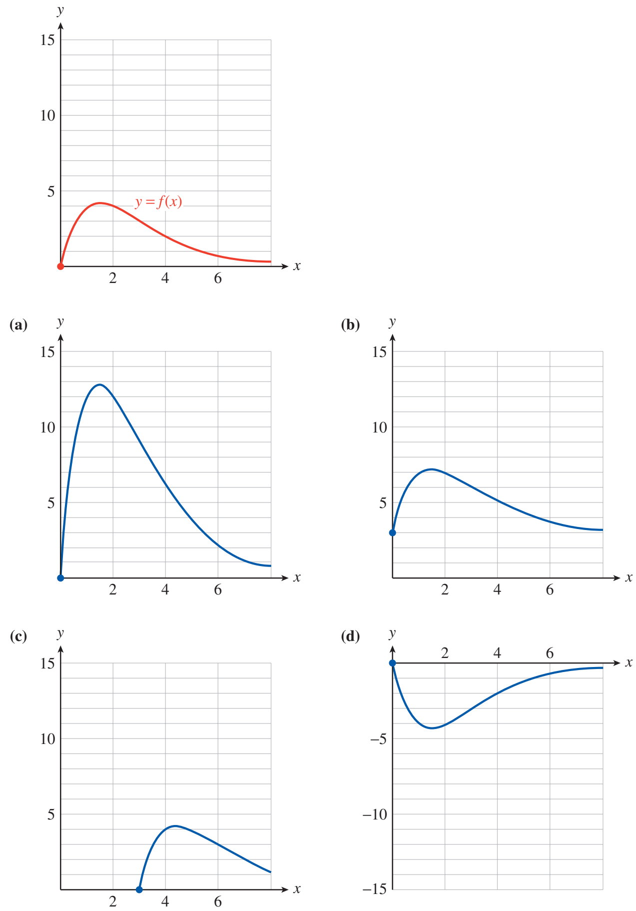
a Reflection about the v-axis and vertical stretch by a factor of 2: $T = -2h(v)$

b Vertical stretch by a factor of 3: $T = 3h(v)$

c Translation 3 units up: $T = h(v) + 3$

d Translation 3 units left: $T = h(v + 3)$

38.



In Problems 39–42, each table in parts (a)–(d) describes a transformation of $f(x)$. Identify the transformation and write a formula for the new function in terms of f.

39.

x	1	2	3	4	5	6
$f(x)$	8	6	4	2	0	2

a

x	1	2	3	4	5	6
y	10	8	6	4	2	4

b

x	1	2	3	4	5	6
y	4	2	0	-2	-4	-2

c

x	1	2	3	4	5	6
y	4	3	2	1	0	1

d

x	1	2	3	4	5	6
y	10	8	6	4	2	0

Answer.

a Translation 2 units up: $y = f(x) + 2$

b Translation 4 units down: $y = f(x) - 4$

c Vertical compression by a factor of $\frac{1}{2}$: $y = \frac{1}{2}f(x)$

d Translation 1 unit right: $y = f(x - 1)$

40.

x	-3	-2	-1	0	1	2
$f(x)$	13	3	-3	-5	-3	3

a

x	-3	-2	-1	0	1	2
y	-26	-6	6	10	6	-6

b

x	-3	-2	-1	0	1	2
y	18	8	2	0	2	8

c

x	-3	-2	-1	0	1	2
y	-3	-5	-3	3	13	27

d

x	-3	-2	-1	0	1	2
y	2.6	0.6	-0.6	-1	-0.6	0.6

41.

x	-2	-1	0	1	2	3
$f(x)$	-9	-8	-7	-6	1	20

a

x	-2	-1	0	1	2	3
y	-34	-9	-8	-7	-6	1

b

x	-2	-1	0	1	2	3
y	-4	21	22	23	24	31

c

x	-2	-1	0	1	2	3
y	18	16	14	12	-2	-40

d

x	-2	-1	0	1	2	3
y	8	6	4	2	-12	-50

Answer.

a Translation 1 unit right: $y = f(x-1)$

b Part (a) is translated 30 units up: $y = f(x-1) + 30$

c f is reflected about the x-axis and stretched vertically by a factor of 2: $y = -2f(x)$

d Part (c) is translated 10 units down: $y = -2f(x) - 10$

42.

x	1	2	3	4	5	6
$f(x)$	60	30	20	15	12	10

a

x	1	2	3	4	5	6
y	30	15	10	7.5	6	5

b

x	1	2	3	4	5	6
y	35	20	15	12.5	11	10

c

x	1	2	3	4	5	6
y	-12	-6	-4	-3	-2.4	-2

d

x	1	2	3	4	5	6
y	-10	-4	-2	-1	1.4	0

For Problems 43-50, write the function in the form $y = kf(x)$, where $f(x)$ is one of the basic functions. Describe how the graph differs from that of the basic function.

43. $y = \dfrac{1}{2x^2}$

Answer. $y = \dfrac{1}{2} \cdot \dfrac{1}{x^2}$ is a vertical compression with factor $dfrac12$ of $y = \dfrac{1}{x^2}$.

44. $y = \sqrt{9x}$

45. $y = \sqrt[3]{8x}$

Answer. $y = 2\sqrt[3]{x}$ is a vertical stretch with factor 2 of $y = \sqrt[3]{x}$.

46. $y = \dfrac{1}{4x}$

47. $y = |3x|$

Answer. $y = 3\,|x|$ is a vertical stretch with factor 3 of $y = |x|$.

48. $y = \left(\dfrac{x}{2}\right)^2$

49. $y = \left(\dfrac{x}{2}\right)^3$

Answer. $y = \dfrac{1}{8}x^3$ is a vertical compression with factor $\dfrac{1}{8}$ of $y = x^3$.

50. $y = \left|\dfrac{x}{5}\right|$

For Problems 51–62,

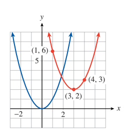
a The graph of each function can be obtained from one of the basic graphs by two or more transformations. Describe the transformations.

b Sketch the basic graph and the graph of the given function by hand on the same axes. Label the coordinates of three points on the graph of the given function.

51. $f(x) = 2 + (x - 3)^2$

Answer.

a Translation by 2 units up and 3 units right

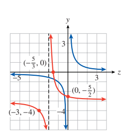
b 

52. $f(x) = (x + 4)^2 + 1$

53. $g(z) = \dfrac{1}{z + 2} - 3$

Answer.

a Translation by 2 units left and 3 units down.

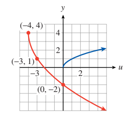
b 

54. $g(z) = \dfrac{1}{z - 1} + 1$

55. $F(u) = -3\sqrt{u+4}+4$

Answer.

a Reflection across the u-axis, vertical stretch by a factor of 3, translation by 4 units left and 4 units up

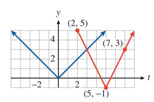
b 

56. $F(u) = 4\sqrt{u-3}-5$

57. $G(t) = 2\,|t-5| - 1$

Answer.

a Vertical stretch by a factor of 2, translation by 5 units right and 1 down

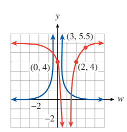
b 

58. $G(t) = 2 - |t+4|$

59. $H(w) = 6 - \dfrac{2}{(w-1)^2}$

Answer.

a Reflection across the w-axis, vertical stretch by a factor of 2, translation by 6 units up and 1 unit right

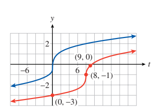
b 

60. $H(w) = \dfrac{3}{(w+2)^2} - 1$

61. $f(t) = \sqrt[3]{t-8} - 1$

Answer.

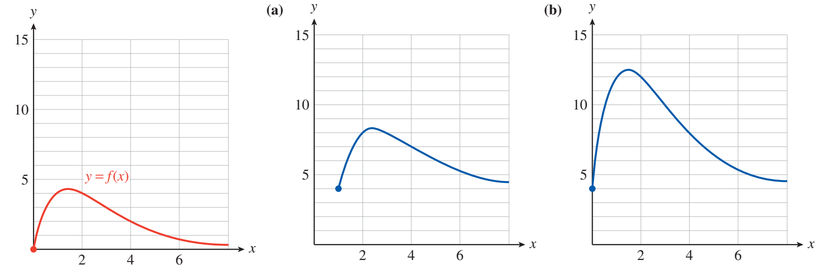
a Translation by 8 units right and 1 unit down

b 

62. $f(t) = \sqrt[3]{t+1} + 8$

In Problems 63 and 64, each graph can be obtained by two transformations of the given graph. Describe the transformations and write a formula for the new graph in terms of f.

63.

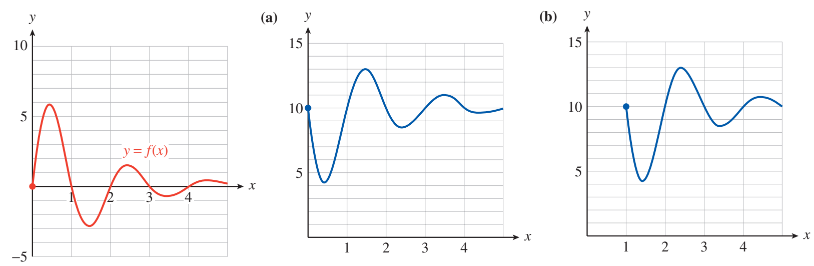


Answer.

a Translation by 4 units up and 1 unit right: $y = f(x-1)+4$

b Vertical stretch by a factor of 2 and a translation by 4 units up: $y = 2f(x)+4$

64.

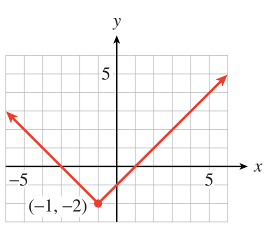


For Problems 65–70,

a Describe the graph as a transformation of a basic function.

b Give an equation for the function shown.

65.

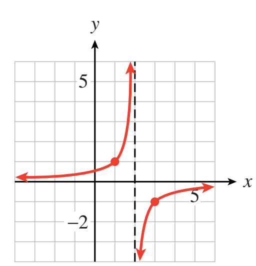


Answer.

a $y = |x|$ translated by 1 unit left and 2 units down

b $y = |x+1| - 2$

66.



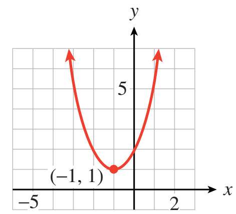
67.

Answer.

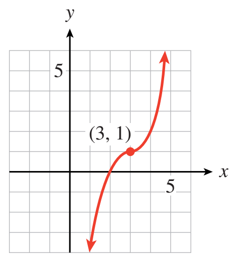
a $y = \sqrt{x}$ reflected about the x-axis and shifted 3 units up

b $y = -\sqrt{x} + 3$

68.



69.

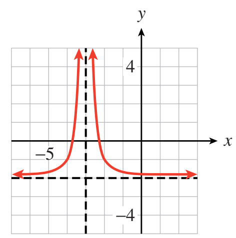


Answer.

a $y = x^3$ translated by 3 units right and 1 unit up

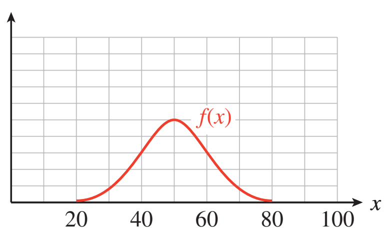
b $y = (x - 3)^3 + 1$

70.

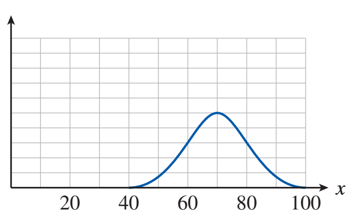


71.

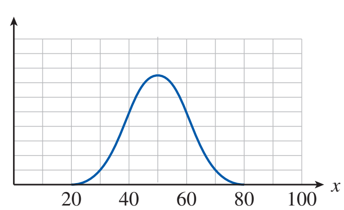
The graph of $f(x)$ shows the number of students in Professor Hilbert's class who scored x points on a quiz. Write a formula for each transformation. Explain how the quiz results given in (a) and (b) compare to the results in Professor Hilbert's class.



(a)



(b)



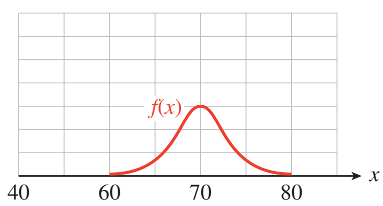
Answer.

a $y = f(x - 20)$: Students scored 20 points higher than Professor Hilbert's class.

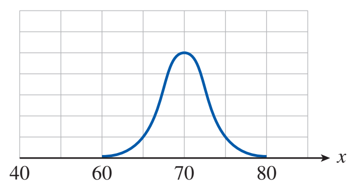
b $y = 1.5f(x)$: The class is about 50% larger than Hilbert's, but the classes scored the same.

72.

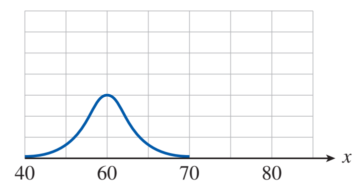
The graph of $f(x)$ shows the number of men at Tyler College who are x inches tall. Write a formula for each transformation of f ; then explain how the heights in that population compare to the Tyler College men.



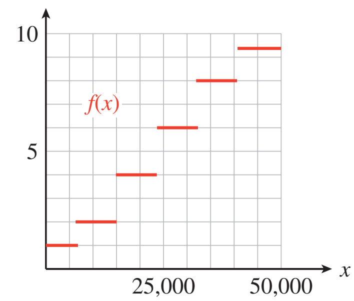
(a)



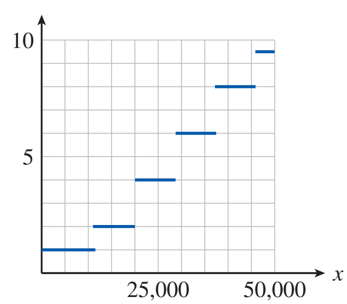
(b)



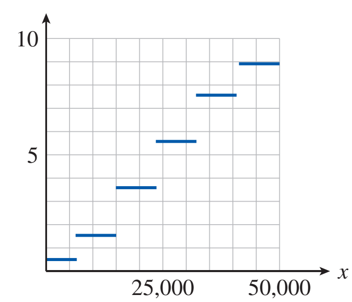
73. The graph of $f(x)$ shows the California state income tax rate, in percent, for a single taxpayer whose annual taxable income is x dollars. Write a formula for each transformation of f; then explain what it tells you about the income tax scheme in that state.



(a)



(b)

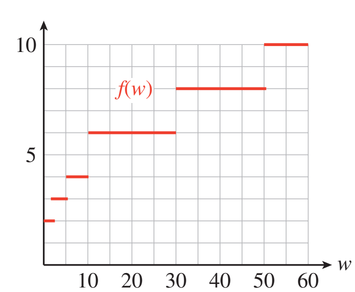


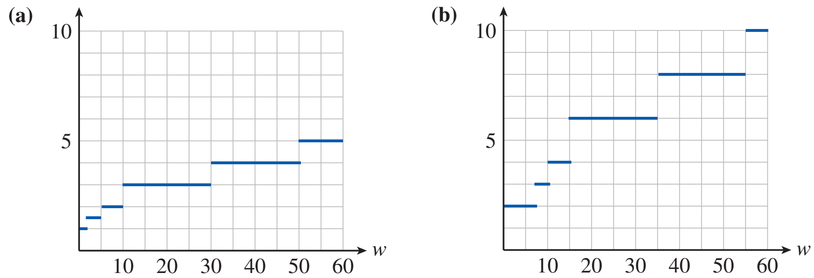
Answer.

a $y = f(x - 5000)$: Taxpayers earn \$5000 more than Californians in each tax rate

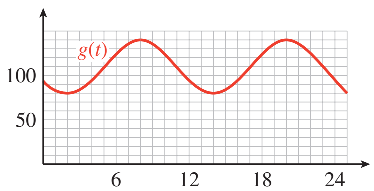
b $y = f(x) - 0.2$: Taxpayers pay 0.2% less tax than Californians on the same income.

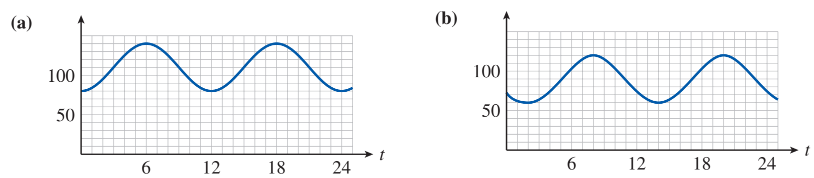
74. The graph of $f(w)$ shows the shipping rate at SendIt for a package that weighs w pounds. Write a formula for each transformation of f and explain how the shipping rates compare to the rates at SendIt.





75. The graph of $g(t)$ shows the population of marmots in a national park t months after January 1. Write a formula for each transformation of g and explain how the population of that species compares to the population of marmots.



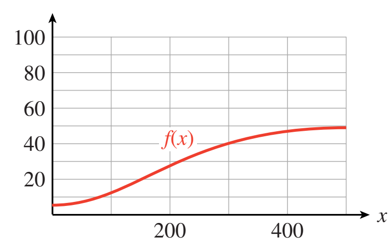


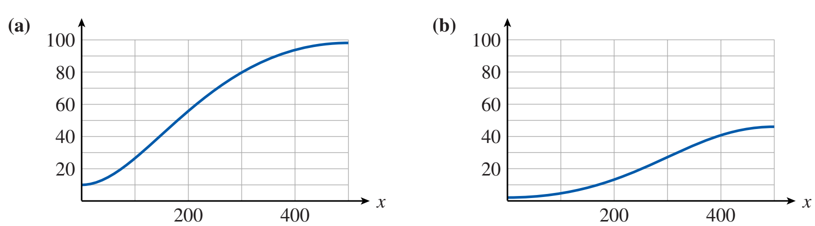
Answer.

a $y = g(t+2)$: This population has its maximum and minimum two months before the marmots.

b $y = g(t) - 20$: This population remains 20 fewer than that of the marmots.

76. The graph of $f(x)$ is a dose-response curve. It shows the intensity of the response to a drug as a function of the dosage x milligrams administered. The intensity is given as a percentage of the maximum response. Write a formula for each transformation of f and explain what it tells you about the response to that drug





2.4 Functions as Mathematical Models

2.4.1 The Shape of the Graph

Creating a good model for a situation often begins with deciding what kind of function to use. An appropriate model can depend on very qualitative considerations, such as the general shape of the graph. What sort of function has the right shape to describe the process we want to model? Should it be increasing or decreasing, or some combination of both? Is the slope constant or is it changing?

In Examples 2.4.1, p. 215 and 2.4.3, p. 216, we investigate how the shape of a graph illustrates the nature of the process it models.

Example 2.4.1 Forrest leaves his house to go to school. For each of the following situations, sketch a possible graph of Forrest's distance from home as a function of time.

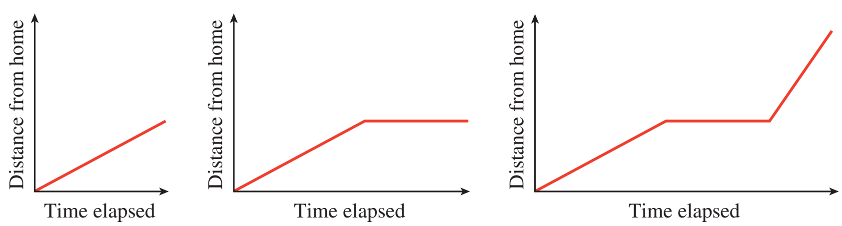
a Forrest walks at a constant speed until he reaches the bus stop.

b Forrest walks at a constant speed until he reaches the bus stop; then he waits there until the bus arrives.

c Forrest walks at a constant speed until he reaches the bus stop, waits there until the bus arrives, and then the bus drives him to school at a constant speed.

Solution.

a The graph is a straight-line segment, as shown in figure (a). It begins at the origin because at the instant Forrest leaves the house, his distance from home is 0. (In other words, when $t = 0, y = 0$.) The graph is a straight line because Forrest has a constant speed. The slope of the line is equal to Forrest's walking speed.



(a) **(b)** **(c)**

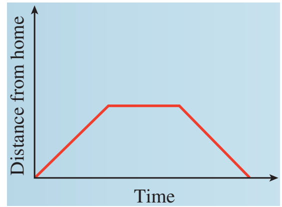
b The graph begins like the graph in part (a). But while Forrest waits for the bus, his distance from home remains constant, so the graph at that time is a horizontal line, as shown in figure (b). The line has slope 0 because while Forrest is waiting for the bus, his speed is 0.

c The graph begins like the graph in part (b). The last section of the graph represents the bus ride. It has a constant slope because the bus is moving at a constant speed. Because the bus (probably) moves faster than Forrest walks, the slope of this segment is greater than the slope for the walking section. The graph is shown in figure (c).

□

Checkpoint 2.4.2 Erin walks from her home to a convenience store, where she buys some cat food, and then walks back home. Sketch a possible graph of her distance from home as a function of time.

Answer.



The graphs in Example 2.4.1, p. 215 are piecewise linear, because Forrest traveled at a constant rate in each segment. In addition to choosing a graph that is increasing, decreasing, or constant to model a process, we can consider graphs that bend upward or downward. The bend is called the **concavity** of the graph.

Example 2.4.3 The two functions described in this example are both increasing functions, but they increase in different ways. Match each function to its graph and to the appropriate table of values.

a The number of flu cases reported at an urban medical center during an epidemic is an increasing function of time, and it is growing at a faster and faster rate.

