Chapter 3

Power Functions

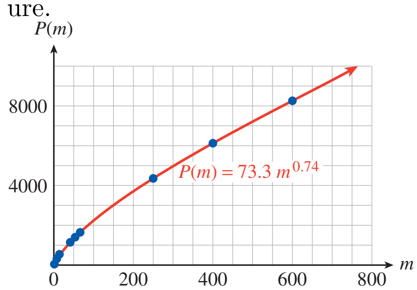
We next turn our attention to a large and useful family of functions called **power functions**. This family includes transformations of several of the basic functions, such as

$$F(d) = \frac{k}{d^2} \quad \text{and} \quad S(T) = 20.06\sqrt{T}$$

The first function gives the gravitational force, F, exerted by the sun on an object at a distance, d. The second function gives the speed of sound, S, in terms of the air temperature, T.

By extending our definition of exponent to include negative numbers and fractions, we will be able to express such functions in the form $f(x) = kx^n$. Here is an example of a power function with a fractional exponent.

In 1932, Max Kleiber published a remarkable equation for the metabolic rate of an animal as a function of its mass. The table at right shows the mass of various animals in kilograms and their metabolic rates, in kilocalories per day. A plot of the data, resulting in the famous ``mouse-to-elephant'' curve, is shown in the figure.



Animal	Mass (kg)	Metabolic rate (kcal/day)
Baboon	6.2	300
Cat	3.0	150
Chimpanzee	38	1110
Cow	400	6080
Dog	15.5	520
Elephant	3670	48, 800
Guinea pig	0.8	48
Human	65	1660
Mouse	0.02	3.4
Pig	250	4350
Polar bear	600	8340
Rabbit	3.5	165
Rat	0.2	28
Sheep	50	1300

Kleiber modeled his data by the power function

$$P(m) = 73.3m^{0.74}$$

where P is the metabolic rate and m is the mass of the animal. Kleiber's rule initiated the use of **allometric equations**, or power functions of mass, in physiology.

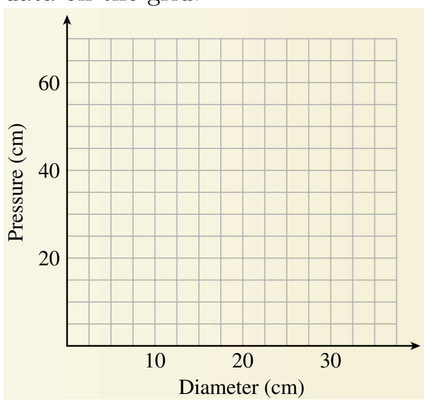
Investigation 13 Inflating a Balloon. If you blow air into a balloon, what do you think will happen to the air pressure inside the balloon as it expands? Here is what two physics books have to say:

> "The greater the pressure inside, the greater the balloon's volume."
>
> —Boleman, Jay
> *Physics, a Window on Our World*

> "Contrary to the process of blowing up a toy balloon, the pressure required to force air into a bubble decreases with bubble size."
>
> —Sears, Francis
> *Mechanics, Heat, and Sound*

1. Based on these two quotes and your own intuition, sketch a graph showing how pressure changes as a function of the diameter of the balloon. Describe your graph: Is it increasing or decreasing? Is it concave up (bending upward) or concave down (bending downward)?

2. In 1998, two high school students, April Leonardo and Tolu Noah, decided to see for themselves how the pressure inside a balloon changes as the balloon expands. Using a column of water to measure pressure, they collected the following data while blowing up a balloon. Graph their data on the grid.



Diameter (cm)	Pressure (cm H2O)
5.7	60.6
7.3	57.2
8.2	47.9
10.7	38.1
12.0	37.1
14.6	31.9
17.5	28.1
20.5	26.4
23.5	28
25.2	31.4
26.1	34.0
27.5	37.2
28.4	37.9
29.0	40.7
30.0	43.3
30.6	46.6
31.3	50.0
32.2	61.9

3. Describe the graph of April and Tolu's data. How does it compare to your graph in part (2)? Do their data confirm the predictions of the physics books? (We will return to April and Tolu's experiment in Section 3.4, p. 353.)

3.1 Variation

Two types of functions are widely used in modeling and are known by special names: **direct variation** and **inverse variation**.

3.1.1 Direct Variation

Two variables are **directly proportional** (or just **proportional**) if the ratios of their corresponding values are always equal. Consider the functions described in the tables below. The first table shows the price of gasoline as a function of the number of gallons purchased.