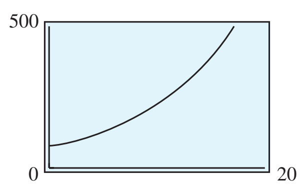
b The temperature of a potato placed in a hot oven increases rapidly at first, then more slowly as it approaches the temperature of the oven.

(1)

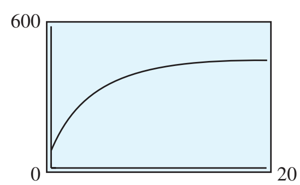
x	0	2	5	10	15
y	70	89	123	217	383

(2)

x	0	2	5	10	15
y	70	219	341	419	441



(A)



(B)

Solution.

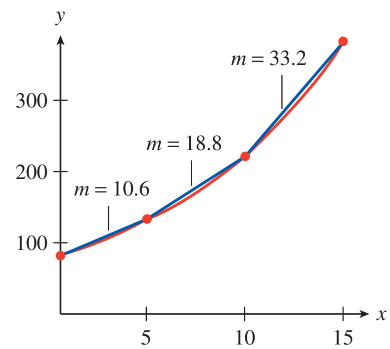
a The number of flu cases is described by graph(A) and table (1). The function values in table (1) increase at an increasing rate. We can see this by computing the rate of change over successive time intervals.

$$x = 0 \text{ to } x = 5: \qquad m = \frac{\Delta y}{\Delta x} = \frac{123-70}{5-0} = 10.6$$

$$x = 5 \text{ to } x = 10: \qquad m = \frac{\Delta y}{\Delta x} = \frac{217-123}{10-5} = 18.8$$

$$x = 10 \text{ to } x = 15: \qquad m = \frac{\Delta y}{\Delta x} = \frac{383-217}{15-10} = 33.2$$

The increasing rates can be seen inthe figure below; the graph bends upward as the slopes increase.



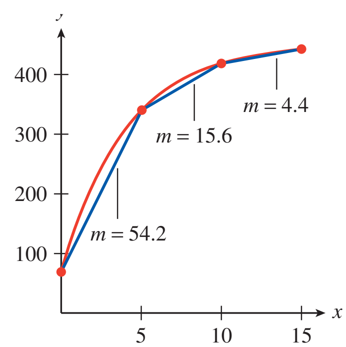
b The temperature of the potato is described by graph(B) and table (2). The function values in table (2) increase, but at a decreasing rate.

$$x = 0 \text{ to } x = 5: \qquad m = \frac{\Delta y}{\Delta x} = \frac{341 - 70}{5 - 0} = 54.2$$

$$x = 5 \text{ to } x = 10: \qquad m = \frac{\Delta y}{\Delta x} = \frac{419 - 341}{10 - 5} = 15.6$$

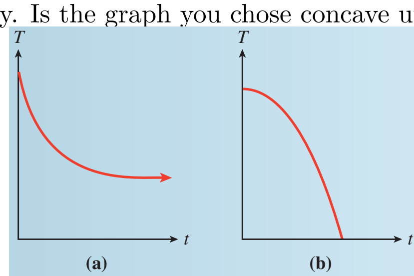
$$x = 10 \text{ to } x = 15: \qquad m = \frac{\Delta y}{\Delta x} = \frac{441 - 419}{15 - 10} = 4.4$$

The decreasing slopes can be seen in the figure below. The graph is increasing but bends downward.



□

Checkpoint 2.4.4 Francine bought a cup of cocoa at the cafeteria. The cocoa cooled off rapidly at first, and then gradually approached room temperature. Which graph more accurately reflects the temperature of the cocoa as a function of time? Explain why. Is the graph you chose concave up or concave down?



Answer. (a): The graph has a steep negative slope at first, corresponding to an initial rapid drop in the temperature of the cocoa. The graph becomes closer to a horizontal line, corresponding to the cocoa approaching room temperature.

The graph is concave up.

2.4.2 Using the Basic Functions as Models

In this section, we consider some situations that can be modeled by the basic functions. Example 2.4.5, p. 218 illustrates an application of the function $f(x) = \sqrt{x}$.

Example 2.4.5 The speed of sound is a function of the temperature of the air in kelvins. (The temperature, T, in kelvins is given by $T = C + 273$, where C is the temperature in degrees Celsius.) The table shows the speed of sound, s, in meters per second, at various temperatures, T.

T (°K)	0	20	50	100	200	400
T (m/sec)	0	89.7	141.8	200.6	283.7	401.2

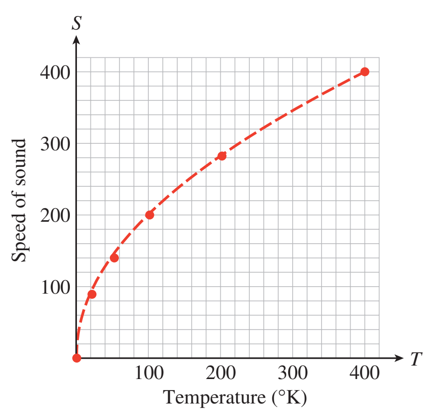
a Plot the data to obtain a graph. Which of the basic functions does your graph most resemble?

b Find a value of k so that $s = kf(T)$ fits the data.

c On a summer night when the temperature is 20° Celsius, you see a flash of lightning, and 6 seconds later you hear the thunderclap. Use your function to estimate your distance from the thunderstorm.

Solution.

a The graph of the data is shown below. The shape of the graph reminds us of the square root function, $y = \sqrt{x}$.



b We are looking for a value of k so that the function $f(T) = k\sqrt{T}$ fits the data. We substitute one of the data points into the formula and solve for k. If we choose the point $(100, 200.6)$, we obtain

$$200.6 = k\sqrt{100}$$

and solving for k yields $k = 20.06$. We can check that the formula $s = 20.06\sqrt{T}$ is a good fit for the rest of the data points as well. Thus, we suggest the function

$$f(T) = 20.06\sqrt{T}$$

as a model for the speed of sound.

c First, we use the model to calculate the speed of sound at a temperature of 20° Celsius. The Kelvin temperature is

$$T = 20 + 273 = 293$$

so we evaluate $s = f(T)$ for $T = 293$.

$$f(293) = 20.06\sqrt{293} \approx 343.4$$

Thus, s is approximately 343.4 meters per second.

The lightning and the thunderclap occur simultaneously, and the speed of light is so fast (about 30,000,000 meters per second) that we see the lightning flash as it occurs. So if the sound of the thunderclap takes 6 seconds after the flash to reach us, we can use our calculated speed of sound to find our distance from the storm.

$$\begin{aligned} \text{distance} &= \text{speed} \times \text{time} \\ &= (343.4 \text{ m/sec})(6 \text{ sec}) = 2060.4 \text{ meters} \end{aligned}$$

The thunderstorm is 2060 meters, or about 1.3 miles, away.

□

Checkpoint 2.4.6 The ultraviolet index (UVI) is issued by the National Weather Service as a forecast of the amount of ultraviolet radiation expected to reach Earth around noon on a given day. The data show how much exposure to the sun people can take before risking sunburn.

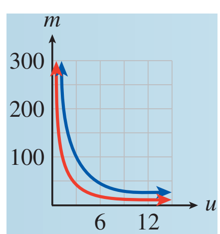
UVI	2	3	4	5	6	8	10	12
Minutes to burn (more sensitive)	30	20	15	12	10	7.5	6	5
Minutes to burn (more sensitive)	150	100	75	60	50	37.5	30	25

a Plot m, the minutes to burn, against u, the UVI, to obtain two graphs, one for people who are more sensitive to sunburn, and another for people less sensitive to sunburn. Which of the basic functions do your graphs most resemble?

b For each graph, find a value of k so that $m = kf(u)$ fits the data.

Answer.

a



The graphs resemble $f(x) = \dfrac{1}{x}$.

b More sensitive: $k = 60$, Less sensitive: $k = 300$

At this point, a word of caution is in order. There is more to choosing a model than finding a curve that fits the data. A model based purely on the data is called an **empirical model**. However, many functions have similar shapes over small intervals of their input variables, and there may be several candidates that model the data. Such a model simply describes the general shape of the data set; the parameters of the model do not necessarily correspond to any actual process.

In contrast, **mechanistic models** provide insight into the biological, chemical, or physical process that is thought to govern the phenomenon under study. Parameters derived from mechanistic models are quantitative estimates of real system properties. Here is what GraphPad Software has to say about modeling:

"Choosing a model is a scientific decision. You should base your choice on your understanding of chemistry or physiology (or genetics, etc.). The choice should not be based solely on the shape of the graph.

"Some programs . . . automatically fit data to hundreds or thousands of equations and then present you with the equation(s) that fit the data best. Using such a program is appealing because it frees you from the need to choose an equation. The problem is that the program has no understanding of the scientific context of your experiment. The equations that fit the data best are unlikely to correspond to scientifically meaningful models. You will not be able to interpret the best-fit values of the variables, and the results are unlikely to be useful for data analysis."

(Source: *Fitting Models to Biological Data Using Linear and Nonlinear Regression*, Motulsky & Christopoulos, GraphPad Software, 2003)

2.4.3 Modeling with Piecewise Functions

Recall that a piecewise function is defined by different formulas on different portions of the x-axis.

Example 2.4.7 In 2005, the income tax $T(x)$ for a single taxpayer with a taxable income x under \$150,000 was given by the following table.

If taxpayer's income is...		Then the estimated tax is...		
Over	But not over	Base tax	+Rate	Of the amount over
\$0	\$7300	\$0	10%	\$0
\$7300	\$29,700	\$730	15%	\$7300
\$29,700	\$71,950	\$4090	25%	\$29,700
\$71,950	\$150,150	\$14,652.50	28%	\$71,950

a Calculate the tax on incomes of \$500, \$29,700, and \$40,000.

b Write a piecewise function for $T(x)$.

c Graph the function $T(x)$.

Solution.

a An income of $x = \mathbf{500}$ is in the first tax bracket, so the tax is

$$T(\mathbf{500}) = 0 + 0.10(\mathbf{500}) = 50$$

The income $x = 29,700$ is just on the upper edge of the second tax bracket. The amount over \$7300 is \$29,700 - \$7300, so

$$T(\mathbf{29,700}) = 730 + 0.15(\mathbf{29,700} - 7300) = 4090$$

The income $x = 40,000$ is in the third bracket, so the tax is

$$T(\mathbf{40,000}) = 4090 + 0.25(\mathbf{40,000} - 29,700) = 6665$$

b The first two columns of the table give the tax brackets, or the x-intervals on which each piece of the function is defined. In each bracket, the tax $T(x)$ is given by

$$\text{Base tax} + \text{Rate} \cdot (\text{Amount over bracket base})$$

For example, the tax in the second bracket is

$$T(x) = 730 + 0.15(x - 7300)$$

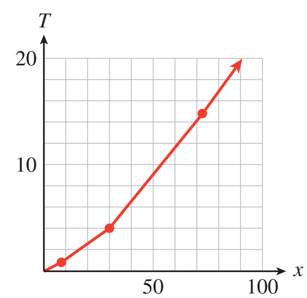
Writing the formulas for each of the four tax brackets gives us

$$T(x) = \begin{cases} 0.10x & 0 \le x \le 7300 \\ 730 + 0.15(x - 7300) & 7300 < x \le 29,700 \\ 4090 + 0.25(x - 29,700) & 29,700 < x \le 71,950 \\ 14,652.50 + 0.28(x - 71,950) & 71,950 < x \le 150,150 \end{cases}$$

c The graph of T is piecewise linear.

- The first piece starts at the origin and has slope 0.10.
- The second piece is in point-slope form, $y = y_1 + m(x - x_1)$, so it has slope 0.15 and passes through the point $(7300,\ 730)$.
- Similarly, the third piece has slope 0.25 and passes through $(29,700,\ \ 40,490)$.
- The fourth piece has slope 0.28 and passes through $(71,950,\ \ 14,652.5)$.

You can check that for this function, all four pieces are connected at their endpoints, as shown below.



□

Checkpoint 2.4.8 As part of a water conservation program, the utilities commission in Arid, New Mexico, establishes a two-tier system of monthly billing for residential water usage: The commission charges a \$30 service fee plus \$2 per hundred cubic feet (HCF) of water if you use 50 HCF or less, and a \$50 service fee plus \$3 per HCF of water if you use over 50 HCF (1 HCF of water is about 750 gallons).

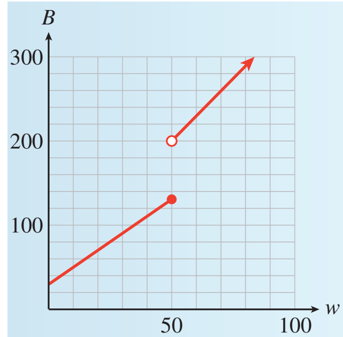
a Write a piecewise formula for the water bill, $B(w)$, as a function of the amount of water used, w, in HCF.

b Graph the function B.

Answer.

a $B(w) = \begin{cases} 30 + 2w & 0 \le w \le 50 \\ 50 + 3w & w > 50 \end{cases}$

b



2.4.4 Section Summary

2.4.4.1 Vocabulary

Look up the definitions of new terms in the Glossary.

- Increasing
- Decreasing
- Concave up
- Concave down
- Empirical model
- Mechanistic model

2.4.4.2 CONCEPTS

1 The shape of a graph describes how the output variable changes.

2 A nonlinear graph may be concave up or concave down. If a graph is concave up, its slope is increasing. If it is concave down, its slope is decreasing.

3 The basic functions can be used to model physical situations.

4 Some situations can be modeled by piecewise functions

5 Fitting a curve to the data is not enough to produce a useful model; appropriate scientific principles should also be considered.

2.4.4.3 STUDY QUESTIONS

1 Sketch the graph of a function whose slope is positive and increasing.

2 Sketch the graph of a function whose slope is positive and decreasing.

3 Which basic function is increasing but bending downward?

4 Which basic function is decreasing but bending upward?

5 Why is it bad practice to choose a model purely on the shape of the data plot?

2.4.4.4 SKILLS

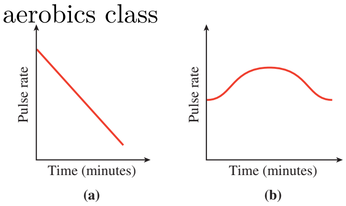
Practice each skill in the Homework 2.4.5, p. 223 problems listed.

1. Sketch a graph whose shape models a situation: #1–18
2. Choose one of the basic graphs to fit a situation or a set of data: #19–24, 35–44
3. Decide whether the graph of a function is increasing or decreasing, concave up or concave down from a table of values: #25–28
4. Write and sketch a piecewise define function to model a situation: #45–48

2.4.5 Functions as Mathematical Models (Homework 2.4)

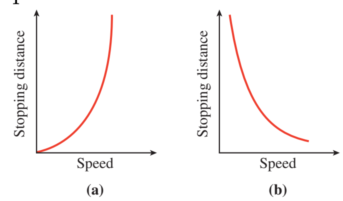
In Problems 1–4, which graph best illustrates each of the following situations?

1. Your pulse rate during an aerobics class

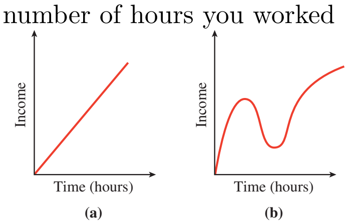


Answer. (b)

2. The stopping distances for cars traveling at various speeds

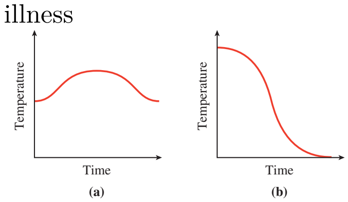


3. Your income in terms of the number of hours you worked



Answer. (a)

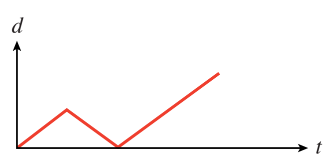
4. Your temperature during an illness



In Problems 5–8, sketch graphs to illustrate the following situations

5. Halfway from your English class to your math class, you realize that you left your math book in the classroom. You retrieve the book, then walk to your math class. Graph the distance between you and your English classroom as a function of time, from the moment you originally leave the English classroom until you reach the math classroom.

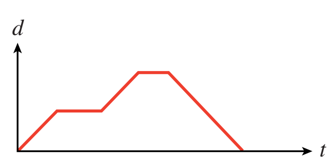
Answer.



6. After you leave your math class, you start off toward your music class. Halfway there you meet an old friend, so you stop and chat for a while. Then you continue to the music class. Graph the distance between you and your math classroom as a function of time, from the moment you leave the math classroom until you reach the music classroom.

7. Toni drives from home to meet her friend at the gym, which is halfway between their homes. They work out together at the gym; then they both go to the friend's home for a snack. Finally Toni drives home. Graph the distance between Toni and her home as a function of time, from the moment she leaves home until she returns.

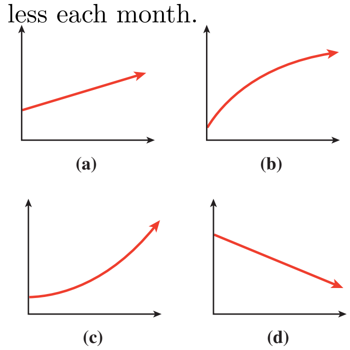
Answer.



8. While bicycling from home to school, Greg gets a flat tire. He repairs the tire in just a few minutes but decides to backtrack a few miles to a service station, where he cleans up. Finally, he bicycles the rest of the way to school. Graph the distance between Greg and his home as a function of time, from the moment he leaves home until he arrives at school.

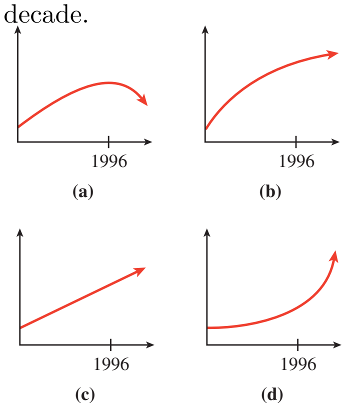
Choose the graph that depicts the function described in Problems 9 and 10.

9. Inflation is still rising, but by less each month.



Answer. (b)

10. The price of wheat was rising more rapidly in 1996 than at any time during the previous decade.



In Problems 11 and 12, match each graph with the function it illustrates.

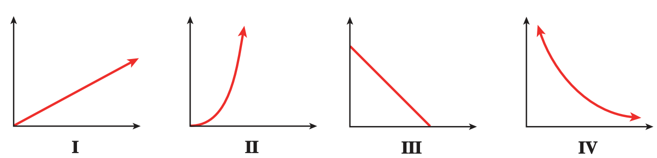
11.

a The volume of a cylindrical container of constant height as a function of its radius

b The time it takes to travel a fixed distance as a function of average speed

c The simple interest earned at a given interest rate as a function of the investment

d The number of Senators present versus the number absent in the U.S. Senate



Answer.

a II   b IV   c I   d III

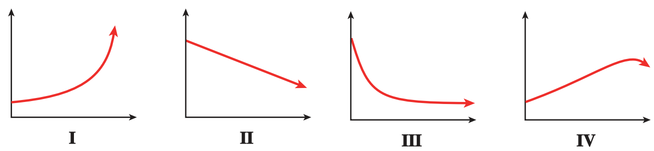
12.

a Unemployment was falling but is now steady.

b Inflation, which rose slowly until last month, is now rising rapidly.

c The birthrate rose steadily until 1990 but is now beginning to fall.

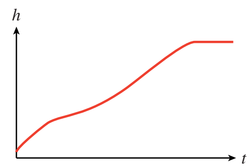
d The price of gasoline has fallen steadily over the past few months.



Sketch possible graphs to illustrate the situations described in Problems 13–18.

13. The height of a man as a function of his age, from birth to adulthood

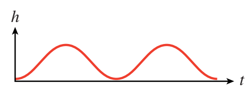
Answer.



14. The number of people willing to buy a new high-definition television, as a function of its price

15. The height of your head above the ground during a ride on a Ferris wheel

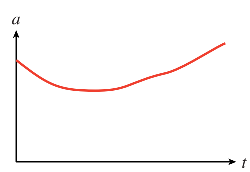
Answer.



16. The height above the ground of a rubber ball dropped from the top of a 10-foot ladder

17. The average age at which women first marry decreased from 1940 to 1960, but it has been increasing since then

Answer.

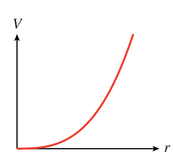


18. When you learn a foreign language, the number of vocabulary words you know increases slowly at first, then increases more rapidly, and finally starts to level off.

Each situation in Problems 19–24 can be modeled by a transformation of a basic function. Name the basic function and sketch a possible graph.

19. The volume of a hot air balloon, as a function of its radius

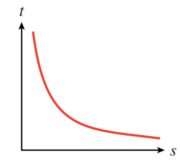
Answer. $y = x^3$ stretched or compressed vertically



20. The length of a rectangle as a function of its width, if its area is 24 square feet

21. The time it takes you to travel 600 miles, as a function of your average speed

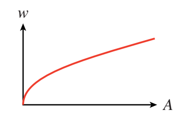
Answer. $y = \dfrac{1}{x}$ stretched or compressed vertically



22. The sales tax on a purchase, as a function of its price

23. The width of a square skylight, as a function of its area

Answer. $y = \sqrt{x}$



24. The sales tax on a purchase, as a function of its price

In Problems 25–28, use the table of values to answer the questions.

a Based on the given values, is the function increasing or decreasing?

b Could the function be concave up, concave down, or linear?