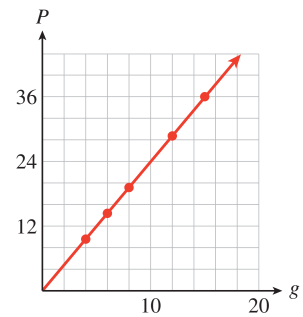
Gallons of gasoline	Total price	Price/Gallons
4	\$9.60	$\frac{9.60}{4} = 2.40$
6	\$14.40	$\frac{14.40}{6} = 2.40$
8	\$19.20	$\frac{19.20}{8} = 2.40$
12	\$28.80	$\frac{28.80}{12} = 2.40$
15	\$36.00	$\frac{36.00}{15} = 2.40$

Years	Population	People/Years
10	432	$\frac{432}{10} \approx 43$
20	932	$\frac{932}{20} \approx 47$
30	2013	$\frac{2013}{30} \approx 67$
40	4345	$\frac{4345}{40} \approx 109$
50	9380	$\frac{9380}{50} \approx 188$
60	$20,251$	$\frac{20,251}{60} \approx 338$

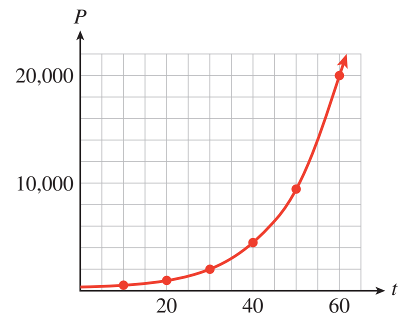
The ratio $\dfrac{\text{total price}}{\text{number of gallons}}$, or price per gallon, is the same for each pair of values in the first table. This agrees with everyday experience: The price per gallon of gasoline is the same no matter how many gallons you buy. Thus, the total price of a gasoline purchase is directly proportional to the number of gallons purchased.

The second table shows the population of a small town as a function of the town's age. The ratio $\dfrac{\text{number of people}}{\text{number of years}}$ gives the average rate of growth of the population in people per year. You can see that this ratio is not constant; in fact, it increases as time goes on. Thus, the population of the town is not proportional to its age.

The graphs of these two functions are shown below.



(a)



(b)

We see that the price, P, of a fill-up is a linear function of the number of gallons, g, purchased. This should not be surprising if we write an equation relating the variables g and P. Because the ratio of their values is constant, we can write

$$\frac{P}{g} = k$$

where k is a constant. In this example, the constant k is 2.40, the price of gasoline per gallon. Solving for P in terms of g, we have

$$P = kg = 2.40g$$

which we recognize as the equation of a line through the origin.

In general, we make the following definition.

Direct Variation.

y varies directly with x if

$$y = kx$$

where k is a positive constant called the **constant of variation**.

Note 3.1.1 From the preceding discussion, we see that *vary directly* means exactly the same thing as *are directly proportional.* The two phrases are interchangeable.

Example 3.1.2

a The circumference, C, of a circle varies directly with its radius, r, because

$$C = 2\pi r$$

The constant of variation is 2π, or about 6.28.

b The amount of interest, I, earned in one year on an account paying 7% simple interest, varies directly with the principal, P, invested, because

$$I = 0.07P$$

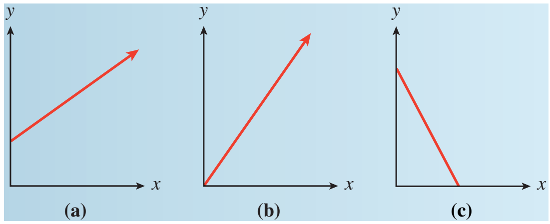
□

Direct variation defines a linear function of the form

$$y = f(x) = kx$$

The positive constant k in the equation $y = kx$ is just the slope of the graph, so it tells us how rapidly the graph increases. Compared to the standard form for a linear function, $y = b + mx$, the constant term, b, is zero, so the graph of a direct variation passes through the origin.

Checkpoint 3.1.3 Which of the graphs below could represent direct variation? Explain why.



Answer. (b): The graph is a straight line through the origin.

3.1.2 The Scaling Property of Direct Variation

The fact that the constant term is zero in a direct variation is significant: If we double the value of x, then the value of y will double also. In fact, increasing x by any factor causes y to increase by the same factor. For example, in the table of gasoline prices, doubling the number of gallons of gas purchased, say, from 4 gallons to 8 gallons or from 6 gallons to 12 gallons, causes the total price to double also.