25.

x	0	1	2	3	4
$f(x)$	1	1.5	2.25	3.375	5.0625

26.

x	0	1	2	3	4
$g(x)$	1	0.8	0.64	0.512	0.4096

Answer.

a Increasing

b Concave up

27.

x	0	1	2	3	4
$f(x)$	0	0.174	0.342	0.5	0.643

28.

x	0	1	2	3	4
$g(x)$	1	0.985	0.940	0.866	0.766

Answer.

a Increasing

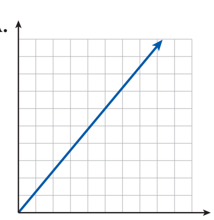
b Concave down

In Problems 29–34,

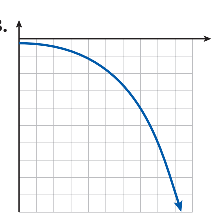
a Is the graph increasing or decreasing, concave up or concave down?

b Match the graph of the function with the graph of its rate of change, shown in Figures A–F.

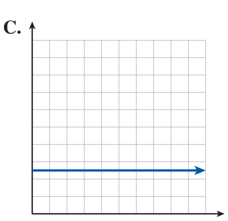
A.



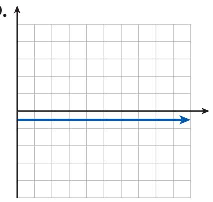
B.



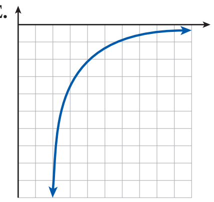
C.



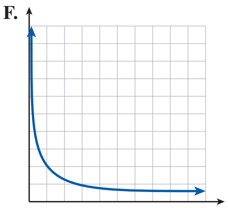
D.



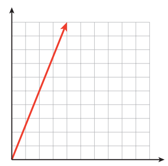
E.



F.



29.

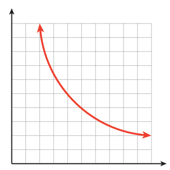


Answer.

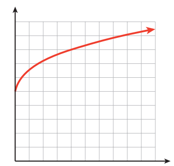
a Increasing, linear (neither concave up nor down)

b C

30.



31.

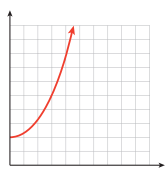


Answer.

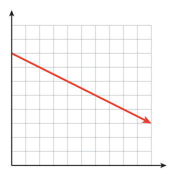
a Increasing, concave down

b F

32.



33.

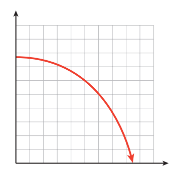


Answer.

a Decreasing, linear (neither concave up nor down)

b D

34.



For Problems 35–40, plot the data; then decide which of the basic functions could describe the data.

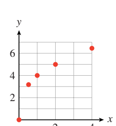
35.

x	0	0.5	1	2	4
y	0	3.17	4	5.04	6.35

36.

x	0	0.5	1	2	4
y	0	5.66	8	11.31	16

Answer.



$y = 4\sqrt[3]{x}$

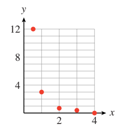
37.

x	0.5	1	2	3	4
y	12	3	0.75	0.33	0.1875

38.

x	0.5	1	2	3	4
y	12	6	3	2	1.5

Answer.



$y = 3 \cdot \dfrac{1}{x^2}$

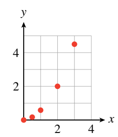
39.

x	0	0.5	1	2	3
y	0	0.125	0.5	2	4.5

40.

x	0	0.5	1	2	3
y	0	0.0125	0.1	0.8	2.7

Answer.



$y = 0.5x^2$

41. Four different functions are described below. Match each description with the appropriate table of values and with its graph.

a As a chemical pollutant pours into a lake, its concentration is a function of time. The concentration of the pollutant initially increases quite rapidly, but due to the natural mixing and self-cleansing action of the lake, the concentration levels off and stabilizes at some saturation level.

b An overnight express train travels at a constant speed across the Great Plains. The train's distance from its point of origin is a function of time.

c The population of a small suburb of a Florida city is a function of time. The population began increasing rather slowly, but it has continued to grow at a faster and faster rate.

d The level of production at a manufacturing plant is a function of capital outlay, that is, the amount of money invested in the plant. At first, small increases in capital outlay result in large increases in production, but eventually the investors begin to experience diminishing returns on their money, so that although production continues to increase, it is at a disappointingly slow rate.

1

x	1	2	3	4	5	6	7	8
y	60	72	86	104	124	149	179	215

2

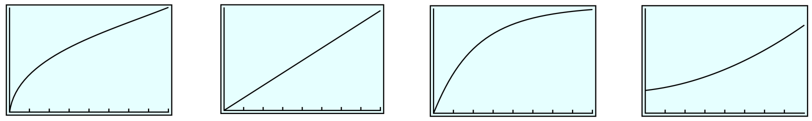
x	1	2	3	4	5	6	7	8
y	60	85	103	120	134	147	159	169

3

x	1	2	3	4	5	6	7	8
y	60	120	180	240	300	360	420	480

4

x	1	2	3	4	5	6	7	8
y	60	96	118	131	138	143	146	147



(A) (B) (C) (D)

Answer.

a Table (4), Graph (C)

b Table (3), Graph (B)

c Table (1), Graph (D)

d Table (2), Graph (A)

42. Four different functions are described below. Match each description with the appropriate table of values and with its graph.

a Fresh water flowing through Crystal Lake has gradually reduced the phosphate concentration to its natural level, and it is now stable.

b The number of bacteria in a person during the course of an illness is a function of time. It increases rapidly at first, then decreases slowly as the patient recovers.

c A squirrel drops a pine cone from the top of a California redwood. The height of the pine cone is a function of time, decreasing ever more rapidly as gravity accelerates its descent.

d Enrollment in Ginny's Weight Reduction program is a function of time. It began declining last fall. After the holidays, enrollment stabilized for a while but soon began to fall off again.

1

x	0	1	2	3	4
y	160	144	96	16	0

2

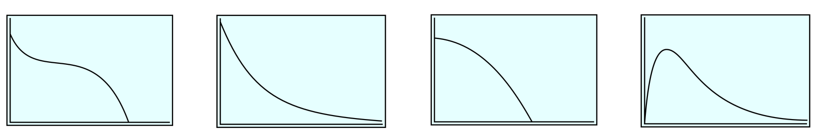
x	0	1	2	3	4
y	20	560	230	90	30

3

x	0	1	2	3	4
y	480	340	240	160	120

4

x	0	1	2	3	4
y	250	180	170	150	80

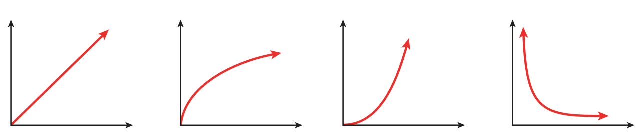


(A) (B) (C) (D)

43. The table shows the radii, r, of several gold coins, in centimeters, and their value, v, in dollars.

Radius	0.5	1	1.5	2	2.5
Value	200	800	1800	3200	5000

a Which graph represents the data?



I II III IV

b Which equation describes the function?

1 $v = k\sqrt{r}$ 2 $v = kr$ 3 $v = kr^2$ 4 $v = \dfrac{k}{r}$

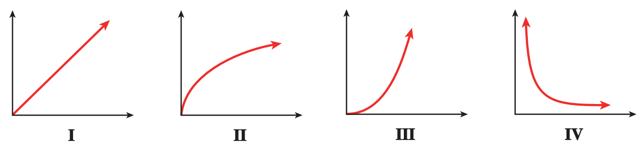
Answer.

a III b 3

44. The table shows how the amount of water, A, flowing past a point on a river is related to the width, W, of the river at that point.

Width (feet)	11	23	34	46
Amount of water (ft^3/sec)	23	34	41	47

a Which graph represents the data?



b Which equation describes the function?

1 $A = k\sqrt{W}$ 2 $A = kW$ 3 $A = kW^2$ 4 $A = \dfrac{k}{W}$

45. If you order from Coldwater Creek, the shipping charges are given by the following table.

Purchase amount	Shipping charge
Up to \$25	\$5.95
\$25.01 to \$50	\$7.95
\$50.01 to \$75	\$9.95
\$75.01 to \$100	\$10.95

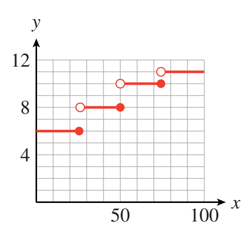
a Write a piecewise formula for $S(x)$, the shipping charge as a function of the purchase amount, x.

b Graph $S(x)$.

Answer.

a $S(x) = \begin{cases} 5.95 & x \le 25 \\ 7.95 & 25 < x \le 50 \\ 9.95 & 50 < x \le 75 \\ 10.95 & 75 < x \le 100 \end{cases}$

b



46. The Bopp-Busch Tool and Die Company markets its products to individuals, to contractors, and to wholesale distributors. The company offers three different price structures for its toggle bolts. If you order 20 or fewer boxes, the price is \$2.50 each. If you order more than 20 but no more than 50 boxes, the price is \$2.25 each. If you order more than 50 boxes, the price is \$2.10 each.

a Write a piecewise formula for $C(x)$, the cost of ordering x boxes of toggle bolts.

b Graph $C(x)$.

47. Bob goes skydiving on his birthday. The function $h(t)$ approximates Bob's altitude t seconds into the trip.

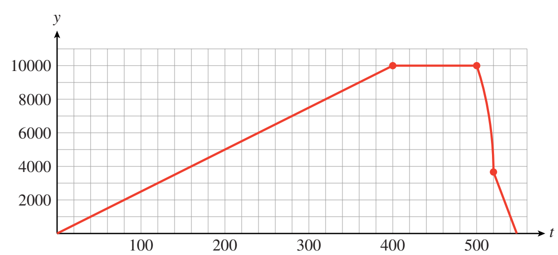
$$h(t) = \begin{cases} 25t & 0 \le t < 400 \\ 10,000 & 400 \le t < 500 \\ 10,000 - 16(t-500)^2 & 500 \le t < 520 \\ 3600 - 120(t-520) & 520 \le t \le 550 \end{cases}$$

a Graph $h(t)$. Describe what you think is happening during each piece of the graph.

b Find two times when Bob is at an altitude of 6000 feet.

Answer.

a



During the first 400 seconds Bob's altitude is climbing with the aircraft; then the aircraft maintains a constant altitude of 10,000 feet for the next 100 seconds; after jumping from the plane, Bob falls for 20 seconds before opening the parachute; he falls at a constant rate after the chute opens.

b 240 seconds (4 minutes) and $500 + \sqrt{250} \approx 515.8$

48. Jenni lives in the San Fernando Valley, where it is hot during summer days but cools down at night. Jenni the air conditioner as little as possible. The function $T(h)$ approximates the temperature in Jenni's house h hours after midnight.

$$T(h) = \begin{cases} 65 & 0 \le h < 8 \\ 25 + 5h & 8 \le h < 14 \\ \dfrac{2240}{h} - 65 & 14 \le h < 16 \\ 75 & 16 \le h < 20 \\ 125 - 2.5h & 20 \le h < 24 \end{cases}$$

a Graph $T(h)$. Describe what you think is happening during each piece of the graph.

b Find two times when the temperature inside the house is 85° Fahrenheit.

49. Lead nitrate and potassium iodide react in solution to produce lead iodide, which settles out, or precipitates, as a yellow solid at the bottom of the container. As you add more lead nitrate to the solution, more lead iodide is produced until all the potassium iodide is used up. The table shows the height of the precipitate in the container as a function of the amount of lead nitrate added. (Source: Hunt and Sykes, 1984)

Lead nitrate solution (cc)	0.5	1.0	1.5	2.0	2.5	3.0	3.5	4.0
Height of precipitate (mm)	2.8	4.8	6.2	7.4	9.5	9.6	9.6	9.6

a Plot the data. Sketch a piecewise linear function with two parts to fit the data points

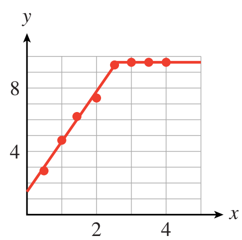
b Calculate the slope of the increasing part of the graph, including units. What is the significance of the slope?

c Write a formula for your piecewise function.

d Interpret your graph in the context of the problem.

Answer.

a

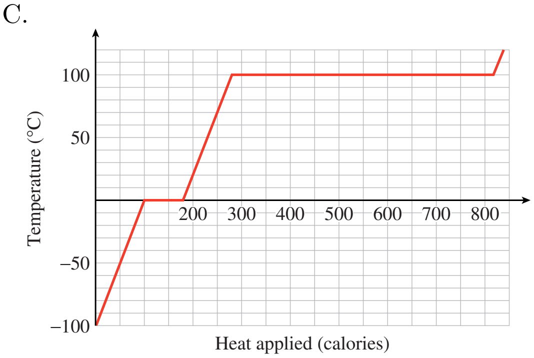


b $m \approx 3.2$ mm/cc: The height of precipitate increases by 1 mm for each additional cc of lead nitrate

c $f(x) = \begin{cases} 1.34 + 3.2x & x < 2.6 \\ 9.6 & x \geq 2.6 \end{cases}$

d The increasing portion of the graph corresponds to the period when the reaction was occurring, and the horizontal section corresponds to when the potassium iodide is used up.

50. The graph shows the temperature of 1 gram of water as a function of the amount of heat applied, in calories. Recall that water freezes at 0°C and boils at 100°C.



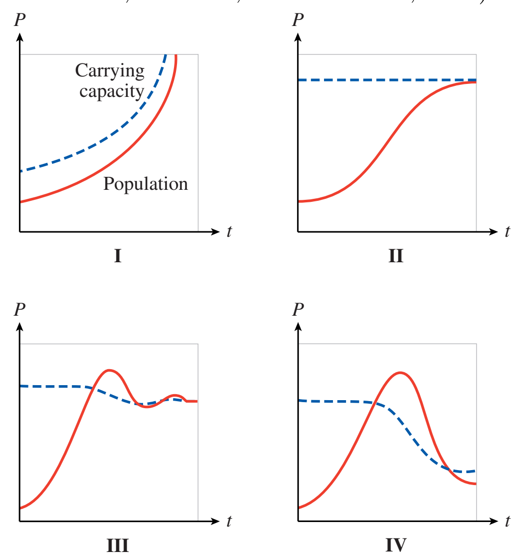
a How much heat is required to raise the temperature of 1 gram of water by 1 degree?

b How much heat is required to convert 1 gram of ice to water?

c How much heat is required to convert 1 gram of water to steam?

d Write a piecewise function to describe the graph.

51. As the global population increases, many scientists believe it is approaching, or has already exceeded, the maximum number the Earth can sustain. This maximum number, or carrying capacity, depends on the finite natural resources of the planet -- water, land, air, and materials -- but also on how people use and preserve the resources. The graphs show four different ways that a growing population can approach its carrying capacity over time. (Source: Meadows, Randers, and Meadows, 2004)



Match each graph to one of the scenarios described in (a)–(d) and explain your choice.

a Sigmoid growth: The population levels off smoothly below the carrying capacity.

b Overshoot and collapse: The population exceeds the carrying capacity with severe damage to the resource base and is forced to decline rapidly to achieve a new balance with a reduced carrying capacity

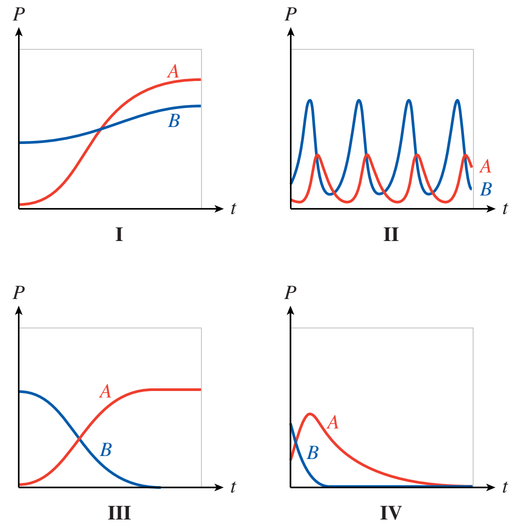
c Continued growth: The carrying capacity is far away, or growing faster than the population.

d Overshoot and oscillation: The population exceeds the carrying capacity without inflicting permanent damage, then oscillates around the limit before leveling off.

Answer.

a II b IV c I d III

52. The introduction of a new species into an environment can affect the growth of an existing species in various ways. The graphs show four hypothetical scenarios after Species A is introduced into an environment where Species B is established.



Match each graph to one of the scenarios described in (a)–(d) and explain your choice.

a Predator-prey (sustained): Species A becomes a predator population that grows when its prey, Species B, is abundant, but declines when the prey population is small. The prey population grows when predators are scarce but shrinks when predators are abundant.

b Predator-prey (extinction): Species A becomes a predator population that annihilates Species B, but then Species A itself declines toward extinction.

c Competition: Species A and B have a common food source, and the Species A replaces Species B in the environment.

d Symbiosis: Species A and B help each other to grow.

53. The Java Stop uses paper cups at a rate of 300 per day. At opening on Tuesday morning Java Stop has on hand 1200 paper cups. On Friday mornings Java Stop takes delivery of a week's worth of cups.

a Write a piecewise function for the number of cups Java Stop has on hand for one week, starting Tuesday morning.

b Graph the function.

c State the domain and range of the function.

2.5 The Absolute Value Function

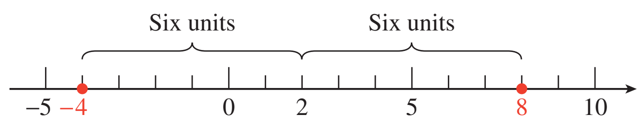
2.5.1 Introduction

The absolute value function is used to model problems involving distance. Recall that the absolute value of a number gives the distance from the origin to that number on the number line.

Distance and Absolute Value.

The distance between two points x and a is given by $|x - a|$.

For example, the equation $|x - 2| = 6$ means "the distance between x and 2 is 6 units." The number x could be to the left or the right of 2 on the number line. Thus, the equation has two solutions, 8 and -4, as shown below.



Example 2.5.1 Write each statement using absolute value notation. Illustrate the solutions on a number line.

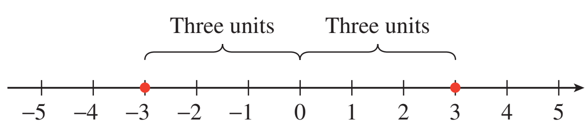
a x is three units from the origin.

b p is two units from 5.

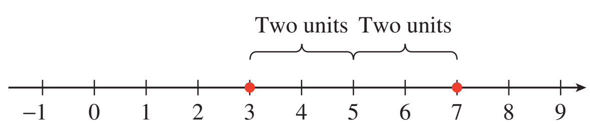
c a is within four units of -2.

Solution. First, restate each statement in terms of distance.

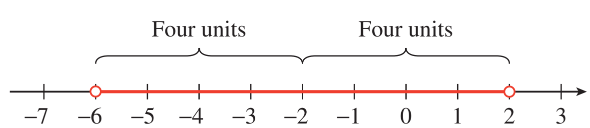
a The distance between x and the origin is three units, or $|x| = 3$. Thus, x can be 3 or -3.



b The distance between p and 5 is two units, or $|p - 5| = 2$. If we count two units on either side of 5, we see that p can be 3 or 7.



c The distance between a and -2 is less than four units, or $\left|a - (-2)\right| < 4$, or $|a + 2| < 4$. Count four units on either side of -2, to find -6 and 2. Then a is between -6 and 2, or $-6 < a < 2$.



□

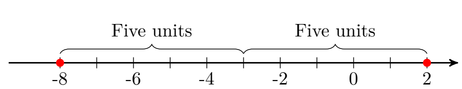
Checkpoint 2.5.2 Write each statement using absolute value notation; then illustrate the solutions on a number line.

a x is five units away from -3.

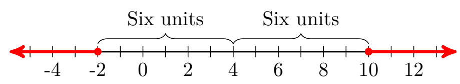
b x is at least six units away from 4.

Answer.

a $|x + 3| = 5$



b $|x-4| \geq 6$



2.5.2 Absolute Value Equations

We can use distances on a number line to solve simple equations such as

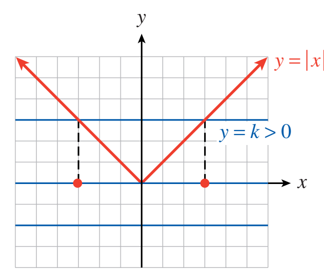
$$|3x-6| = 9$$

First, we factor out the coefficient of x, to get $\left|3(x-2)\right| = 9$. Because of the multiplicative property of the absolute value, namely that $|ab| = |a|\,|b|$, we can write the left side as

$$\begin{aligned} |3|\,|x-2| &= 9 \\ 3\,|x-2| &= 9 \quad \text{Divide both sides by 3.} \\ |x-2| &= 3 \end{aligned}$$

which tells us that the distance between x and 2 is 3 units, so the solutions are $x = -1$ and $x = 5$.

Alternatively, we can use graphs when working with absolute values. For example, we know that the simple equation $|x| = 5$ has two solutions, $x = 5$ and $x = -5$. In fact, we can see from the graph at right that the equation $|x| = k$ has two solutions if $k > 0$, one solution if $k = 0$, and no solution if $k < 0$.

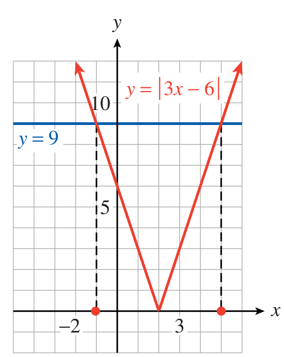


Example 2.5.3

a Use a graph of $y = |3x-6|$ to solve the equation $|3x-6| = 9$.

b Use a graph of $y = |3x-6|$ to solve the equation $|3x-6| = -2$.

Solution.



a The graph shows the graphs of $y = |3x-6|$ and $y = 9$. We see that there are two points on the graph of $y = |3x-6|$ that have $y = 9$, and those points have x-coordinates $x = -1$ and $x = 5$. We can verify algebraically that the solutions are -1 and 5.

$$\begin{aligned} x = \mathbf{-1}: &\quad \left|3(\mathbf{-1}) - 6\right| = |-9| = 9 \\ x = \mathbf{5}: &\quad \left|3(\mathbf{5}) - 6\right| = |9| = 9 \end{aligned}$$

b There are no points on the graph of $y = |3x - 6|$ with $y = -2$, so the equation $|3x - 6| = -2$ has no solutions.

□

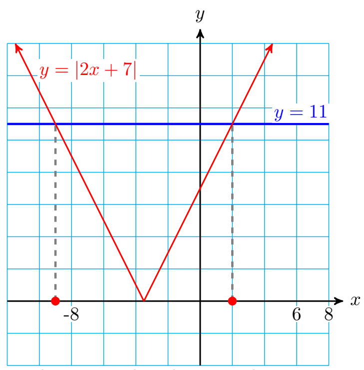
Checkpoint 2.5.4

a Graph $y = |2x + 7|$ for $-12 \le x \le 8$.

b Use your graph to solve the equation $|2x + 7| = 11$.

Answer.

a



b $x = 9$ or $x = 2$

To solve an absolute value equation algebraically, we use the definition of absolute value.

Example 2.5.5 Solve the equation $|3x - 6| = 9$ algebraically.

Solution. We write the piecewise definition of $|3x - 6|$.

$$|3x-6| = \begin{cases} 3x-6 & \text{if } 3x-6 \ge 0, \text{ or } x \ge 2 \\ -(3x-6) & \text{if } 3x-6 < 0, \text{ or } x < 2 \end{cases}$$

Thus, the absolute value equation $|3x - 6| = 9$ is equivalent to two regular equations:

$$3x - 6 = 9 \quad \text{or} \quad -(3x - 6) = 9$$

or, by simplifying the second equation,

$$3x - 6 = 9 \quad \text{or} \quad 3x - 6 = -9$$

Solving these two equations gives us the same solutions we found in Example 2.5.3, p. 236, namely $x = 5$ and -1. □

In general, we have the following strategy for solving absolute value equations.

Absolute Value Equations.

The equation

$$|ax + b| = c \qquad (c > 0)$$

is equivalent to

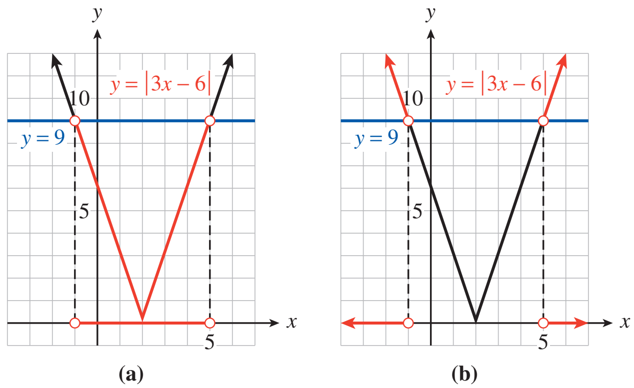
$$ax + b = c \quad \text{or} \quad ax + b = -c$$

Checkpoint 2.5.6 Solve $|2x + 7| = 11$ algebraically.

Answer. $x = -9$ or $x = 2$

2.5.3 Absolute Value Inequalities

We can also use graphs to solve absolute value inequalities. Look again at the graph of $y = |3x - 6|$ in figure (a) below.



Because of the V-shape of the graph, all points with y-values less than 9 lie between the two solutions of $|3x - 6| = 9$, that is, between -1 and 5. Thus, the solutions of the inequality $|3x - 6| < 9$ are $-1 < x < 5$. (In the Homework Problems, you will be asked to show this algebraically.)

On the other hand, to solve the inequality $|3x - 6| > 9$, we look for points on the graph with y-values greater than 9. In figure (b), we see that these points have x-values outside the interval between -1 and 5. In other words, the solutions of the inequality $|3x - 6| > 9$ are $x < -1$ or $x > 5$.

Thus, we can solve an absolute value inequality by first solving the related equation.

Absolute Value Inequalities.

Suppose the solutions of the equation $|ax + b| = c$ are r and s, with $r < s$. Then

1. The solutions of $|ax + b| < c$ are

$$r < x < s$$

2. The solutions of $|ax + b| > c$ are

$$x < r \quad \text{or} \quad x > s$$

Example 2.5.7 Solve $|4x - 15| < 0.01$

Solution. First, we solve the equation $|4x - 15| = 0.01$. There are two cases:

$$\begin{aligned} 4x - 15 &= 0.01 & \text{or} \quad 4x - 15 &= -0.01 \\ 4x &= 15.01 & 4x &= 14.99 \\ x &= 3.7525 & x &= 3.7475 \end{aligned}$$

Because the inequality symbol is $<$, the solutions of the inequality are between these two values: $3.7475 < x < 3.7525$. In interval notation, the solutions are $(3.7475, 3.7525)$. □

Checkpoint 2.5.8

a Solve the inequality $|2x + 7| < 11$

b Solve the inequality $|2x + 7| > 11$

Answer.

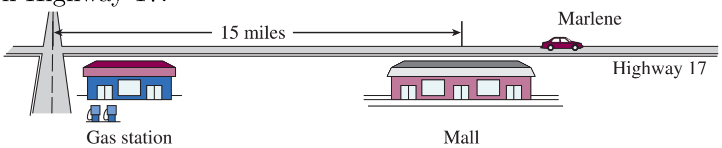
a $(-9, 2)$

b $(-\infty, -9) \cup (2, \infty)$

2.5.4 Using the Absolute Value in Modeling

In the next Example, we use the absolute value function to model a problem about distances.

Example 2.5.9 Marlene is driving to a new outlet mall on Highway 17. There is a gas station at Marlene's on-ramp, where she buys gas and resets her odometer to zero before getting on the highway. The mall is only 15 miles from Marlene's on-ramp, but she mistakenly drives past the mall and continues down the highway. Marlene's distance from the mall is a function of how far she has driven on Highway 17.



a Make a table of values showing how far Marlene has driven on Highway 17 and how far she is from the mall.

b Make a graph of Marlene's distance from the mall versus the number of miles she has driven on the highway. Which of the basic graphs from Section 2.2 does your graph most resemble?

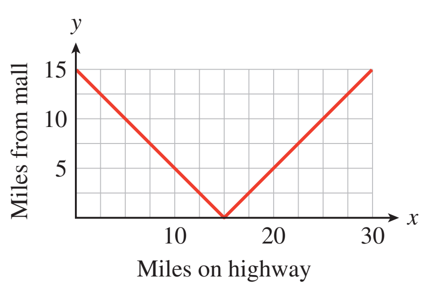
c Find a piecewise defined formula that describes Marlene's distance from the mall as a function of the distance she has driven on the highway.

Solution.

a Marlene gets closer to the mall for each mile that she has driven on the highway until she has driven 15 miles, and after that she gets farther from the mall.

Miles on highway	0	5	10	15	20	25	30
Miles from mall	15	10	5	0	5	10	15

b We plot the points in the table to obtain the graph shown below. This graph looks like the absolute value function defined in Section 2.2, p. 170, except that the vertex is the point $(15, 0)$ instead of the origin.



c Let x represent the number of miles on the highway and $f(x)$ the number of miles from the mall. For x-values less than 15, the graph is a straight line with slope -1 and y-intercept at $(0, 15)$, so its equation is $y = -x + 15$. Thus,

$$f(x) = -x + 15 \quad \text{when} \quad 0 \le x < 15$$

On the other hand, when $x \geq 15$, the graph of f is a straight line with slope 1 that passes through the point $(15, 0)$. The point-slope form of this line is

$$y = 0 + 1(x - 15)$$

so $y = x - 15$. Thus,

$$f(x) = x - 15 \quad \text{when} \quad x \geq 15$$

Combining the two pieces, we obtain

$$f(x) = \begin{cases} -x + 15 & \text{when} \quad 0 \leq x < 15 \\ x - 15 & \text{when} \quad x \geq 15 \end{cases}$$

The graph of $f(x)$ is a part of the graph of $y = |x - 15|$. If we think of the highway as a portion of the real number line, with Marlene's on-ramp located at the origin, then the outlet mall is located at 15. Marlene's coordinate as she drives along the highway is x, and the distance from Marlene to the mall is given by $f(x) = |x - 15|$.

□

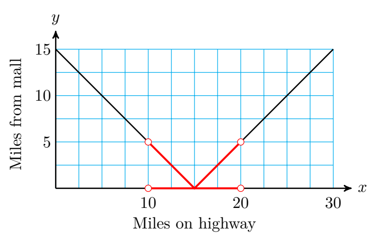
Checkpoint 2.5.10

a Use the graph in Example 2.5.9, p. 239 to determine how far Marlene has driven when she is within 5 miles of the mall. Write and solve an absolute value inequality to verify your answer.

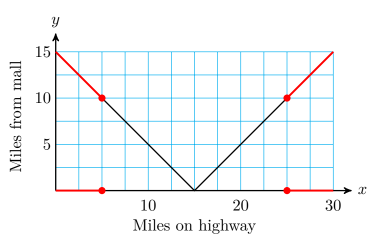
b Write and solve an absolute value inequality to determine how far Marlene has driven when she is at least 10 miles from the mall.

Answer.

a $|x - 15| < 5$; $10 < x < 20$



b $|x - 15| \geq 10$; $x \leq 5$ or $x \geq 25$



2.5.5 Measurement Error

If you weigh a sample in chemistry lab, the scale's digital readout might show 6.0 grams. But it is unlikely that the sample weighs *exactly* 6 grams; there is always some error in measured values.

Because the scale shows the weight as 6.0 grams, we know that the true weight of the sample must be between 5.95 grams and 6.05 grams: If the weight were less than 5.95 grams, the scale would round down to 5.9 grams, and if the weight were more than 6.05 grams, the scale would round up to 6.1 grams. We should report the mass of the sample as 6 ± 0.05 grams, which tells the reader that the error in the measurement is no more than 0.05 grams.

We can also describe this measurement error, or **error tolerance**, using an absolute value inequality. Because the measured mass m can be no more than 0.05 from 6, we write

$$|m - 6| \leq 0.05$$

Note that the solution of this inequality is $5.95 \leq m \leq 6.05$.

Example 2.5.11

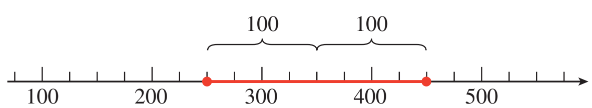
a The specifications for a computer chip state that its thickness in millimeters must satisfy $|t - 0.023| < 0.001$. What are the acceptable values for the thickness of the chip?

b The safe dosage of a new drug is between 250 and 450 milligrams, inclusive. Write the safe dosage as an error tolerance involving absolute values.

Solution.

a The error tolerance can also be stated as $t = 0.023 \pm 0.001$ millimeters, so the acceptable values are between 0.022 and 0.024 millimeters.

b The safe dosage d satisfies $250 \leq d \leq 450$, as shown below.



The center of this interval is 350, and the endpoints are each 100 units from the center. Thus, the safe values are within 100 units of 350, or

$$|d - 350| \leq 100$$

□

Checkpoint 2.5.12 The temperature, T, in a laboratory must remain between 9°C and 12°C.

a Write the error tolerance as an absolute value inequality.

b For a special experiment, the temperature in degrees celsius must satisfy $|T - 6.7| \leq 0.03$. Give the interval of possible temperatures.

Answer.

a $|T - 10.5| < 1.5$ b $6.67 \leq T \leq 6.73$

2.5.6 Section Summary

2.5.6.1 Vocabulary

Look up the definitions of new terms in the Glossary.

- Absolute value equation
- Absolute value inequality
- Error tolerance

2.5.6.2 CONCEPTS

1 The absolute value is used to model distance: The **distance** between two points x and a is given by $|x - a|$.

2 Absolute Value Equations.

The equation

$$|ax + b| = c \qquad (c > 0)$$

is equivalent to

$$ax + b = c \quad \text{or} \quad ax + b = -c$$

3 Absolute Value Inequalities.

Suppose the solutions of the equation $|ax + b| = c$ are r and s, with $r < s$. Then

(a) The solutions of $|ax + b| < c$ are

$$r < x < s$$

(b) The solutions of $|ax + b| > c$ are

$$x < r \quad \text{or} \quad x > s$$

4 The **error tolerance** e in a measurement M can be expressed as $|x - M| < e$, or as $x = M \pm e$. Both indicate that $M - e < x < M + e$.

2.5.6.3 STUDY QUESTIONS

1 Write a function that models the distance between x and a fixed point k on the number line.

2 For what values of c does the equation $|ax + b| = c$ have one solution? No solution?

3 If you know that the solutions of $|ax + b| < c$ are $-3 < x < 6$, what are the solutions of $|ax + b| > c$?

4 What is the center of the interval $[220, 238]$?

5 What is the center of the interval $[a, b]$?

2.5.6.4 SKILLS

Practice each skill in the Homework 2.5.7, p. 243 problems listed.

1 Use absolute value notation to write statements about distance: #1–8

2 Use graphs to solve absolute value equations and inequalities: #9–12

3 Solve absolute value equations: #13–24

4 Solve absolute value inequalities: #25–40

5 Express error tolerances using absolute value notation: #41–48

6 Analyze absolute value functions: #49–56

7 Model problems about distance using the absolute value function: #57–60

2.5.7 The Absolute Value Function (Homework 2.5)

In Problems 1–8,

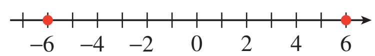
a Use absolute value notation to write each expression as an equation or an inequality. (It may be helpful to restate each sentence using the word *distance*.)

b Illustrate the solutions on a number line.

1. x is six units from the origin.

Answer.

a $|x| = 6$

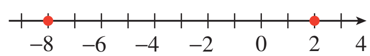
b 

2. a is seven units from the origin.

3. The distance from p to -3 is five units.

Answer.

a $|p + 3| = 5$

b 

4. The distance from q to -7 is two units.

5. t is within three units of 6.

Answer.

a $|t - 6| < 3$

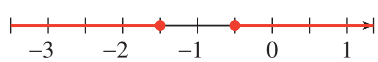
b 

6. w is no more than one unit from -5.

7. b is at least 0.5 unit from -1.

Answer.

a $|b + 1| \geq 0.5$

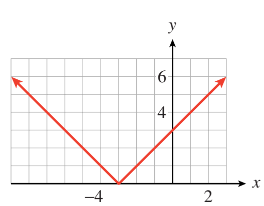
b 

8. m is more than 0.1 unit from 8.

9. Graph $y = |x + 3|$. Use your graph to solve the following equations and inequalities.

a $|x + 3| = 2$ b $|x + 3| \leq 4$ c $|x + 3| > 5$

Answer.



a $x = -5$ or $x = -1$

b $-7 \le x \le 1$

c $x < -8$ or $x > 2$

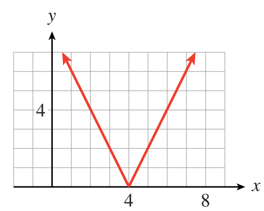
10. Graph $y = |x - 2|$. Use your graph to solve the following equations and inequalities.

a $|x - 2| = 5$ b $|x - 2| < 8$ c $|x - 2| \ge 4$

11. Graph $y = |2x - 8|$. Use your graph to solve the following equations and inequalities.

a $|2x - 8| = 0$ b $|2x - 8| = -2$ c $|2x - 8| < -6$

Answer.



a $x = 4$

b No solution

c No solution

12. Graph $y = |4x + 8|$. Use your graph to solve the following equations and inequalities.

a $|4x + 8| = 0$ b $|4x + 8| < 0$ c $|4x + 8| > -3$

For Problems 13-24, solve.

13. $|2x - 1| = 4$

Answer.
$x = \dfrac{-3}{2}$ or $x = \dfrac{5}{2}$

14. $|3x - 1| = 5$

15. $0 = |7 + 3q|$

Answer.
$q = \dfrac{-7}{3}$

16. $|-11 - 5t| = 0$

17. $4 = \dfrac{|b + 2|}{3}$

Answer.
$b = -14$ or $b = 10$

18. $6\,|n + 2| = 9$

19. $\big|2(w - 7)\big| = 1$

Answer.
$w = \dfrac{13}{2}$ or $w = \dfrac{15}{2}$

20. $2 = \left|\dfrac{a - 4}{5}\right|$

21. $|c - 2| + 3 = 1$

Answer. No solution

22. $5 = 4 - |h + 3|$

23. $-7 = |2m + 3|$

Answer. No solution

24. $|5r - 3| = -2$

For Problems 25-36, solve.

25. $|2x + 6| < 3$

Answer.
$\dfrac{-9}{2} < x < \dfrac{-3}{2}$

26. $|5 - 3x| \le 1$

27. $7 \le |3 - 2d|$

Answer.
$d \le -2$ or $d \ge 5$

28. $10 < |3r + 2|$

29. $|6s + 15| > -3$

Answer. All real numbers

30. $|8b - 12| < -4$

31. $|t - 1.5| < 0.1$

Answer. $1.4 < t < 1.6$

32. $|z - 2.6| \leq 0.1$

33. $|T - 3.25| \geq 0.05$

Answer. $T \leq 3.2$ or $T \geq 3.3$

34. $|P - 0.6| > 0.01$

35. $-1 \geq \left|\dfrac{n-3}{2}\right|$

Answer. No solution

36. $-0.1 \leq \left|9(p + 2)\right|$

In Problems 37–40, give an interval of possible values for the measurement.

37. The length, l, of a rod is given by $|l - 4.3| < 0.001$, in centimeters.

Answer. $4.299 < l < 4.301$

38. The mass, m, of the device shall be $|m - 450| < 4$, in grams.

39. The candle will burn for t minutes, where $|t - 300| \leq 50$.

Answer. $250 \leq t \leq 350$

40. The ramp will have angle of inclination α, and $|\alpha - 10°| \leq 0.5°$.

In Problems 41–44, write the error tolerance using absolute values.

41. The chemical compound must be maintained at a temperature, T, between 4.7° and 5.3°C.

Answer. $|T - 5| < 0.3$

42. The diameter, d, of the hole shall be in the range of 24.98 to 25.02 centimeters.

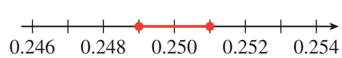
43. The subject will receive a dosage D from 95 to 105 milligrams of the drug.

Answer. $|D - 100| \leq 5$

44. The pendulum swings out and back in a time period t between 0.9995 and 1.0005 seconds.

45. An electrical component of a high-tech sensor requires 0.25 ounce of gold. Assume that the actual amount of gold used, g, is not in error by more than 0.001 ounce. Write an absolute value inequality for the possible error and show the possible values of g on a number line.

Answer. $|g - 0.25| \leq 0.001$



46. In a pasteurization process, milk is to be irradiated for 10 seconds. The actual period t of irradiation cannot be off by more than 0.8 second. Write an absolute value inequality for the possible error and show the possiblev alues of t on a number line.

47. In a lab assignment, a student reports that a chemical reaction required 200 minutes to complete. Let t represent the actual time of the reaction.

a Write an absolute value inequality for t, assuming that the student rounded his answer to the nearest 100 minutes. Give the smallest and largest possible value for t.

b Write an absolute value inequality for t, assuming that the student rounded his answer to the nearest minute. Give the smallest and largest possible value for t.

c Write an absolute value inequality for t, assuming that the student rounded his answer to the nearest 0.1 minute. Give the smallest and largest possible value for t.

Hint. What is the shortest time that would round to 200 minutes? The greatest time?

Answer.

a $|t-200| < 50, \;\; 150 \le t < 250$

b $|t-200| < 0.5, \;\; 199.5 \le t < 200.5$

c $|t-200| < 0.05, \;\; 199.95 \le t < 200.05$

48. An espresso machine has a square metal plate. The side of the plate is 2 ± 0.01 cm.

a Write an absolute value inequality for the length of the side, x. Give the smallest and largest possible value for s.

b Compute the smallest and largest possible area of the plate, including units.

c Write an absolute value inequality for the area, A.

49.

a Write the piecewise definition for $|3x-6|$.

b Use your answer to part (a) to write two inequalities that together are equivalent to $|3x-6| < 9$.

c Solve the inequalities in part (b) and check that the solutions agree with the solutions of $|3x-6| < 9$.

d Show that $|3x-6| < 9$ is equivalent to the compound inequality $-9 < 3x-6 < 9$.

Answer.

a $|3x-6| = \begin{cases} -(3x-6) & \text{if } x < 2 \\ 3x-6 & \text{if } x \ge 2 \end{cases}$

b $-(3x-6) \le 9, \;\; 3x-6 < 9$

c $-1 < x < 5$

d The solutions are the same.

50.

a Write the piecewise definition for $|3x-6|$.

b Use your answer to part (a) to write two inequalities that together are equivalent to $|3x-6| > 9$.

c Solve the inequalities in part (b) and check that the solutions agree with the solutions of $|3x-6| > 9$.

d Show that $|3x-6| > 9$ is equivalent to the compound inequality $3x-6 < -9 \;$ or $\; 3x-6 > 9$.