Or, consider investing \$800 for one year at 7% simple interest, as in Example 3.1.2, p. 292b. The interest earned is

$$I = 0.07(800) = \$56$$

If we increase the investment by a factor of 1.6 to 1.6(800), or \$1280, the interest will be

$$I = 0.07(1280) = \$89.60$$

You can check that multiplying the original interest of \$56 by a factor of 1.6 does give the same figure for the new interest, \$89.60.

Example 3.1.4

a Tuition at Woodrow University is \$400 plus \$30 per unit. Is the tuition proportional to the number of units you take?

b Imogen makes a 15% commission on her sales of environmentally friendly products marketed by her co-op. Do her earnings vary directly with her sales?

Solution.

a Let u represent the number of units you take, and let $T(u)$ represent your tuition. Then

$$T(u) = 400 + 30u$$

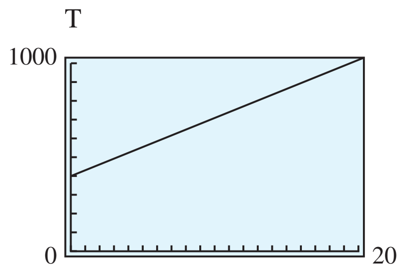
Thus, $T(u)$ is a linear function of u, but the T-intercept is 400, not 0. Your tuition is *not* proportional to the number of units you take, so this is not an example of direct variation. You can check that doubling the number of units does not double the tuition. For example,

$$T(6) = 400 + 30(6) = 580$$

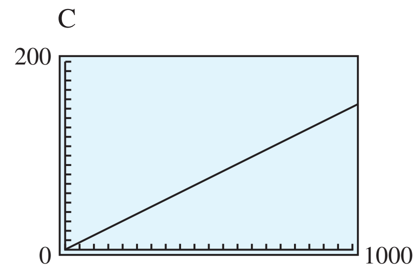
and

$$T(12) = 400 + 30(12) = 760$$

Tuition for 12 units is not double the tuition for 6 units. The graph of $T(u)$ in figure (a) does not pass through the origin.



(a)



(b)

b Let S represent Imogen's sales, and let $C(S)$ represent her commission. Then

$$C(S) = 0.15S$$

Thus, $C(S)$ is a linear function of S with a C-intercept of zero, so Imogen's earnings do vary directly with her sales. This is an example of direct variation. (See figure (b).)

□

Checkpoint 3.1.5 Which table could represent direct variation? Explain why. (Hint: What happens to$y$ if you multiply x by a constant?)

a

x	1	2	3	6	8	9
y	2.5	5	7.5	15	20	22.5

b

x	2	3	4	6	8	9
y	2	3.5	5	7	8.5	10

Answer. (a): If we multiply x by c, y is also multiplied by c.

3.1.3 Finding a Formula for Direct Variation

If we know any one pair of values for the variables in a direct variation, we can find the constant of variation. We can then use the constant to write a formula for one of the variables as a function of the other.

Example 3.1.6 If an object is dropped from a great height, its speed, v, varies directly with the time, t, the object has been falling. A rock dropped off the rim of the Grand Canyon is falling at a speed of 39.2 meters per second when it passes a lizard on a ledge 4 seconds later.

a Express v as a function of t.

b What is the speed of the rock after it has fallen for 6 seconds?

c Sketch a graph of $v(t)$ versus t.

Solution.

a Because v varies directly with t, there is a positive constant k for which $v = kt$. We substitute $v = 39.2$ when $t = 4$ and solve for k to find

$$\begin{aligned} 39.2 &= k(4) \quad \text{Divide both sides by 4.} \\ k &= 9.8 \end{aligned}$$

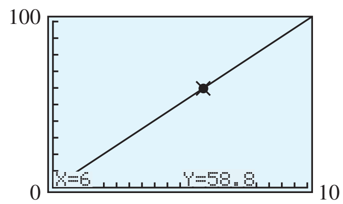
Thus, $v(t) = 9.8t$.

b We evaluate the function found in part (a) for $t = 6$.

$$v(\mathbf{6}) = 9.8(\mathbf{6}) = 58.8$$

At $t = 6$ seconds, the rock is falling at a speed of 58.8 meters per second.

c You can use your calculator to graph the function $v(t) = 9.8t$. The graph is shown below.