51.

a Write the piecewise definition for $|2x+5|$.

b Use your answer to part (a) to write two inequalities that together are equivalent to $|2x+5| > 7$.

c Solve the inequalities in part (b) and check that the solutions agree with the solutions of $|2x+5| > 7$.

d Show that $|2x+5| > 7$ is equivalent to the compound inequality $2x+5 < -7$ or $2x+5 > 7$.

Answer.

a $|2x+5| = \begin{cases} -(2x+5) & \text{if } x < \dfrac{-5}{2} \\ 2x+5 & \text{if } x \geq \dfrac{-5}{2} \end{cases}$

b $-(2x+5) > 7,\ \ 2x+5 > 7$

c $x < -6$ or $x > 1$

d The solutions are the same.

52.

a Write the piecewise definition for $|2x+5|$.

b Use your answer to part (a) to write two inequalities that together are equivalent to $|2x+5| < 7$.

c Solve the inequalities in part (b) and check that the solutions agree with the solutions of $|2x+5| < 7$.

d Show that $|2x+5| < 7$ is equivalent to the compound inequality $-7 < 2x+5 < 7$.

For Problems 53–56, graph the function and answer the questions.

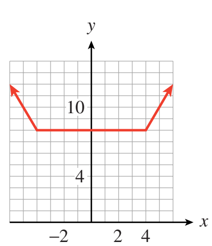
53. $f(x) = |x+4| + |x-4|$

a Using your graph, write a piecewise formula for $f(x)$.

b Experiment by graphing $g(x) = |x+p| + |x-q|$ for different positive values of p and q. Make a conjecture about how the graph depends on p and q.

c Write a piecewise formula for $g(x) = |x+p| + |x-q|$.

Answer.



a $f(x) = \begin{cases} -2x, & x < -4 \\ 8, & -4 \leq x \leq 4 \\ 2x, & x > 4 \end{cases}$

b The graphs looks like like a trough. The middle horizontal section is $y = p + q$ for $-p \le x \le q$, the left side, $x < -p$, has slope -2 and the right side, $x > q$, has slope 2.

c $g(x) = \begin{cases} -2x + q - p, & x < -p \\ p + q, & -p \le x \le q \\ 2x + p - q, & x > q \end{cases}$

54. $f(x) = |x + 4| - |x - 4|$

a Using your graph, write a piecewise formula for $f(x)$.

b Experiment by graphing $g(x) = |x + p| - |x - q|$ for different positive values of p and q. Make a conjecture about how the graph depends on p and q.

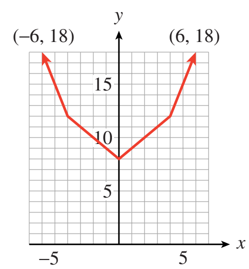
55. $f(x) = |x + 4| + |x| + |x - 4|$

a Using your graph, write a piecewise formula for $f(x)$.

b What is the minimum value of $f(x)$?

c If $p, q \ge 0$, what is the minimum value of $g(x) = |x + p| + |x| + |x - q|$?

Answer.



a $f(x) = \begin{cases} -3x, & x < -4 \\ -x + 8, & -4 \le x \le 0 \\ x + 8, & 0 < x < 4 \\ 3x, & x \ge 4 \end{cases}$

b 8

c $p + q$

56. $f(x) = |x + 4| - |x| + |x - 4|$

a Using your graph, write a piecewise formula for $f(x)$.

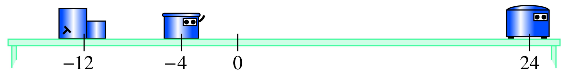
b What is the minimum value of $f(x)$?

c If $p, q \ge 0$, what is the minimum value of $g(x) = |x + p| - |x| + |x - q|$?

Problems 57–60 use the absolute value function to model distance. Use the strategy outlined in Problems 57 and 58 to solve Problems 59 and 60.

57. A small pottery is setting up a workshop to produce mugs. Three machines are located on a long table, as shown in the figure. The potter must use each machine once in the course of producing a mug.

Let x represent the coordinate of the potter's station.



a Write expressions for the distance from the potter's station to each of the machines.

b Write a function that gives the sum of the distances from the potter's station to the three machines.

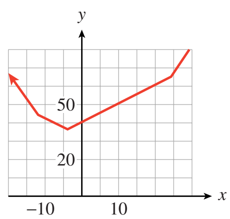
c Graph your function for $-20 \le x \le 30$. Where should the potter stand in order to minimize the distance she must walk to the machines?

Answer.

a $|x+12|$, $|x+4|$, $|x-24|$

b $f(x) = |x+12| + |x+4| + |x-24|$

c



At x-coordinate -4

58. Suppose the pottery in Problem 57 adds a fourth machine to the procedure for producing a mug, located at $x = 16$ in the figure.

a Write and graph a new function for the sum of the potter's distances to the four machines.

b Where should the potter stand now to minimize the distance she has to walk while producing a mug?

59. Richard and Marian are moving to Parkville to take jobs after they graduate. The main road through Parkville runs east and west, crossing a river in the center of town. Richard's job is located 10 miles east of the river on the main road, and Marian's job is 6 miles west of the river. There is a health club they both like located 2 miles east of the river. If they plan to visit the health club every workday, where should Richard and Marian look for an apartment to minimize their total daily driving distance?

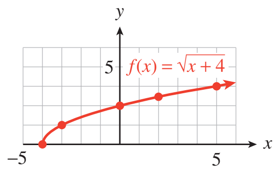
Answer. 2 miles east of the river

60. Romina's Bakery has just signed contracts to provide baked goods for three new restaurants located on Route 28 outside of town. The Coffee Stop is 2 miles north of town center, Sneaky Pete's is 8 miles north, and the Sea Shell is 12 miles south. Romina wants to open a branch bakery on Route 28 to handle the new business. Where should she locate the bakery in order to minimize the distance she must drive for deliveries?

2.6 Domain and Range

2.6.1 Definitions of Domain and Range

In Example 1.3.5, p. 60 of Section 1.3, p. 57, we graphed the function $f(x) = \sqrt{x+4}$ and observed that $f(x)$ is undefined for x-values less than -4. For this function, we must choose x-values in the interval $[-4, \infty)$.



All the points on the graph have x-coordinates greater than or equal to -4, as shown at left. The set of all permissible values of the input variable is called the **domain** of the function f.

We also see that there are no points with negative $f(x)$-values on the graph of f: All the points have $f(x)$-values greater than or equal to zero. The set of all outputs or function values corresponding to the domain is called the **range** of the function. Thus, the domain of the function $f(x) = \sqrt{x+4}$ is the interval $[-4, \infty)$, and its range is the interval $[0, \infty)$. In general, we make the following definitions.

Domain and Range.

The **domain** of a function is the set of permissible values for the input variable. The **range** is the set of function values (that is, values of the output variable) that correspond to the domain values.

Using the notions of domain and range, we restate the definition of a function as follows.

Definition of Function.

A relationship between two variables is a **function** if each element of the domain is paired with exactly one element of the range.

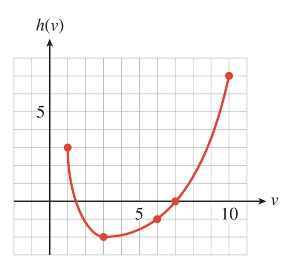
2.6.2 Finding Domain and Range from a Graph

We can identify the domain and range of a function from its graph. The domain is the set of x-values of all points on the graph, and the range is the set of y-values.

Example 2.6.1

a Determine the domain and range of the function h graphed at right.

b For the indicated points, show the domain values and their corresponding range values in the form of ordered pairs.

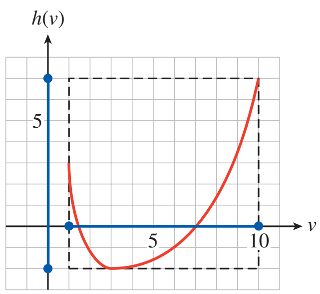


Solution.

a All the points on the graph have v-coordinates between 1 and 10, inclusive, so the domain of the function h is the interval $[1, 10]$. The $h(v)$-coordinates have values between -2 and 7, inclusive, so the range of the function is the interval $[-2, 7]$.

b Recall that the points on the graph of a function have coordinates $(v, h(v))$. In other words, the coordinates of each point are made up of a domain value and its corresponding range value. Read the coordinates of the indicated points to obtain the ordered pairs $(1, 3)$, $(3, -2)$, $(6, -1)$, $(7, 0)$, and $(10, 7)$.

□



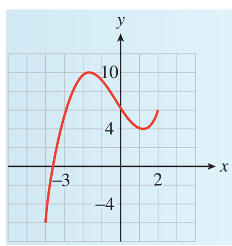
The figure at left shows the graph of the function h in Example 2.6.1, p. 250 with the domain values marked on the horizontal axis and the range values marked on the vertical axis. Imagine a rectangle whose length and width are determined by those segments, as shown in the figure. All the points $(v, h(v))$ on the graph of the function lie within this rectangle.

The rectangle described above is a convenient window in the plane for viewing the function. Of course, if the domain or range of the function is an infinite interval, we can never include the whole graph within a viewing rectangle and must be satisfied with studying only the important parts of the graph.

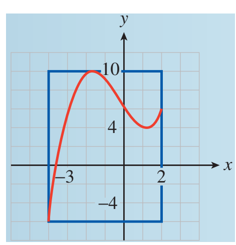
Checkpoint 2.6.2

a Draw the smallest viewing window possible around the graph shown below.

b Find the domain and range of the function.



Answer. domain: $[-4, 2]$; range: $[-6, 10.1]$



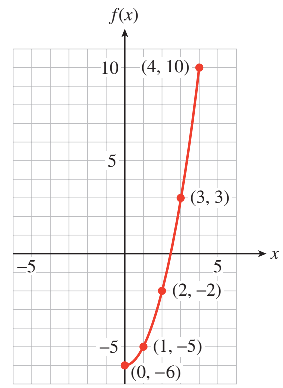
Sometimes the domain is given as part of the definition of a function.

Example 2.6.3 Graph the function $f(x) = x^2 - 6$ on the domain $0 \le x \le 4$ and give its range.

Solution. The graph is part of a parabola that opens upward. Ww obtain several points on the graph by evaluating the function at convenient x-values in the domain.

x	$f(x)$	
0	-6	since $f(\mathbf{0}) = \mathbf{0}^2 - 6 = -6$
1	-5	since $f(\mathbf{1}) = \mathbf{1}^2 - 6 = -5$
2	-2	since $f(\mathbf{2}) = \mathbf{2}^2 - 6 = -2$
3	3	since $f(\mathbf{3}) = \mathbf{3}^2 - 6 = 3$
4	10	since $f(\mathbf{4}) = \mathbf{4}^2 - 6 = 10$

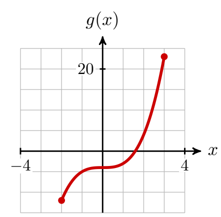
The range of the function is the set of all $f(x)$-values that appear on the graph. We can see that the lowest point on the graph is $(0, -6)$, so the smallest $f(x)$-value is -6. The highest point on the graph is $(4, 10)$, so the largest $f(x)$-value is 10. Thus, the range of the function f is the interval $[-6, 10]$.



□

Checkpoint 2.6.4 Graph the function $g(x) = x^3 - 4$ on the domain $[-2, 3]$ and give its range.

Answer. range: $[-12, 23]$



Not all functions have domains and ranges that are intervals.

Example 2.6.5

a The table gives the postage for sending printed material by first-class mail in 2016. Graph the postage function $p = g(w)$.

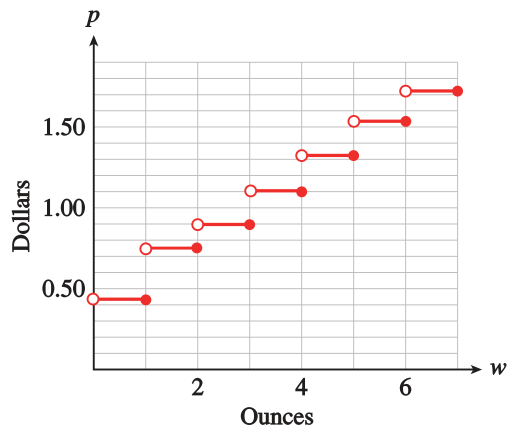
Weight in ounces (w)	Postage (p)
$0 < w \leq 1$	\$0.47
$1 < w \leq 2$	\$0.68
$2 < w \leq 3$	\$0.89
$3 < w \leq 4$	\$1.10
$4 < w \leq 5$	\$1.31
$5 < w \leq 6$	\$1.52
$6 < w \leq 7$	\$1.73

b Determine the domain and range of the function.

Solution.

a From the table, we see that articles of any weight up to 1 ounce require \$0.47 postage. This means that for all w-values greater than 0 but less than or equal to 1, the p-value is 0.47. Thus, the graph of $p = g(w)$ between $w = 0$ and $w = 1$ looks like a small piece of the horizontal line $p = 0.47$.

Similarly, for all w-values greater than 1 but less than or equal to 2, the p-value is 0.68, so the graph on this interval looks like a small piece of the line $p = 0.68$. Continue in this way to obtain the graph shown below.



The open circles at the left endpoint of each horizontal segment indicate that that point is not included in the graph; the closed circles are points on the graph. For instance, if $w = 3$, the postage, p, is \$0.89, not \$1.10. Consequently, the point $(3, 0.89)$ is part of the graph of g, but the point $(3, 1.10)$ is not.

b Postage rates are given for all weights greater than 0 ounces up to and including 7 ounces, so the domain of the function is the half-open interval $(0, 7]$. (The domain is an interval because there is a point on the graph for every w-value from 0 to 7.)

The range of the function is not an interval, however, because the possible values for p do not include all the real numbers between 0.3 and 1.75. The range is the set of discrete values 0.47, 0.68, 0.89, 1.10, 1.31, 1.52, and 1.73.

□

Checkpoint 2.6.6 In Checkpoint 2.4.8, p. 221 of Section 2.4, p. 215, you wrote a formula for residential water bills, $B(w)$, in Arid, New Mexico:

$$B(w) = \begin{cases} 30 + 2w, & 0 \leq w \leq 50 \\ 50 + 3w, & w > 50 \end{cases}$$

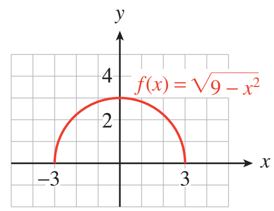
If the utilities commission imposes a cap on monthly water consumption at 120 HCF, find the domain and range of the function $B(w)$.

Answer. domain: $[0, 120]$; range: $[30, 130] \cup (200, 410]$

2.6.3 Finding the Domain from a Formula

If the domain of a function is not given as part of its definition, we assume that the domain is as large as possible. We include in the domain all x-values that make sense when substituted into the function's formula.

For example, the domain of the function $f(x) = \sqrt{9 - x^2}$ is the interval $[-3, 3]$, because x-values less than -3 or greater than 3 result in square roots of negative numbers. You may recognize the graph of f as the upper half of the circle $x^2 + y^2 = 9$, as shown at right.



Example 2.6.7 Find the domain of the function $g(x) = \dfrac{1}{x-3}$

Solution. We must omit any x-values that do not make sense in the function's formula. Because division by zero is undefined, we cannot allow the denominator of $\dfrac{1}{x-3}$ to be zero. Since $x - 3 = 0$ when $x = 3$, we exclude $x = 3$ from the domain of g. Thus, the domain of g is the set of all real numbers except 3. □

Checkpoint 2.6.8

a Find the domain of the function $h(x) = \dfrac{1}{(x-4)^2}$.

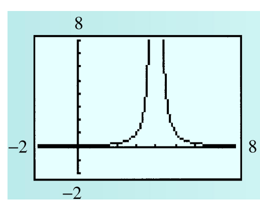
b Graph the function in the window

$$\begin{array}{ll} \text{Xmin} = -2 & \text{Xmax} = 8 \\ \text{Ymin} = -2 & \text{Ymax} = 8 \end{array}$$

Use your graph and the function's formula to find its range.

Answer.

a Domain: $x \neq 4$

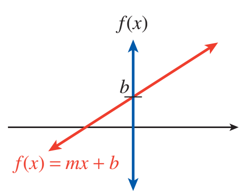


b Range: $y > 0$

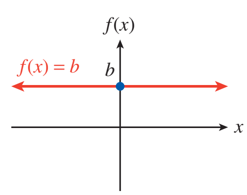
For the functions we have studied so far, there are only two operations we must avoid when finding the domain: division by zero and taking the square root of a negative number.

Many common functions have as their domain the entire set of real numbers. In particular, a linear function $f(x) = b + mx$ can be evaluated at any real number value of x, so its domain is the set of all real numbers. This set is represented in interval notation as $(-\infty, \infty)$.

The range of the linear function $f(x) = b + mx$ (if $m \neq 0$) is also the set of all real numbers, because the graph continues infinitely at both ends, as shown in figure (a). If $m = 0$, then $f(x) = b$, and the graph of f is a horizontal line. In this case, the range consists of a single number, b.



(a)



(b)

2.6.4 Restricting the Domain

In many applications, we may restrict the domain of a function to suit the situation at hand.

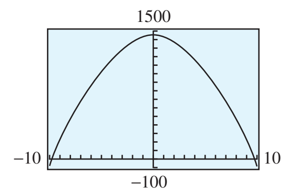
Example 2.6.9 The function $h = f(t) = 1454 - 16t^2$ gives the height of an algebra book dropped from the top of the Sears Tower as a function of time. Give a suitable domain for this application, and the corresponding range.

Solution.

You can use the window

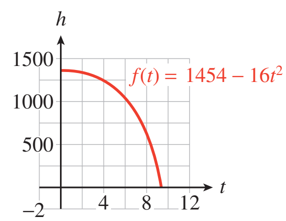
$$\text{Xmin} = -10 \quad \text{Xmax} = 10$$
$$\text{Ymin} = -100 \quad \text{Ymax} = 1500$$

to obtain the graph shown at right.



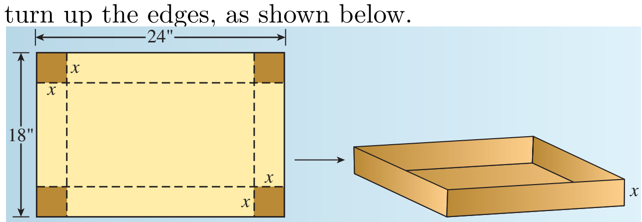
Because t represents the time in seconds after the book was dropped, only positive t-values make sense for the problem. The book stops falling when it hits the ground, at $h = 0$. You can verify that this happens at approximately $t = 9.5$ seconds. Thus, only t-values between 0 and 9.5 are realistic for this application, so we restrict the domain of the function f to the interval $[0, 9.5]$.

During that time period, the height, h, of the book decreases from 1454 feet to 0 feet. The range of the function on the domain $[0, 9.5]$ is $[0, 1454]$. The graph is shown at right.



□

Checkpoint 2.6.10 The children in Francine's art class are going to make cardboard boxes. Each child is given a sheet of cardboard that measures 18 inches by 24 inches. To make a box, the child will cut out a square from each corner and turn up the edges, as shown below.



a Write a formula $V = f(x)$ for the volume of the box in terms of x, the side of the cut-out square. (See the geometric formulas inside the front cover for the formula for the volume of a box.)

b What is the domain of the function? (What are the largest and smallest possible values of x?)

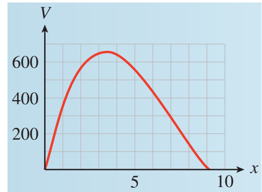
c Graph the function and estimate its range.

Answer.

a $V = f(x) = x(24 - 2x)(18 - 2x)$

b $(0, 9)$

c $(0, 655)$



2.6.5 Section Summary

2.6.5.1 Vocabulary

Look up the definitions of new terms in the Glossary.

- Domain
- Range
- Restricted domain

2.6.5.2 CONCEPTS

1 The **domain** of a function is the set of permissible values for the input variable.

2 The **range** is the set of function values (that is, values of the output variable) that correspond to the domain values.

3 A relationship between two variables is a **function** if each element of the domain is paired with only one element of the range.

4 We can identify the domain and range of a function from its graph. The domain is the set of x-values of all points on the graph, and the range is the set of y-values.

5 If the domain of a function is not given as part of its definition, we assume that the domain is as large as possible.

6 In applications, we may restrict the domain and range of a function to suit the situation at hand.

2.6.5.3 STUDY QUESTIONS

1 Explain how to find the domain and range of a function from its graph.

2 What is the domain of the function $f(x) = 4$? What is its range?

3 Which of the eight basic functions are increasing on their entire domain? Which are decreasing on their entire domain?

4 Which of the eight basic functions are concave up on their entire domain? Which are concave down on their entire domain?

5 Which of the eight basic functions can be evaluated at any real number? Which can take on any real number as a function value?

6 Which of the eight basic functions can be graphed in one piece, without lifting the pencil from the paper?

2.6.5.4 SKILLS

Practice each skill in the Homework 2.6.6, p. 257 problems listed.

1 Find the domain and range of a function from its graph: #1–16

2 Restrict the domain of a function to suit an application: #17–24

3 Find the domain of a function from its algebraic formula: #25–30

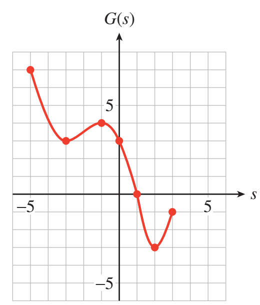
4 Find the corresponding domain value for a given range value: #31–38

5 Find the range of a function on a given domain: #39–50

2.6.6 Domain and Range (Homework 2.6)

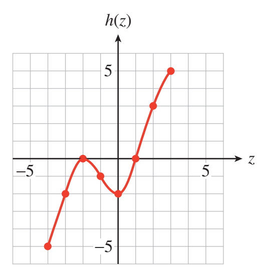
For Problems 1-8, find the domain and range of the function from its graph. Write answers in interval notation.

1.

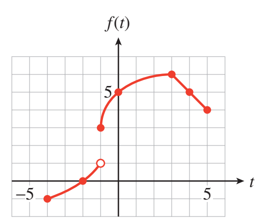


Answer. Domain: $[-5, 3]$; Range: $[-3, 7]$

2.

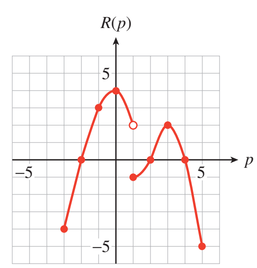


3.

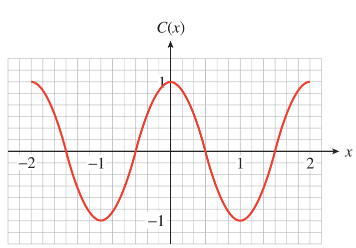


Answer. Domain: $[-4, 5]$; Range: $[-1, 1) \cup [3, 6]$

4.

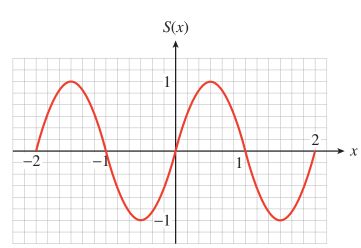


5.

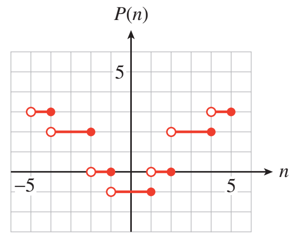


Answer. Domain: $[-2, 2]$; Range: $[-1, 1]$

6.

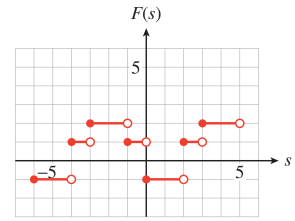


7.



Answer. Domain: $(-5, 5]$; Range: $\{-1, 0, 2, 3\}$

8.



For Problems 9–2, state the domain and range of the basic function.

9.

a $f(x) = x^3$

b $g(x) = x^2$

Answer.

a Domain: all real numbers; Range: all real numbers

b Domain: all real numbers; Range: $[0, \infty)$

10.

a $F(x) = |x|$

b $G(x) = x$

11.

a $H(x) = \dfrac{1}{x^2}$

b $M(x) = \dfrac{1}{x}$

Answer.

a Domain: all real numbers except zero; Range: $(0, \infty)$

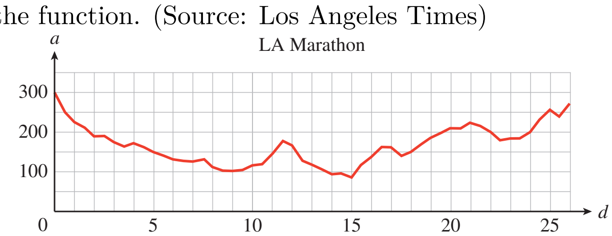
b Domain: all real numbers except zero; Range: all real numbers except zero

12.

a $p(x) = \sqrt[3]{x}$

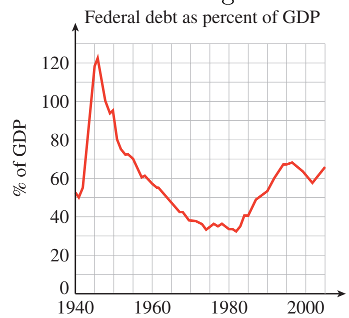
b $q(x) = \sqrt{x}$

13. The graph shows the elevation of the Los Angeles Marathon course as a function of the distance into the race, $a = f(d)$. Estimate the domain and range of the function. (Source: Los Angeles Times)

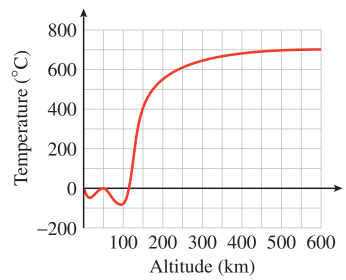


Answer. Domain: $[0, 26.2]$; Range: $[90, 300]$

14. The graph shows the federal debt as a percentage of the gross domestic product, as a function of time, $D = f(t)$. Estimate the domain and range of the function. (Source: Office of Management and Budget)

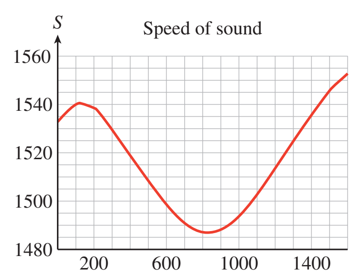


15. The graph shows the average air temperature as a function of altitude, $T = f(h)$. Estimate the domain and range of the function. (Source: Ahrens, 1998)



Answer. Domain: $[0, 600]$; Range: $[-90, 700]$

16. The graph shows the speed of sound in the ocean as a function of depth, $S = f(d)$. Estimate the domain and range of the function. (Source: Scientific American)



17. Clinton purchases \$6000 of photographic equipment to set up his studio. He estimates a salvage value of \$500 for the equipment in 10 years, and for tax purposes he uses straight-line depreciation.

a Write a formula for the value of the equipment, $V(t)$, after t years.

b State the domain and range of the function $V(t)$.

Answer.

a $V(t) = 6000 - 550t$

b Domain: $[0, 10]$; Range: $[500, 6000]$

18. Leslie plans to invest some money in two CD accounts. The first account pays 3.6% interest per year, and the second account pays 2.8% interest per year. Leslie would like to earn \$500 per year on her investment.

a Write a linear equation in general form that relates x, the amount

Leslie invests at 3.6%, and y, the amount she invests at 2.8%.

b Use your equation from part (a) to write y as a function of x, $y = f(x)$.

c Find the domain and range of f.

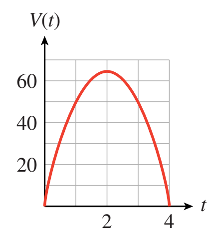
19. The height of a golfball, in feet, t seconds after being hit is given by the function $h = f(t) = -16(t-2)^2 + 64$.

a Graph the function.

b State the domain and range of the function and explain what they tell us about the golfball.

Answer.

a



b Domain: $[0, 4]$; Range: $[0, 64]$. The ball reaches a height of 64 feet and hits the ground 4 seconds after being hit.

20. Gameworld is marketing a new boardgame called Synaps. If Gameworld charges p dollars for the game, their revenue is given by the function $R = f(p) = -50(p-10)^2 + 5000$.

a Graph the function.

b State the domain and range of the function and explain what they tell us about the revenue.

21. In New York City, taxi cabs charge \$2.50 for distances up to $\dfrac{1}{3}$ mile, plus \$0.40 for each additional $\dfrac{1}{5}$ mile or portion thereof. (Source: www.visitnyc.com)

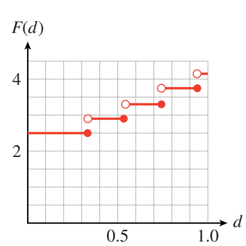
a Sketch a graph of $F(d)$, which gives taxi fare as a function of distance traveled, on the domain $0 < d < 1$.

b State the range of $F(d)$ on that domain.

c How much will it cost Renee to travel by taxi from Columbia University to Rockefeller Center, a distance of 5.7 miles?

Answer.

a



b Range: $\{2.50, 2.90, 3.30, 3.70, 4.10\}$

c \$13.30

22. If you order from Coldwater Creek, the shipping charges are given by the following table.

Purchase amount	Shipping charge
Up to \$25	\$5.95
\$25.01 to \$50	\$7.95
\$50.01 to \$75	\$9.95
\$75.01 to 4100	\$10.95

State the domain and range of $S(x)$, the shipping charge as a function of the purchase amount, x.

23. The Bopp-Busch Tool and Die Company markets its products to individuals, to contractors, and to wholesale distributors. The company offers three different price structures for its toggle bolts. If you order 20 or fewer boxes, the price is \$2.50 each. If you order more than 20 but no more than 50 boxes, the price is \$2.25 each. If you order more than 50 boxes, the price is \$2.10 each. State the domain and range of $C(x)$, the cost of ordering x boxes of toggle bolts.

Answer. Domain: nonnegative integers; The range includes all whole number multiples of 2.50 up to $20 \times 2.50 = 50$, all integer multiples of 2.25 from $21 \times 2.25 = 47.25$ to $50 \times 2.25 = 112.50$ and all integer multiples of 2.10 from $51 \times 2.10 = 107.10$ onwards: $0, 2.50, 5.00, 7.50, \ldots, 50,$ $47.25, 49.50, 51.75, \ldots, 112.50,\ 107.10, 109.20, 111.30, \ldots$

24. The Java Stop uses paper cups at a rate of 300 per day. At opening on Tuesday morning Java Stop has on hand 1200 paper cups. On Friday mornings Java Stop takes delivery of a week's worth of cups.

a Write a piecewise function for the number of cups Java Stop has on hand for one week, starting Tuesday morning.

b Graph the function.

c State the domain and range of the function.

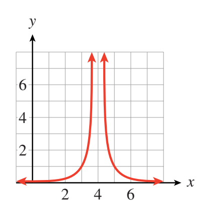
For Problems 25-30, find the domain of each function algebraically. Then graph the function, and use the graph to help you find the range.

25.

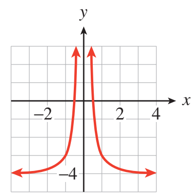
a $f(x) = \dfrac{1}{(x-4)^2}$ b $h(x) = \dfrac{1}{x^2} - 4$

Answer.

a $f(x)$ domain: $x \neq 4$; Range: $(0, \infty)$



b $h(x)$ domain: $x \neq 0$; Range: $(-4, \infty)$



26.

a $g(t) = \dfrac{1}{t} + 2$

b $F(t) = \dfrac{1}{t+2}$

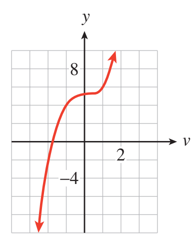
27.

a $G(v) = v^3 + 35$

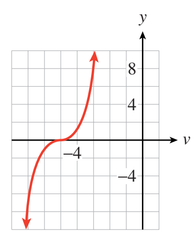
b $H(v) = (v+5)^3$

Answer.

a $G(v)$ domain: all real numbers; Range: all real numbers



b $H(v)$ domain: all real numbers; Range: all real numbers



28.

a $h(n) = 3 + (n-1)^2$

b $g(n) = 3 - (n+1)^2$

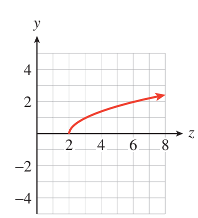
29.

a $T(z) = \sqrt{z-2}$

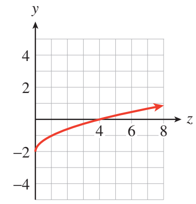
b $S(z) = \sqrt{z} - 2$

Answer.

a $G(v)$ domain: $[2, \infty)$; Range: $[0, \infty)$



b $H(v)$ domain: $[0, \infty)$; Range: $[-2, \infty)$



30.

a $Q(x) = 4 - |x|$

b $P(x) = |4 - x|$

For Problems 31-38, decide whether the given value is in the range of the function. If so, find the domain value(s) that produce each range value.

31. $f(x) = 6 - |2x + 4|$

a $f(x) = 8$

b $f(x) = -2$

Answer.

a Not in range

b $x = -6$ or $x = 2$

32. $g(x) = (x - 5)^3 + 1$

a $g(x) = 0$

b $g(x) = -7$

33. $h(t) = 4 + 2\sqrt[3]{t}$

a $h(t) = -4$

b $h(t) = 0$

Answer.

a $t = -64$

b $t = -8$

34. $F(t) = 12 + 0.5(t - 2)^2$

a $F(t) = 10$

b $F(t) = 20$

35. $G(w) = 3 + \dfrac{2}{w - 1}$

a $G(w) = -1$

b $G(w) = 3$

Answer.

a $w = \dfrac{1}{2}$

b Not in range

36. $H(n) = \dfrac{4}{(n + 2)^2} - 5$

a $H(n) = -6$

b $H(n) = -1$

37. $Q(h) = 2 + \sqrt{h + 5}$

a $Q(h) = 1$

b $Q(h) = 5$

Answer.

a Not in range

b $h = 4$

38. $P(q) = 8 - \sqrt{4 - q}$

a $P(q) = 4$

b $P(q) = 12$

For Problems 39–50,

a Use a graphing calculator to graph each function on the given domain.

Using the TRACE key, adjust **Ymin** and **Ymax** until you can estimate the range of the function.

b Verify your answer algebraically by evaluating the function. State the domain and range in interval notation.

39. $f(x) = x^2 - 4x; \quad -2 \le x \le 5$

Answer. Domain: $[-2, 5]$; Range: $[-4, 12]$

40. $g(x) = 6x - x^2; \quad -1 \le x \le 5$

41. $g(t) = -t^2 - 2t; \quad -5 \le t \le 3$

Answer. Domain: $[-5, 3]$; Range: $[-15, 1]$

42. $f(t) = -t^2 - 4t; \quad -6 \le t \le 2$

43. $h(x) = x^3 - 1; \quad -2 \le x \le 2$

Answer. Domain: $[-2, 2]$; Range: $[-9, 7]$

44. $q(x) = x^3 + 4; \quad -3 \le x \le 2$

45. $F(t) = \sqrt{8 - t}; \quad -1 \le t \le 8$

Answer. Domain: $[-1, 8]$; Range: $[0, 3]$

46. $G(t) = \sqrt{t + 6}; \quad -6 \le t \le 3$

47. $G(x) = \dfrac{1}{3 - x}; \quad -1.25 \le x \le 2.75$

Answer. Domain: $[-1.25, 2.75]$; Range: $\left[\dfrac{4}{17}, 4\right]$

48. $H(x) = \dfrac{1}{x - 1}; \quad -3.25 \le x \le -1.25$

49. $G(x) = \dfrac{1}{3 - x}; \quad 3 < x \le 6$

Answer. Domain: $(3, 6]$; Range: $\left[-\infty, \dfrac{-1}{3}\right]$

50. $H(x) = \dfrac{1}{x - 1}; \quad 1 < x \le 4$

51.

a Show that the graph of $y = \sqrt{16 - x^2}$ is a semicircle.

b State the domain and range of the function.

c Graph the function in the window

$$\begin{array}{ll} \text{Xmin} = -6 & \text{Xmax} = 6 \\ \text{Ymin} = 0 & \text{Ymax} = 8 \end{array}$$

In what way is the calculator's graph misleading?

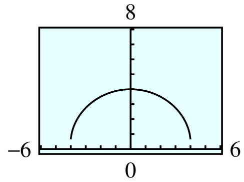
Hint. (Hint: Write the equation in the form $x^2 + y^2 = r^2$. See Algebra Skills Refresher Section A.11, p. 944 to review circles.)

Answer.

a Squaring both sides of the equation gives the equation of the circle centered on the origin with radius 4, but the points in the third and fourth quadrants are extraneous solutions introduced by squaring. (The original equation allowed only $y \ge 0$.)

b Domain: $[-4, 4]$; Range: $[0, 4]$

c



The calculator does not show the graph extending down to the x-axis.

52.

a For what values of x is the function $y = \dfrac{2x-8}{x-2}$ undefined?

b Graph the function in the standard window. In what way is the calculator's graph misleading?

c Graph the function in the window

Xmin $= -9.4$	Xmax $= 9.4$
Ymin $= -10$	Ymax $= 10$

State the domain and range of the function.

In Problems 53–60, find the domain and range of each transformation of the given function.

53. $f(x) = \dfrac{1}{x^2}$

a $y = f(x-2)$

b $y = f(x) - 2$

c $y = f(x-3) - 5$

Answer.

a Domain: $x \neq 2$; Range: $(0, \infty)$

b Domain: $x \neq 0$; Range: $(-2, \infty)$

c Domain: $x \neq 3$; Range: $(-5, \infty)$

54. $f(x) = \sqrt{x}$

a $y = -f(x)$

b $y = 4 + f(x)$

c $y = 4 - f(x)$

55. $f(x) = x^2$

a $y = -2f(x)$

b $y = 6 - 2f(x)$

c $y = 6 - 2f(x + 3)$

Answer.

a Domain: all real numbers; Range: $(-\infty, 0)$

b Domain: all real numbers; Range: $(-\infty, 6]$

c Domain: all real numbers; Range: $(-\infty, 6]$

56. $f(x) = \dfrac{1}{x}$

a $y = 3f(x)$

b $y = 3 + f(x - 1)$

c $y = 3 - f(x - 1)$

57. The domain of f is $[0, 10]$ and the range is $[-2, 2]$.

a $y = f(x - 3)$

b $y = 3f(x)$

c $y = 2f(x - 5)$

Answer.

a Domain: $[3, 13]$; Range: $[-2, 2]$

b Domain: $[0, 10]$; Range: $[-6, 6]$

c Domain: $[5, 15]$; Range: $[-4, 4]$

58. The domain of f is $[-4, 4]$ and the range is $[3, 10]$.

a $y = f(x) + 10$

b $y = f(x + 10)$

c $y = f(x - 1) + 4$

59. The domain of f is $(0, +\infty)$ and the range is $(0, 1)$.

a $y = 5f(x)$

b $y = 3f(x + 2)$

c $y = 2f(x - 3) + 2$

Answer.

a Domain: $(0, \infty)$; Range: $(0, 5)$

b Domain: $(-2, \infty)$; Range: $(0, 3)$

c Domain: $(3, \infty)$; Range: $(2, 4)$

60. The domain of f is $(-1, 1)$ and the range is $(-\infty, 0)$.

a $y = f(x + 1)$

b $y = 3 - f(x + 1)$

c $y = 4 + 2f(x - 1)$

In Problems 61–64, use a graphing calculator to explore some properties of the basic functions.

61.

a Graph $f(x) = x^2$ and $g(x) = x^3$ on the domain $[0, 1]$ and state the range of each function. On the interval $(0, 1)$, which is greater, $f(x)$ or $g(x)$?

b Graph $f(x) = x^2$ and $g(x) = x^3$ on the domain $[1, 10]$ and state the range of each function. On the interval $(1, 100)$, which is greater, $f(x)$ or $g(x)$?

Answer.

a $f(x)$ b $g(x)$

62.

a Graph $f(x) = \sqrt{x}$ and $g(x) = \sqrt[3]{x}$ on the domain $[0, 1]$ and state the range of each function. On the interval $(0, 1)$, which is greater, $f(x)$ or $g(x)$?

b Graph $f(x) = \sqrt{x}$ and $g(x) = \sqrt[3]{x}$ on the domain $[1, 100]$ and state the range of each function. On the interval $(1, 100)$, which is greater, $f(x)$ or $g(x)$?

63.

a Graph $f(x) = \dfrac{1}{x}$ and $g(x) = \dfrac{1}{x^2}$ on the domain $[0.01, 1]$ and state the range of each function. On the interval $(0, 1)$, which is greater, $f(x)$ or $g(x)$?

b Graph $f(x) = \dfrac{1}{x}$ and $g(x) = \dfrac{1}{x^2}$ on the domain $[1, 10]$ and state the range of each function. On the interval $(1, \infty)$, which is greater, $f(x)$ or $g(x)$?

Answer.

a $g(x)$ b $f(x)$

64.

a Graph $F(x) = \left|x^3\right|$ in the **ZDecimal** window. How does the graph compare to the graph of $y = x^3$?

b Graph $G(x) = \left|\dfrac{1}{x}\right|$ in the **ZDecimal** window. How does the graph compare to the graph of $y = \dfrac{1}{x}$?

65. The number of hours of daylight on the summer solstice is a function of latitude in the northern hemisphere. Give the domain and range of the function.

Answer. Domain: $[0°, 90°]$; Range: $[12, 24]$

66. A semicircular window has a radius of 2 feet. The area of a sector of the window (a pie-shaped wedge) is a function of the angle at the center of the circle. Give the domain and range of this function.

2.7 Chapter Summary and Review

2.7.1 Key Concepts

1 We can solve equations of the form $a(px+q)^2+r=0$ by extraction of roots.

2 The formula for compound interest is $A=P(1+r)^n$.

3 Simple nonlinear equations can be solved by undoing the operations on the variable.

4 .

The absolute value of x is defined by

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

5 .

The absolute value has the following properties:

$\lvert a+b\rvert \leq \lvert a\rvert + \lvert b\rvert$	Triangle inequality
$\lvert ab\rvert = \lvert a\rvert\,\lvert b\rvert$	Multiplicative property

6 Many situations can be modeled by one of eight basic functions:

$$\begin{array}{llll} y=x & y=|x| & y=x^2 & y=x^3 \\ y=\dfrac{1}{x} & y=\dfrac{1}{x^2} & y=\sqrt{x} & y=\sqrt[3]{x} \end{array}$$

7 Functions can be defined piecewise, with different formulas on different intervals.

8 **Transformations of Functions.**

- The graph of $y=f(x)+k$ is **shifted vertically** compared to the graph of $y=f(x)$.
- The graph of $y=f(x+h)$ is **shifted horizontally** compared to the graph of $y=f(x)$.
- The graph of $y=af(x)$ is **stretched or compressed vertically** compared to the graph of y = f (x).
- The graph of $y=-f(x)$ is **reflected about the x-axis** compared to the graph of $y=f(x)$.