Checkpoint 3.1.7 The volume of a bag of rice, in cups, is directly proportional to the weight of the bag. A 2-pound bag contains 3.5 cups of rice.

a Express the volume, V, of a bag of rice as a function of its weight, w.

b How many cups of rice are in a 15-pound bag?

Answer.

a $V = 1.75w$

b 26.25

3.1.4 Direct Variation with a Power of x

We can generalize the notion of direct variation to include situations in which y is proportional to a power of x, instead of x itself.

Direct Variation with a Power.

y **varies directly** with a power of x if

$$y = kx^n$$

where k and n are positive constants.

Example 3.1.8 The surface area of a sphere varies directly with the *square* of its radius. A balloon of radius 5 centimeters has surface area 100π square centimeters, or about 314 square centimeters. Find a formula for the surface area of a sphere as a function of its radius.

Solution. If S stands for the surface area of a sphere of radius r, then

$$S = f(r) = kr^2$$

To find the constant of variation, k, we substitute the values of S and r.

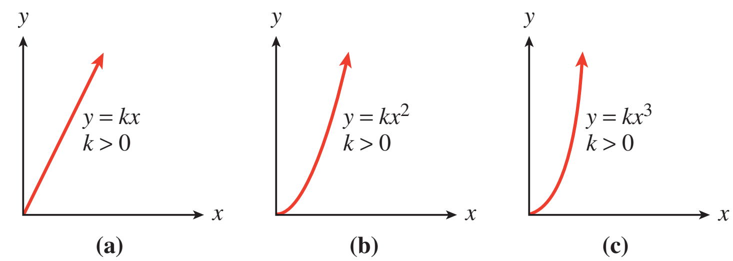
$$100\pi = k(5)^2$$
$$4\pi = k$$

Thus, $S = f(r) = 4\pi r^2$. □

Checkpoint 3.1.9 The volume of a sphere varies directly with the *cube* of its radius. A balloon of radius 5 centimeters has volume $\dfrac{500\pi}{3}$ cubic centimeters, or about 524 cubic centimeters. Find a formula for the volume of a sphere as a function of its radius.

Answer. $V = \dfrac{4}{3}\pi r^3$

In any example of direct variation, as the input variable increases through positive values, the output variable increases also. Thus, a direct variation is an increasing function, as we can see when we consider the graphs of some typical direct variations shown below.



Caution 3.1.10 The graph of a direct variation always passes through the origin, so when the input is zero, the output is also zero. Thus, the functions $y = 3x + 2$ and $y = 0.4x^2 - 2.3$, for example, are not direct variation, even though they are increasing functions for positive x.

Even without an equation, we can check whether a table of data describes direct variation or merely an increasing function. If y varies directly with x^n , then $y = kx^n$, or, equivalently, $\dfrac{y}{x^n} = k$.

Test for Direct Variation.

If the ratio $\dfrac{y}{x^n}$ is constant, then y varies directly with x^n.

Example 3.1.11 Delbert collects the following data and would like to know if y varies directly with the square of x. What should he calculate?

x	2	5	8	10	12
y	6	16.5	36	54	76

Solution. If y varies directly with x^2, then $y = kx^2$, or $\dfrac{y}{x^2} = k$. Delbert should calculate the ratio $\dfrac{y}{x^2}$ for each data point.

x	2	5	8	10	12
y	6	16.5	36	54	76
$\dfrac{y}{x^2}$	1.5	0.66	0.56	0.54	0.53

Because the ratio $\dfrac{y}{x^2}$ is not constant, y does not vary directly with x^2. □

Checkpoint 3.1.12 Does B vary directly with the cube of r? Explain your decision.

r	0.1	0.3	0.5	0.8	1.2
B	0.072	1.944	9.0	16.864	124.416

Answer. Yes, $\dfrac{B}{r^3}$ is constant.

3.1.5 Scaling

Recall that if y varies directly with x, then doubling x causes y to double also. But:

- Is the area of a 16-inch circular pizza double the area of an 8-inch pizza?
- If you double the dimensions of a model of a skyscraper, will its weight double also?

You probably know that the answer to both of these questions is *No.* The area of a circle is proportional to the *square* of its radius, and the volume (and hence the weight) of an object is proportional to the *cube* of its linear dimension. Variation with a power of x produces a different scaling effect.

Example 3.1.13 The Taipei 101 building is 1671 feet tall, and in 2006 it was the tallest skyscraper in the world. Show that doubling the dimensions of a model of the Taipei 101 building produces a model that weighs 8 times as much.