9 A nonlinear graph may be **concave up** or **concave down**. If a graph is concave up, its slope is increasing. If it is concave down, its slope is decreasing.

10 The absolute value is used to model distance: The distance between two points x and a is given by $|x-a|$.

11 **Absolute Value Equations and Inequalities.**

- The equation $|ax+b| = c \ \ (c > 0)$ is equivalent to

$$ax + b = c \text{ or } ax + b = -c$$

- If the solutions of the equation $|ax+b| = c$ are r and s, with $r < s$, then the solutions of $|ax+b| < c$ are $r < x < s$.
- If the solutions of the equation $|ax+b| = c$ are r and s, with $r < s$, then the solutions of $|ax+b| > c$ are $x < r$ or $x > s$.

12 We can use absolute value notation to express error tolerances in measurements.

13 The **domain** of a function is the set of permissible values for the input variable. The **range** is the set of function values (that is, values of the output variable) that correspond to the domain values.

14 A relationship between two variables is a **function** if each element of the domain is paired with only one element of the range.

15 We can identify the domain and range of a function from its graph. The domain is the set of input values of all points on the graph, and the range is the set of output values.

16 If the domain of a function is not given as part of its definition, we assume that the domain is as large as possible. In many applications, however, we may restrict the domain and range of a function to suit the situation at hand.

2.7.2 Chapter 2 Review Problems

For Problems 1-4, solve by extraction of roots.

1. $(2x-5)^2 = 9$

Answer. $x = 1$ or $x = 4$

2. $(7x-1)^2 = 15$

3. $6\left(\dfrac{w-1}{3}\right)^2 - 4 = 2$

Answer. $w = -2$ or $w = 4$

4. $\left(\dfrac{2p}{5}\right)^2 = -3$

For problems 5-6, solve the formula for the specified variable.

5. $A = P(1+r)^2$, for r

Answer. $r = -1 \pm \sqrt{\dfrac{A}{P}}$

6. $V = \dfrac{4}{3}\pi r^3$, for r

7. Lewis invested \$2000 in an account that compounds interest annually. He made no deposits or withdrawals after that. Two years later, he closed the account, withdrawing \$2464.20. What interest rate did Lewis earn?

Answer. 11%

8. Earl borrowed \$5500 from his uncle for two years with interest compounded annually. At the end of two years, he owed his uncle \$6474.74. What was the interest rate on the loan?

For Problems 9-14, solve.

9. $\sqrt[3]{P-1} = 0.1$

Answer. $P = 1.001$

10. $\dfrac{1}{1-t} = \dfrac{2}{3}$

11. $\dfrac{3}{\sqrt{m+7}} = \dfrac{1}{2}$

Answer. $m = 29$

12. $15 = 3\sqrt{w+1}$

13. $4r^3 - 8 = 100$

Answer. $r = 3$

14. $5s^2 + 6 = 3s^2 + 31$

For Problems 15-16, use the Pythagorean theorem to write and solve an equation.

15. A widescreen television measures 96 cm by 54 cm. How long is the diagonal?

Answer. $\sqrt{12,132} \approx 110$ cm

16. A 15-foot ladder leans to the top of a 12-foot fence. How far is the foot of the ladder from the base of the fence?

For Problems 17-20, simplify.

17. $|-18| - |20|$

Answer. -2

18. $\left|-2 \cdot (3-18)\right|$

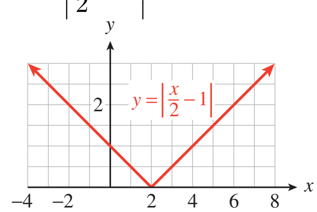
19. $|-2 \cdot 3 - 18|$

Answer. 24

20. $-2 \cdot |3-18|$

For Problems 21-24, use the graph to solve the equation or inequality.

21. Refer to the graph of $y = \left|\dfrac{x}{2} - 1\right|$



(a) Solve $\left|\dfrac{x}{2} - 1\right| = 2$

(b) Solve $\left|\dfrac{x}{2} - 1\right| < 2$

(c) Solve $\left|\dfrac{x}{2} - 1\right| \geq 2$

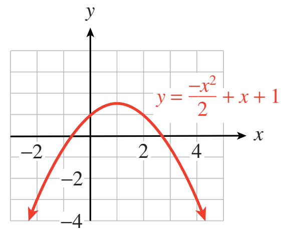
Answer.

a $x = -2$ or $x = 6$

b $(-2, 6)$

c $(-\infty, -2] \cup [6, +\infty)$

22. Refer to the graph of $y = \dfrac{-x^2}{2} + x + 1$

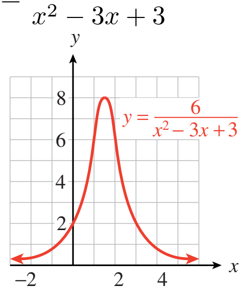


(a) Solve $\dfrac{-x^2}{2} + x + 1 = -3$

(b) Solve $\dfrac{-x^2}{2} + x + 1 \geq -3$

(c) Solve $\dfrac{-x^2}{2} + x + 1 \leq -3$

23. Refer to the graph of $y = \dfrac{6}{x^2 - 3x + 3}$



(a) Solve $2 = \dfrac{6}{x^2 - 3x + 3}$

(b) Solve $2 > \dfrac{6}{x^2 - 3x + 3}$

(c) Solve $2 < \dfrac{6}{x^2 - 3x + 3}$

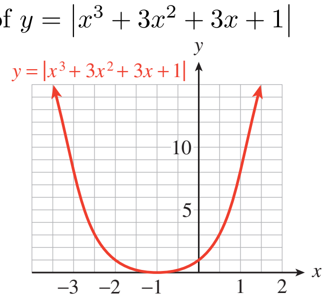
Answer.

a $x = 0$ or $x = 3$

b $(-\infty, 0) \cup (3, \infty)$

c $(0, 3)$

24. Refer to the graph of $y = \left|x^3 + 3x^2 + 3x + 1\right|$



(a) Solve $8 = \left|x^3 + 3x^2 + 3x + 1\right|$

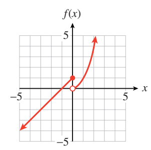
(b) Solve $8 > \left|x^3 + 3x^2 + 3x + 1\right|$

(c) Solve $8 < \left|x^3 + 3x^2 + 3x + 1\right|$

For Problems 25-30, graph the piecewise defined function.

25. $f(x) = \begin{cases} x+1 & \text{if } x \le 0 \\ x^2 & \text{if } x > 0 \end{cases}$

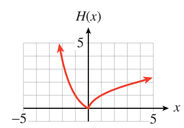
Answer.



26. $g(x) = \begin{cases} x-1 & \text{if } x \le 1 \\ x^3 & \text{if } x > 1 \end{cases}$

27. $H(x) = \begin{cases} x^2 & \text{if } x \le 0 \\ \sqrt{x} & \text{if } x > 0 \end{cases}$

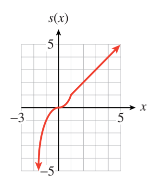
Answer.



28. $F(x) = \begin{cases} |x| & \text{if } x \le 0 \\ \dfrac{1}{x} & \text{if } x > 0 \end{cases}$

29. $S(x) = \begin{cases} x^3 & \text{if } x \le 1 \\ |x| & \text{if } x > 1 \end{cases}$

Answer.



30. $T(x) = \begin{cases} \dfrac{1}{x^2} & \text{if } x < 0 \\ \sqrt{x} & \text{if } x \ge 0 \end{cases}$

For Problems 31–38,

a Describe each function as transformation of a basic function.

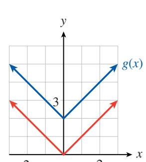
b Sketch a graph of the basic function and the given function on the same axes.

31. $g(x) = |x| + 2$

Answer.

a $y = |x|$ shifted up 2 units

b

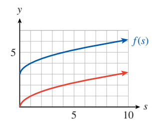


32. $F(t) = \dfrac{1}{t} - 2$

33. $f(s) = \sqrt{s} + 3$

Answer.

a $y = \sqrt{x}$ shifted up 3 units

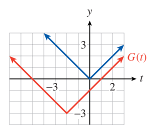
b 

34. $g(u) = \sqrt{u+2} - 3$

35. $G(t) = |t+2| - 3$

Answer.

a $y = |x|$ shifted left 2 units and down 3 units

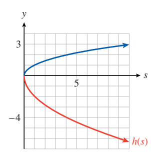
b 

36. $H(t) = \dfrac{1}{(t-2)^2} + 3$

37. $h(s) = -2\sqrt{s}$

Answer.

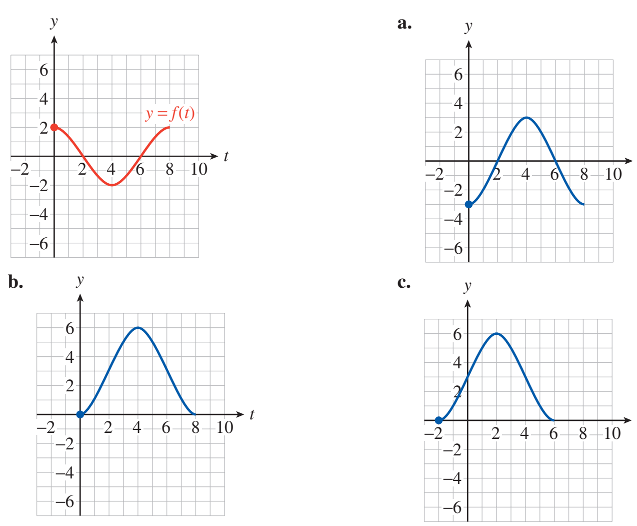
a $y = \sqrt{x}$ reflected across the horizontal axis and stretched vertically by a factor of 2

b 

38. $H(t) = \dfrac{1}{2}|s|$

In Problems 39–42, write a formula for each transformation of the given function.

39.



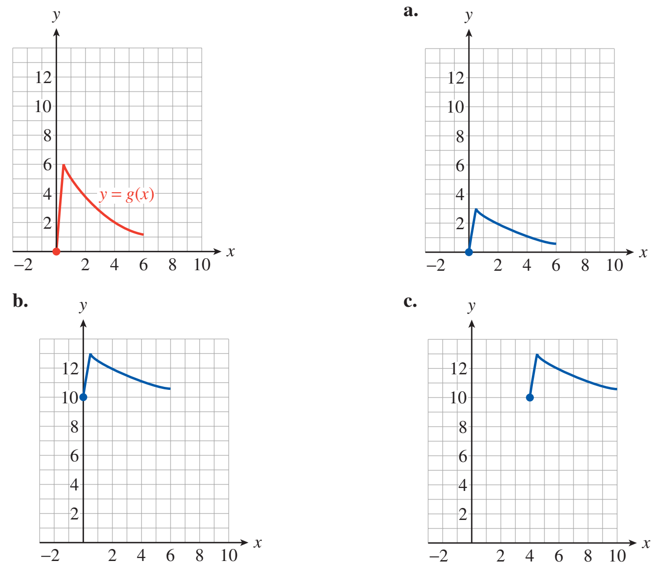
Answer.

a $y = \dfrac{-3}{2} f(t)$

b $y = \dfrac{-3}{2} f(t) + 3$

c $y = \dfrac{-3}{2} f(t+2) + 3$

40.



41.

t	0	1	2	3	4	5
$f(t)$	243	81	27	9	3	1

a

t	1	2	3	4	5	6
y	243	81	27	9	3	1

b

t	1	2	3	4	5	6
y	-243	-81	-27	-9	-3	-1

c

t	1	2	3	4	5	6
y	57	219	273	291	297	299

Answer.

a $y = f(t-1)$

b $y = -f(t-1)$

c $y = -f(t-1) + 300$

42.

x	1	2	3	4	5	6
$f(x)$	25	24	21	16	9	0

a

x	-1	0	1	2	3	4
y	25	24	21	16	9	0

b

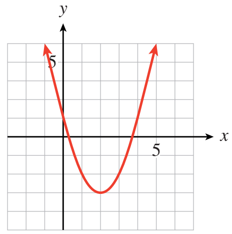
x	-1	0	1	2	3	4
y	50	48	42	32	18	0

c

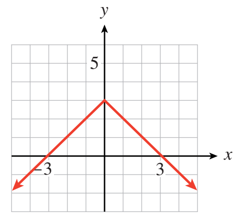
x	-1	0	1	2	3	4
y	70	68	62	52	38	20

For Problems 43-44, give an equation for the function graphed.

43.



44.

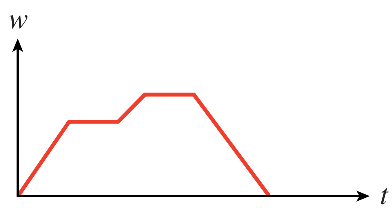


Answer. $y = (x - 2)^2 - 4$

Sketch graphs to illustrate the situations in Problems 45 and 46.

45. Inga runs hot water into the bathtub until it is about half full. Because the water is too hot, she lets it sit for a while before getting into the tub. After several minutes of bathing, she gets out and drains the tub. Graph the water level in the bathtub as a function of time, from the moment Inga starts filling the tub until it is drained.

Answer.



46. David turns on the oven and it heats up steadily until the proper baking temperature is reached. The oven maintains that temperature during the time David bakes a pot roast. When he turns the oven off, David leaves the oven door open for a few minutes, and the temperature drops fairly rapidly during that time. After David closes the door, the temperature continues to drop, but at a much slower rate. Graph the temperature of the oven as a function of time, from the moment David first turns on the oven until shortly after David closes the door when the oven is cooling.

For Problems 47-48, match each table with its graph.

47.

I

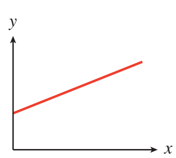
x	0	2	4	6	8
y	10	14	21	30	43

III

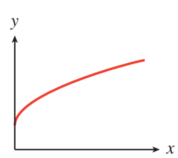
x	0	1	2	3	4
y	140	190	240	290	340

II

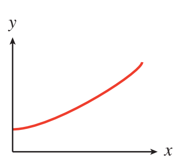
x	0	10	20	30	40
y	20	52	65	75	83



(a)



(b)



(c)

Answer. I (c), II (b), III (a)

48.

I

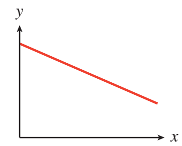
x	0	0.1	0.2	0.3	0.4
y	100	95	80	55	20

III

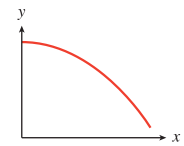
x	0	10	20	30	40
y	50	37	27	20	15

II

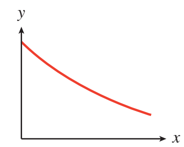
x	0	1	2	3	4
y	8.5	7.1	5.7	4.3	2.9



(a)



(b)

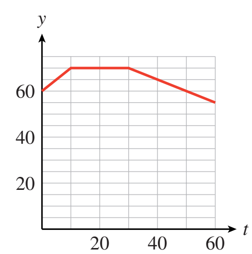


(c)

Write and graph a piecewise function for Problems 49 and 50.

49. The fluid level in a tank is a function of the number of days since the year began. The level was initially at 60 inches and rose an inch a day for 10 days, remained constant for the next 20 days, then dropped a half-inch each day for 30 days.

Answer. $g(t) = \begin{cases} 60+t, & 0 \le t < 10 \\ 70, & 10 \le t < 30 \\ 70 - \frac{1}{2}(t-30), & 30 \le t \le 60 \end{cases}$



50. The temperature at different locations in a large room is a function of distance from the window. Within 2 feet of the window, the temperature is 66° Fahrenheit, but the temperature rises by 0.5° for each of the next 10 feet, then maintains the temperature at 12 feet for the rest of the room.

For Problems 51-54, use absolute value notation to write the expression as an equation or inequality.

51. x is four units from the origin.

Answer. $|x| = 4$

52. The distance from y to -5 is three units.

53. p is within four units of 7.

Answer. $|p-7| < 4$

54. q is at least $\dfrac{3}{10}$ unit from -4.

For Problems 55-64, solve.

55. $|9-5t|=3$

Answer. $t=\dfrac{6}{5}$ or $t=\dfrac{12}{5}$

56. $1=|4q-7|$

57. $-29=|2w+3|$

Answer. No solutions

58. $\left|\dfrac{8n+3}{5}\right|=-11$

59. $1=\left|\dfrac{7-2p}{5}\right|$

Answer. $p=1$ or $p=6$

60. $\big|6(r-10)\big|=30$

61. $|3x-2|<4$

Answer. $\left(\dfrac{-2}{3},2\right)$

62. $|2x+0.3|\le 0.5$

63. $|3y+1.2|\ge 1.5$

Answer. $(-\infty,-0.9]\cup[0.1,\infty)$

64. $\left|3z+\dfrac{1}{2}\right|>\dfrac{1}{3}$

For Problems 65-66, express the error tolerance using absolute value.

65. The height, H, of a female trainee must be between 56 inches and 75 inches.

Answer. $|H-65.5|<9.5$

66. The time, t, in freefall must be at least 3.5 seconds but no more than 8.1 seconds.

For Problems 67-68, give an interval of possible values for the measurement.

67. The mass, M, of the sample must satisfy $|M-2.1|\le 0.05$.

Answer. $[2.05, 2.15]$

68. The temperature, T, of the refrigerator is specified by $|T-4.0|<0.5$.

In Problems 69 and 70,

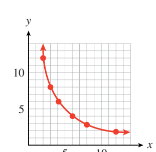
a Plot the points and sketch a smooth curve through them.

b Use your graph to help you discover the equation that describes the function.

69.

x	$g(x)$
2	12
3	8
4	6
6	4
8	3
12	2

Answer.

a 

b $g(x)=\dfrac{24}{x}$

70.

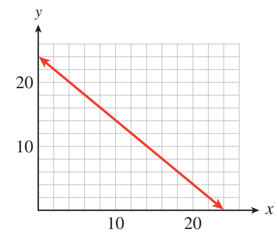
x	$F(x)$
-2	8
-1	1
0	0
1	-1
2	-8
3	-27

In Problems 71–76,

a Use the graph to complete the table of values.

b By finding a pattern in the table of values, write an equation for the graph.

71.



x	0	4	8		16	
y				10		2

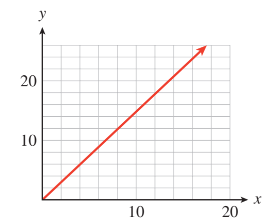
Answer.

a

x	0	4	8	14	16	22
y	24	20	16	10	8	2

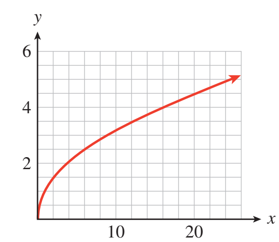
b $y = 24 - x$

72.



x	0	4	10		14	
y				18		24

73.



x	0		4		16	25
y		1		3		

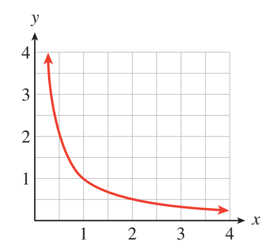
Answer.

a

x	0	1	4	9	16	25
y	0	1	2	3	4	5

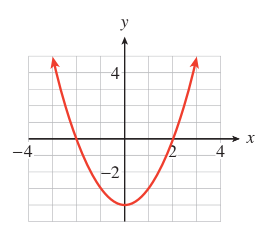
b $y = \sqrt{x}$

74.



x		0.5	1	1.5		4
y	4				0.5	

75.



x	-3	-2		0	1	2
y			-3			

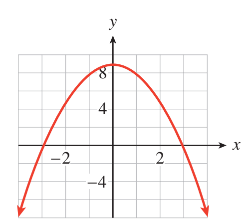
Answer.

a

x	-3	-2	-1	0	1	2
y	5	0	-3	-4	-3	0

b $y = x^2 - 4$

76.



x	-3	-2		0	1	
y			8			-7

For Problems 77-80, use a graphing calculator to graph the function on the given domain. Adjust **Ymin** and **Ymax** until you can determine the range of the function using the `TRACE` key. Then verify your answer algebraically by evaluating the function. State the domain and corresponding range in interval notation.

77. $f(t) = -t^2 + 3t; \quad -2 \le t \le 4$

Answer. Domain: $[-2, 4]$;
Range: $[-10, -4]$

78. $g(x) = \sqrt{s-2}; \quad 2 \le s \le 6$

79. $F(x) = \dfrac{1}{x+2}; \quad -2 < x \le 4$

Answer. Domain: $(-2, 4]$;
Range: $\left[\dfrac{1}{6}, \infty\right)$

80. $H(x) = \dfrac{1}{2-x}; \quad -4 \le x < 2$

2.8 Projects for Chapter 2: Periodic Functions

A **periodic function** is one whose values repeat at evenly spaced intervals, or **periods**, of the input variable. Periodic functions are used to model phenomena that exhibit cyclical behavior, such as growth patterns in plants and animals, radio waves, and planetary motion. In this project, we consider some applications of periodic functions.

Example 2.8.1 Which of the functions in Figure 2.8.2, p. 279 are periodic? If the function is periodic, give its period.

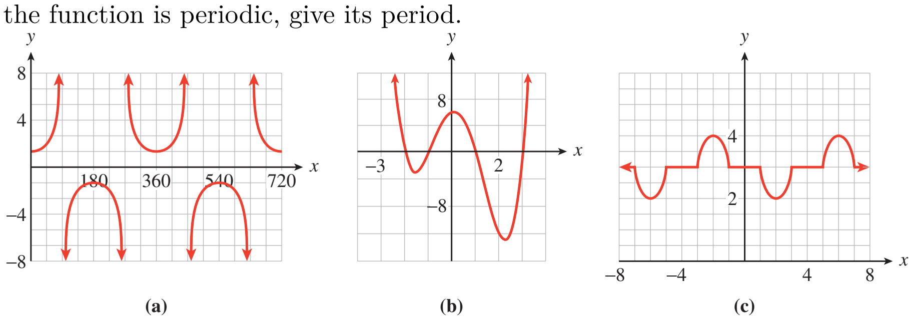


Figure 2.8.2

Solution.

a This graph is periodic with period 360.