Solution. The Taipei 101 skyscraper is approximately box shaped, so its volume is given by the product of its linear dimensions, $V = lwh$. The weight of an object is proportional to its volume, so the weight, W, of the model is

$$W = klwh$$

where the constant k depends on the material of the model. If we double the length, width, and height of the model, then

$$\begin{aligned} W_{\text{new}} &= k(2l)(2w)(2h) \\ &= 2^3(klwh) = 8W_{\text{old}} \end{aligned}$$

The weight of the new model is $2^3 = 8$ times the weight of the original model. □

Checkpoint 3.1.14 Use the formula for the area of a circle to show that doubling the diameter of a pizza quadruples its area.

Answer. $A = \pi r^2$, so $A_{\text{new}} = \pi(2r)^2 = 4\pi r^2 = 4A_{\text{old}}$

In general, if y varies directly with a power of x, that is, if $y = kx^n$, then doubling the value of x causes y to increase by a factor of 2^n. In fact, if we multiply x by any positive number c, then

$$\begin{aligned} y_{\text{new}} &= k(cx)^n \\ &= c^n(kx^n) = c^n(y_{\text{old}}) \end{aligned}$$

so the value of y is multiplied by c^n.

We will call n the **scaling exponent**, and you will often see variation described in terms of scaling. For example, we might say that "the area of a circle scales as the square of its radius." (In many applications, the power n is called the *scale factor*, even though it is not a factor but an exponent.)

3.1.6 Inverse Variation

How long does it take to travel a distance of 600 miles? The answer depends on your average speed. If you are on a bicycle trip, your average speed might be 15 miles per hour. In that case, your traveling time will be

$$T = \frac{D}{R} = \frac{600}{15} = 40 \text{ hours}$$

(Of course, you will have to add time for rest stops; the 40 hours are just your travel time.)

If you are driving your car, you might average 50 miles per hour. Your travel time is then

$$T = \frac{D}{R} = \frac{600}{50} = 12 \text{ hours}$$

If you take a commercial air flight, the plane's speed might be 400 miles per hour, and the flight time would be

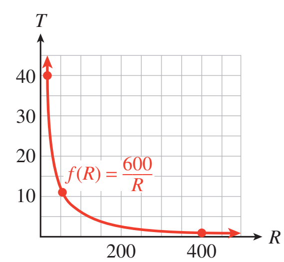
$$T = \frac{D}{R} = \frac{600}{400} = 1.5 \text{ hours}$$

You can see that for higher average speeds, the travel time is shorter. In other words, the time needed for a 600-mile journey is a decreasing function of average speed. In fact, a formula for the function is

$$T = f(R) = \frac{600}{R}$$

This function is an example of **inverse variation**. A table of values and a graph of the function are shown below.

R	T
10	60
15	40
20	30
50	12
200	3
400	1.5



Inverse Variation.

y **varies inversely** with x if

$$y = \frac{k}{x},\ x \neq 0$$

where k is a positive constant.

Caution 3.1.15 Inverse variation describes a decreasing function, but not every decreasing function represents inverse variation. People sometimes mistakenly use the phrase *varies inversely* to describe any decreasing function, but if y varies inversely with x, the variables must satisfy an equation of the form $y = \frac{k}{x}$, or $xy = k$.

To decide whether two variables truly vary inversely, we can check whether their product is constant. For instance, in the preceding travel-time example, we see from the table that $RT = 600$.

R	10	15	20	50	200	400
T	60	40	30	12	3	1.5
RT	600	600	600	600	600	600

We can also define inverse variation with a power of the variable.

Inverse Variation with a power.

y **varies inversely** with x^n if

$$y = \frac{k}{x^n},\ x \neq 0$$

where k and n are positive constants.

We may also say that y is **inversely proportional** to x^n.

Example 3.1.16 The weight, w, of an object varies inversely with the square of its distance, d, from the center of the Earth. Thus,

$$w = \frac{k}{d^2}$$

If you double your distance from the center of the Earth, what happens to your weight? What if you triple the distance?

Solution. Suppose you weigh W pounds at distance D from the center of the Earth. Then $W = \dfrac{k}{D^2}$. At distance $2D$, your weight will be

$$w = \frac{k}{(2D)^2} = \frac{k}{4D^2} = \frac{1}{4} \cdot \frac{k}{D^2} = \frac{1}{4}W$$

Your new weight will be $\dfrac{1}{4}$ of your old weight. By a similar calculation, you can check that by tripling the distance, your weight will be reduced to $\dfrac{1}{9}$ of its original value. ☐

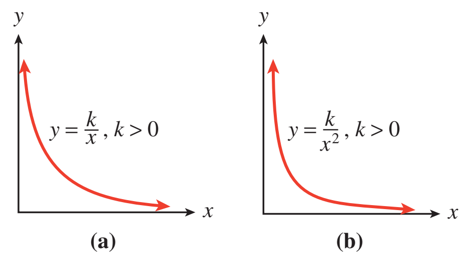
Checkpoint 3.1.17 The amount of force, F, (in pounds) needed to loosen a rusty bolt with a wrench is inversely proportional to the length, l, of the wrench. Thus,

$$F = \frac{k}{l}$$

If you increase the length of the wrench by 50% so that the new length is $1.5l$, what happens to the amount of force required to loosen the bolt?

Answer. $F_{\text{new}} = \dfrac{2}{3}F_{\text{old}}$

In Example 3.1.16, p. 299 and Checkpoint 3.1.17, p. 299, as the independent variable increases through positive values, the dependent variable decreases. An inverse variation is an example of a decreasing function. The graphs of some typical inverse variations are shown below.



3.1.7 Finding a Formula for Inverse Variation

If we know that two variables vary inversely and we can find one pair of corresponding values for the variables, we can determine k, the constant of variation.

Example 3.1.18 The intensity of electromagnetic radiation, such as light or radio waves, varies inversely with the square of the distance from its source. Radio station KPCC broadcasts a signal that is measured at 0.016 watt per square meter by a receiver 1 kilometer away.

a Write a formula that gives signal strength as a function of distance.

b If you live 5 kilometers from the station, what is the strength of the signal you will receive?

Solution.

a Let I stand for the intensity of the signal in watts per square meter, and d for the distance from the station in kilometers. Then $I = \dfrac{k}{d^2}$. To find the constant k, we substitute 0.016 for I and 1 for d. Solving for k gives us

$$0.016 = \frac{k}{1^2}$$
$$k = 0.016(1^2) = 0.016$$

Thus, $I = \dfrac{0.016}{d^2}$.

b Now we can substitute 5 for d and solve for I.

$$I = \frac{0.016}{5^2} = 0.00064$$

At a distance of 5 kilometers from the station, the signal strength is 0.00064 watt per square meter.

□

Checkpoint 3.1.19 Delbert's officemates want to buy a \$120 gold watch for a colleague who is retiring. The cost per person is inversely proportional to the number of people who contribute.

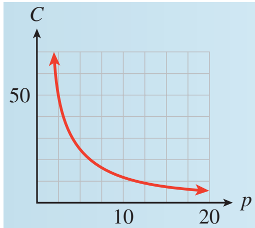
a Express the cost per person, C, as a function of the number of people, p, who contribute.

b Sketch the function on the domain $0 \leq p \leq 20$.

Answer.

a $C = \dfrac{120}{p}$

b



3.1.8 Section Summary

3.1.8.1 Vocabulary

Look up the definitions of new terms in the Glossary.

- Direct variation
- Directly proportional
- Constant of variation
- Scaling exponent
- Inverse variation
- Inversely proportional

3.1.8.2 CONCEPTS

1 **Direct and Inverse Variation.**

- y **varies directly with** x if the ratio $\dfrac{y}{x}$ is constant, that is, if $y = kx$.
- y **varies directly with a power of** x if the ratio $\dfrac{y}{x^n}$ is constant, that is, if $y = kx^n$.
- y **varies inversely with** x if the product xy is constant, that is, if $y = \dfrac{k}{x}$.
- y **varies inversely with a power of** x if the product $x^n y$ is constant, that is, if $y = \dfrac{k}{x^n}$.

2 The graph of a direct variation passes through the origin. The graph of an inverse variation has a vertical asymptote at the origin.

3 If $y = kx^n$, we say that y **scales** as x^n.

3.1.8.3 STUDY QUESTIONS

1 Describe the graph of $y = f(x)$ if y varies directly with x.

2 What is true about the ratio of two variables if they are directly proportional?

3 If y is inversely proportional to x, then the graph of y versus x is a transformation of which basic graph?

4 If y varies directly with a power of x, write a formula for y as a function of x.

5 If y varies inversely with a power of x, write a formula for y as a function of x.

6 If $y = kx^4$, what happens to y if you double x?

7 State a test to determine whether y varies inversely with x^n.

8 If $y = \dfrac{k}{x^2}$, and we double the value of x, what happens to the value of y?

3.1.8.4 SKILLS

Practice each skill in the Homework 3.1.9, p. 302 problems listed.