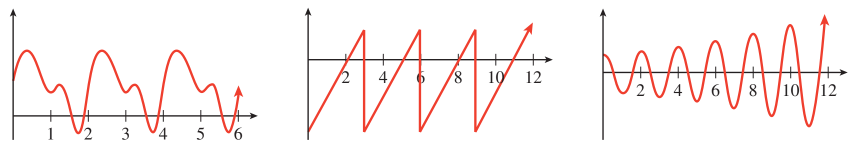
b This graph is not periodic.

c This graph is periodic with period 8.

□

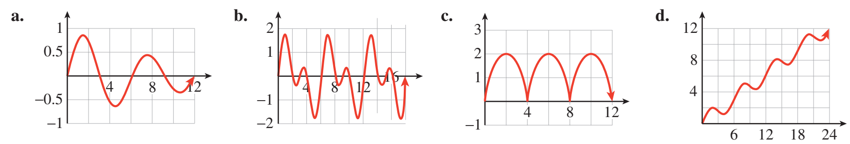
Project 11 Part I.

1 Which of the functions are periodic? If the function is periodic, give its period.

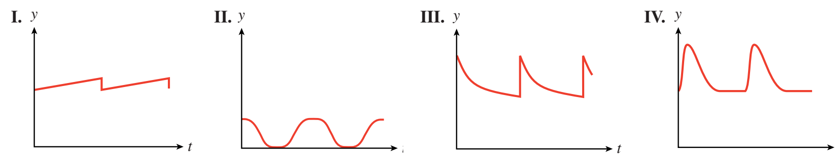


(a) **(b)** **(c)**

2 Which of the functions are periodic? If the function is periodic, give its period.



3 Match each of the following situations with the appropriate graph.



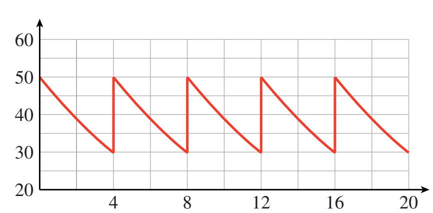
a When the heart contracts, blood pressure in the arteries rises rapidly to a peak (systolic blood pressure) and then falls off quickly to a minimum (diastolic blood pressure). Blood pressure is a periodic function of time.

b After an injection is given to a patient, the amount of the drug present in his bloodstream decreases exponentially. The patient receives injections at regular intervals to restore the drug level to the prescribed level. The amount of the drug present is a periodic function of time.

c The monorail shuttle train between the north and south terminals at Gatwick Airport departs from the south terminal every 12 minutes. The distance from the train to the south terminal is a periodic function of time.

d Delbert gets a haircut every two weeks. The length of his hair is a periodic function of time.

4 A patient receives regular doses of medication to maintain a certain level of the drug in his body. After each dose, the patient's body eliminates a certain percent of the medication before the next dose is administered. The graph shows the amount of the drug, in milliliters, in the patient's body as a function of time in hours.

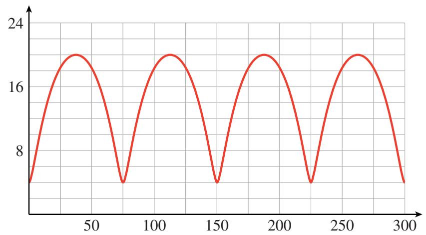


a How much of the medication is administered with each dose?

b How often is the medication administered?

c What percent of the drug is eliminated from the body between doses?

5 You are sitting on your front porch late one evening, and you see a light coming down the road tracing out the path shown below, with distances in inches. You realize that you are seeing a bicycle light, fixed to the front wheel of the bike.

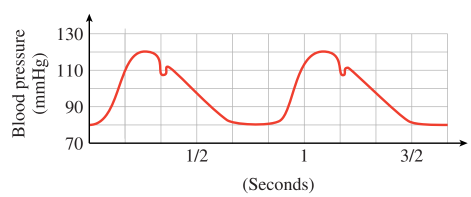


a Approximately what is the period of the graph?

b How far above the ground is the light?

c What is the diameter of the bicycle wheel?

6 The graph shows arterial blood pressure, measured in millimeters of mercury (mmHg), as a function of time.



a What are the maximum (systolic) and minimum (diastolic) pressures? The **pulse pressure** is the difference of systolic and diastolic pressures. What is the pulse pressure?

b The **mean arterial pressure** is the diastolic pressure plus one-third of the pulse pressure. Calculate the mean arterial pressure and draw a horizontal line on the graph at that pressure.

c The blood pressure graph repeats its cycle with each heartbeat. What is the heart rate, in beats per minute, of the person whose blood pressure is shown in the graph?

7 At a ski slope, the lift chairs take 5 minutes to travel from the bottom, at an elevation of 3000 feet, to the top, at elevation 4000 feet. The cable supporting the ski lift chairs is a loop turning on pulleys at a constant speed. At the top and bottom, the chairs are at a constant elevation for a few seconds to allow skiers to get on and off.

a Sketch a graph of $h(t)$, the height of one chair at time t. Show at least two complete up-and-down trips.

b What is the period of $h(t)$?

8 The heater in Paul's house doesn't have a thermostat; it runs on a timer. It uses 300 watts when it is running. Paul sets the heater to run from 6 a.m. to noon, and again from 4 p.m. to 10 p.m.

a Sketch a graph of $P(t)$, the power drawn by the heater as a function of time. Show at least two days of heater use.

b What is the period of $P(t)$?

9 Francine adds water to her fish pond once a week to keep the depth at 30 centimeters. During the week, the water evaporates at a constant rate of 0.5 centimeter per day.

a Sketch a graph of $D(t)$, the depth of the water, as a function of time. Show at least two weeks.

b What is the period of $D(t)$?

10 Erin's fox terrier, Casey, is very energetic and bounces excitedly at dinner time. Casey can jump 30 inches high,and each jump takes him 0.8 second.

a Sketch a graph of Casey's height, $h(t)$, as a function of time. Show at least two jumps.

b What is the period of $h(t)$?

Many periodic functions have a characteristic wave shape like the graph shown in Figure 2.8.3, p. 282. These graphs are called **sinusoidal**, after the trigonometric functions sine and cosine. They are often described by three parameters: the **period**, **midline**, and **amplitude**.

The period of the graph is the smallest interval of input values on which the graph repeats. The midline is the horizontal line at the average of the maximum and minimum values of the output variable. The amplitude is the vertical distance between the maximum output value and the midline.

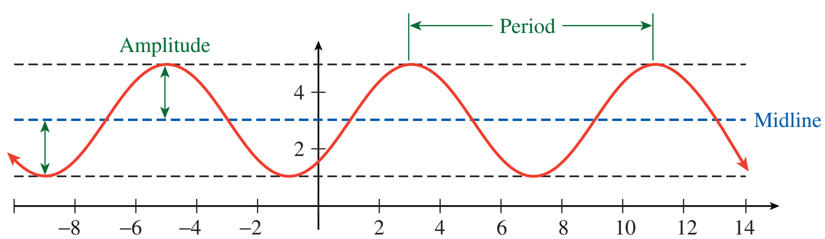


Figure 2.8.3

Example 2.8.4 The table shows the number of hours of daylight in Glasgow, Scotland, on the first of each month.

Month	Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep	Oct	Nov	Dec
Daylight hours	7.1	8.7	10.7	13.1	15.3	17.2	17.5	16.7	13.8	11.5	9.2	7.5

a Sketch a sinusoidal graph of daylight hours as a function of time, with $t = 1$ in January.

b Estimate the period, amplitude, and midline of the graph.

Solution.

a Plot the data points and fit a sinusoidal curve by eye, as shown in Figure 2.8.5, p. 283.

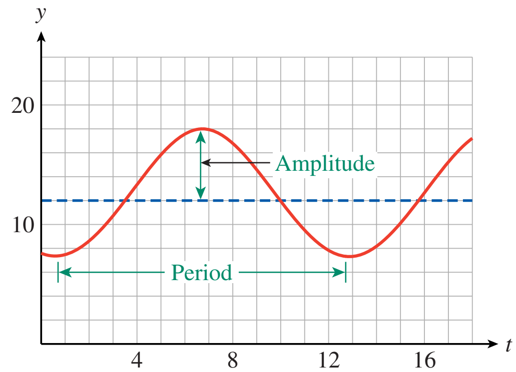


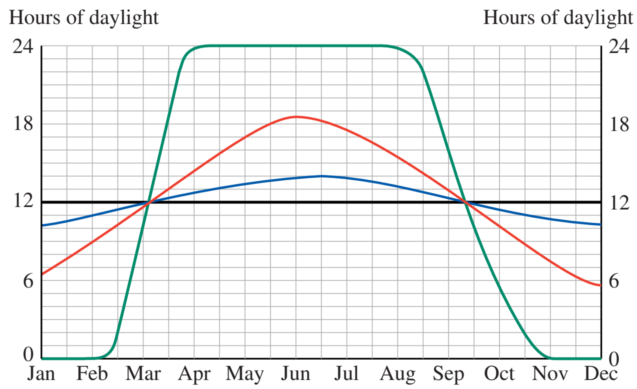
Figure 2.8.5

b The period of the graph is 12 months. The midline is approximately $y = 12.25$, and the amplitude is approximately 5.25.

□

Project 12 Part II.

1 The graph shows the number of daylight hours in Jacksonville, in Anchorage, at the Arctic Circle, and at the Equator.



a Which graph corresponds to each location?

b What are the maximum and minimum number of daylight hours in Jacksonville?

c For how long are there 24 hours of daylight per day at the Arctic Circle?

2 Match each of the following situations with the appropriate graph.

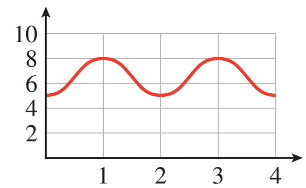
a The number of hours of daylight in Salt Lake City varies from a minimum of 9.6 hours on the winter solstice to a maximum of 14.4 hours on the summer solstice.

b A weight is 6.5 feet above the floor, suspended from the ceiling by a spring. The weight is pulled down to 5 feet above the floor and released, rising past 6.5 feet in 0.5 second before attaining its maximum height of 8 feet. Neglecting the effects of friction, the height of the weight will continue to oscillate between its minimum and maximum height.

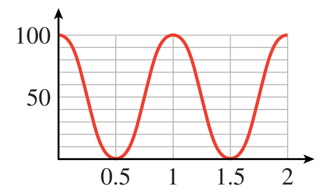
c The voltage used in U.S. electrical current changes from 155 V to -155 V and back 60 times each second.

d Although the moon is spherical, what we can see from Earth looks like a (sometimes only partly visible) disk. The percentage of the moon's disk that is visible varies between 0 (at new moon) to 100 (at full moon).

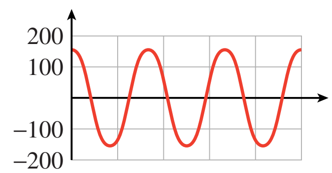
I.



II.



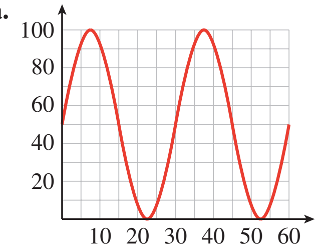
III.



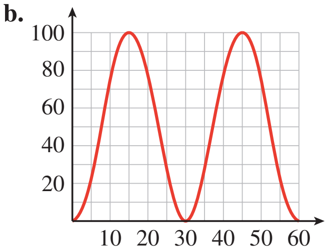
IV.

As the moon revolves around the Earth, the percent of the disk that we see varies sinusoidally with a period of approximately 30 days. There are eight phases, starting with the new moon, when the moon's disk is dark, followed by waxing crescent, first quarter, waxing gibbous, full moon (when the disk is 100% visible), waning gibbous, last quarter, and waning crescent. Which graph best rep-
3 resents the phases of the moon?

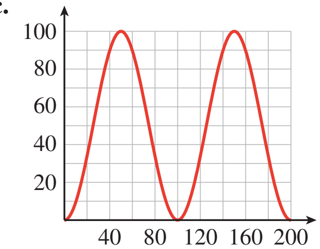
a.



b.



c.



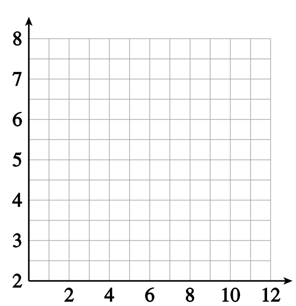
4 The table shows sunrise and sunset times in Los Angeles on the fifteenth of each month.

Month	Oct	Nov	Dec	Jan	Feb	Mar
Sunrise	5:58	6:26	6:51	6:59	6:39	6:04
Sunset	17:20	16:50	16:45	17:07	17:37	18:01

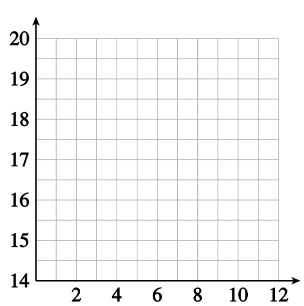
Month	Apr	May	Jun	Jul	Aug	Sep
Sunrise	5 : 22	4 : 52	4 : 42	4 : 43	5 : 15	5 : 37
Sunset	18 : 25	18 : 48	19 : 07	19 : 05	18 : 40	18 : 00

a Use the grid (a) to plot the sunrise times and sketch a sinusoidal graph through the points

b Use the grid (b) to plot the sunset times and sketch a sinusoidal graph through the points.



(a)



(b)

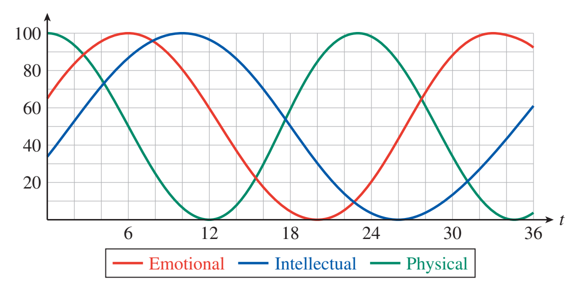
5 a Use the data from Problem 4 to complete the table with the hours of sunlight in Los Angeles on the fifteenth of each month.

Month	Oct	Nov	Dec	Jan	Feb	Mar
Hours of daylight						

Month	Apr	May	Jun	Jul	Aug	Sep
Hours of daylight						

b Plot the daylight hours and sketch a sinusoidal graph through the points.

6 Many people who believe in astrology also believe in biorhythms. The graph shows an individual's three biorhythms -- physical, emotional, and intellectual -- for 36 days, from $t = 0$ on September 30 to November 5.

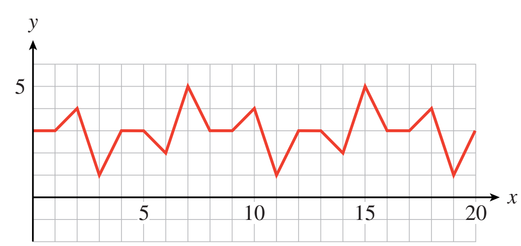


a Find the dates of highest and lowest activity for each biorhythm during the month of October.

b Find the period of each biorhythm in days

c On the day of your birth, all three biorhythms are at their maximum. How old will you be before all three are again at the maximum level?

7 a Is the function shown periodic? If so, what is its period? If not, explain why not.

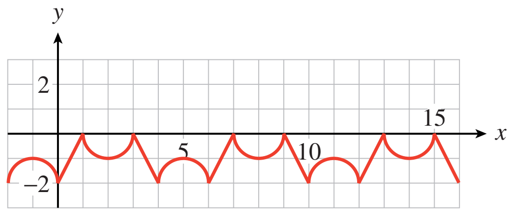


b Compute the difference between the maximum and minimum function values. Sketch in the midline of the graph.

c Find the smallest positive value of k for which $f(x) = f(x + k)$ for all x.

d Find the smallest positive values of a and b for which $f(b) - f(a)$ is a maximum.

8 a Find the period, the maximum and minimum values, and the midline of the graph of $y = f(x)$.

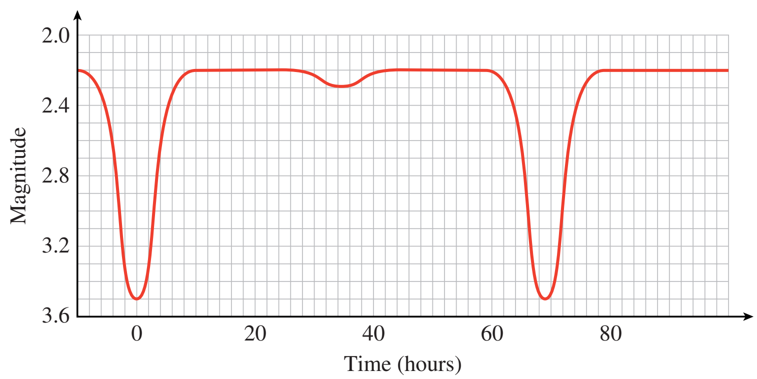


b Sketch a graph of $y = 2f(x)$.

c Sketch a graph of $y = 2 + f(x)$.

d Modify the graph of $f(x)$ so that the period is twice its current value.

9 The apparent magnitude of a star is a measure of its brightness as seen from Earth. Smaller values for the apparent magnitude correspond to brighter stars. The graph below, called a light curve, shows the apparent magnitude of the star Algol as a function of time. Algol is an eclipsing binary star, which means that it is actually a system of two stars, a bright principal star and its dimmer companion, in orbit around each other. As each star passes in front of the other, it eclipses some of the light that reaches Earth from the system. (Source: Gamow, 1965, Brandt & Maran, 1972)

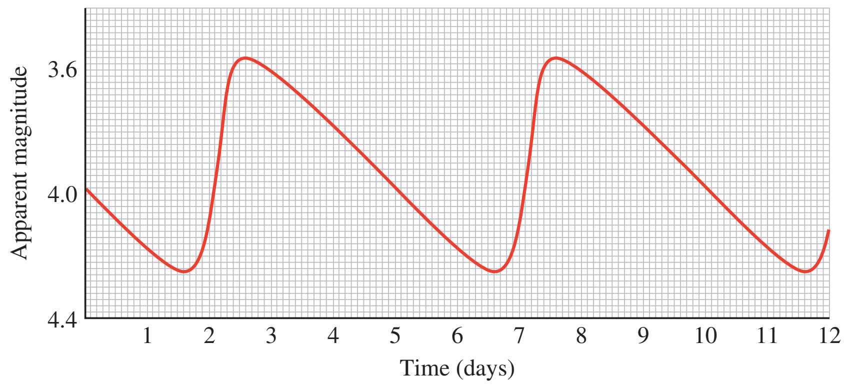


a The light curve is periodic. What is its period?

b What is the range of apparent magnitudes of the Algol system?

c Explain the large and small dips in the light curve. What is happening to cause the dips?

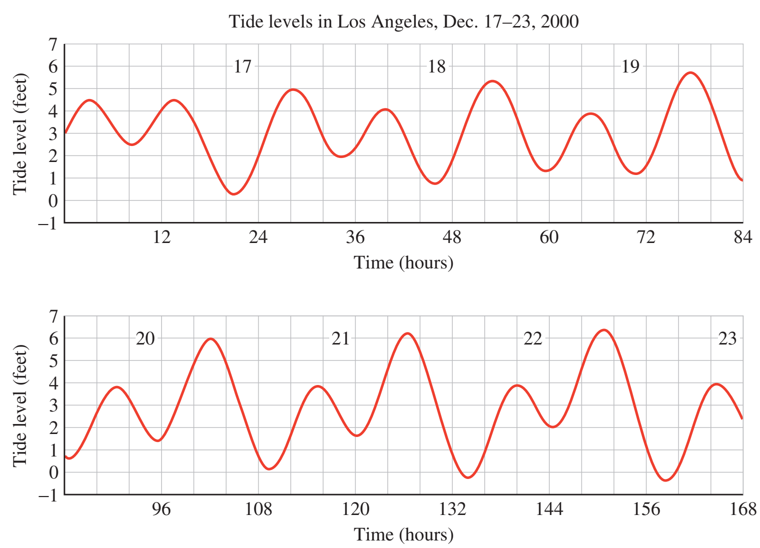
10 Some stars, called Cepheid variable stars, appear to pulse, getting brighter and dimmer periodically. The graph shows the light curve for the star Delta Cephei. (Source: Ingham, 1997)



a What is the period of the graph?

b What is the range of apparent magnitudes for Delta Cephei?

11 The figure is a tide chart for Los Angeles for the week of December 17–23, 2000. The horizontal axis shows time in hours, with $t = 12$ corresponding to noon on December 17. The vertical axis shows the height of the tide in feet above mean sea level.



a High tides occurred at 3:07 a.m. and 2:08 p.m. on December 17, and low tides at 8:41 a.m. and 9:02 p.m. Estimate the heights of the high and low tides on that day.

b Is tide height a periodic function of time? Use the information from part (a) to justify your answer.

c Make a table showing approximate times and heights for the high tides throughout the week. Make a similar table for the low tides

d Describe the trend in the heights of the high tides over the week. Describe the trend in the heights of the low tides.

e What is the largest height difference between consecutive high and low tides during the week shown? When does it occur?