1 Find the constant of variation: #1–4, 13–26

2 Write a formula for direct or inverse variation: #1–4, 13–26, 35–46

3 Recognize direct and inverse variation from a table of values: #27–34, 39–42

4 Recognize direct or inverse variation from a graph: #9–12, 35–38

5 Use scaling in direct and inverse variation: #13–20, 43–46

3.1.9 Variation (Homework 3.1)

1. Delbert's credit card statement lists three purchases he made while on a business trip in the Midwest. His company's accountant would like to know the sales tax rate on the purchases.

Price of item	18	28	12
Tax	1.17	1.82	0.78
Tax/Price			

a Compute the ratio of the tax to the price of each item. Is the tax proportional to the price? What is the tax rate?

b Express the tax, T, as a function of the price, p, of the item.

c Sketch a graph of the function by hand, and label the scales on the axes.

Answer.

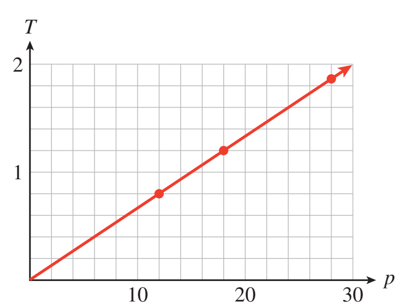
a

Price of item	18	28	12
Tax	1.17	1.82	0.78
Tax/Price	**0.065**	**0.065**	**0.065**

Yes; 6.5%

b $T = 0.065p$

c



2. At constant acceleration from rest, the distance traveled by a race car is proportional to the square of the time elapsed. The highest recorded road-tested acceleration is 0 to 60 miles per hour in 3.07 seconds, which produces the following data.

Time (seconds)	2	2.5	3
Distance (feet)	57.32	89.563	128.97
Distance/Time^2			

a Compute the ratios of the distance traveled to the square of the time elapsed. What was the acceleration, in feet per second squared?

b Express the distance traveled, d, as a function of time in seconds, t.

c Sketch a graph of the function by hand, and label the scales on the axes.

3. The marketing department for a paper company is testing wrapping paper rolls in various dimensions to see which shape consumers prefer. All the rolls contain the same amount of wrapping paper.

Width (feet)	2	2.5	3
Length (feet)	12	9.6	8
Length $\times$ width			

a Compute the product of the length and width for each roll of wrapping paper. What is the constant of inverse proportionality?

b Express the length, L, of the paper as a function of the width, w, of the roll.

c Sketch a graph of the function by hand, and label the scales on the axes.

Answer.

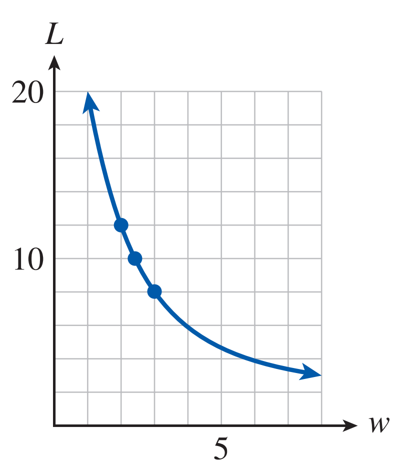
a

Width (feet)	2	2.5	3
Length (feet)	12	9.6	8
Length $\times$ width	**24**	**24**	**24**

24 square feet

b $L = \dfrac{24}{w}$

c



4. The force of gravity on a 1-kilogram mass is inversely proportional to the square of the object's distance from the center of the Earth. The table shows the force on the object, in newtons, at distances that are multiples of the Earth's radius.

Distance (Earth radii)	1	2	4
Force (newtons)	9.8	2.45	0.6125
Force $\times$ distance^2			

a Compute the products of the force and the square of the distance. What is the constant of inverse proportionality

b Express the gravitational force, F, on a 1-kilogram mass as a function of its distance, r, from the Earth's center, measured in Earth radii

c Sketch a graph of the function by hand, and label the scales on the axes.

5.

a How can you tell from a table of values whether y varies directly with x?

b How can you tell from a table of values whether y varies inversely with x?

Answer.

a The ratio $\frac{y}{x}$ is a constant.

b The product xy is a constant.

6.

a How can you tell from a table of values whether y varies directly with a power of x?

b How can you tell from a table of values whether y varies inversely with a power of x?

7. The length of a rectangle is 10 inches, and its width is 8 inches. Suppose we increase the length of the rectangle while holding the width constant.

a Fill in the table.

Length	Width	Perimeter	Area
10	8		
12	8		
15	8		
20	8		

b Does the perimeter vary directly with the length?

c Write a formula for the perimeter of the rectangle in terms of its length.

d Does the area vary directly with the length?

e Write a formula for the area of the rectangle in terms of its length.

Answer.

a

Length	Width	Perimeter	Area
10	8	**36**	**80**
12	8	**40**	**96**
15	8	**46**	**120**
20	8	**56**	**160**

b No

c $P = 16 + 2l$

d Yes

e $A = 8l$

8. The base of an isosceles triangle is 12 centimeters, and the equal sides have length 15 centimeters. Suppose we increase the base of the triangle while holding the sides constant.

a Fill in the table.

Base	Sides	Height	Perimeter	Area
12	15			
15	15			
18	15			
20	15			

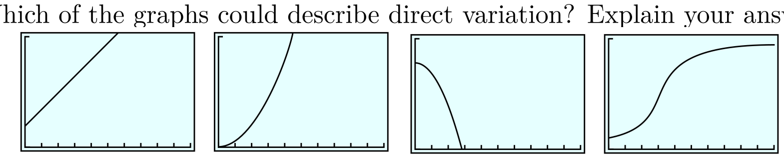
b Does the perimeter vary directly with the base?

c Write a formula for the perimeter of the triangle in terms of its base.

d Write a formula for the area of the triangle in terms of its base.

e Does the area vary directly with the base?

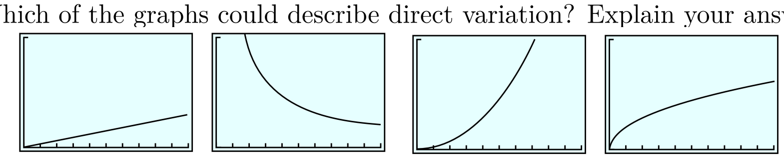
9. Which of the graphs could describe direct variation? Explain your answer.



(a) (b) (c) (d)

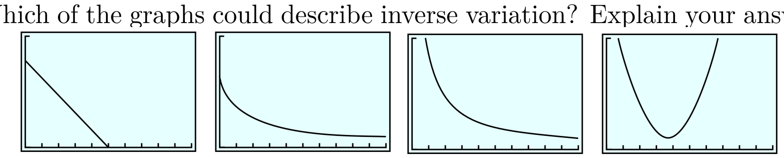
Answer. (b)

10. Which of the graphs could describe direct variation? Explain your answer.



(a) (b) (c) (d)

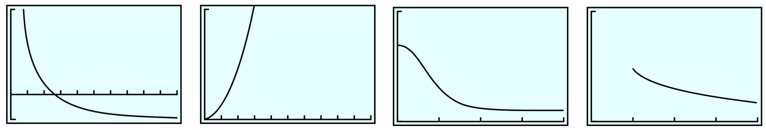
11. Which of the graphs could describe inverse variation? Explain your answer.



(a) (b) (c) (d)

Answer. (c)

12. Which of the graphs could describe inverse variation? Explain your answer.



(a) (b) (c) (d)

13. The weight of an object on the Moon varies directly with its weight on Earth. A person who weighs 150 pounds on Earth would weigh only 24.75 pounds on the Moon.

a Find a function that gives the weight m of an object on the Moon in terms of its weight w on Earth. Complete the table and graph your function in a suitable window.

w	50	100	200	400
m				

b How much would a person weigh on the Moon if she weighs 120 pounds on Earth?

c A piece of rock weighs 50 pounds on the Moon. How much will it weigh back on Earth?

d If you double the weight of an object on Earth, what will happen to its weight on the Moon?

Answer.

a $m = 0.165w$

w	50	100	200	400
m	**8.25**	**16.5**	**33**	**66**

b 19.8 lb

c 303.03 lb

d It will double.

14. Hubble's law says that distant galaxies are receding from us at a rate that varies directly with their distance. (The speeds of the galaxies are measured using a phenomenon called redshifting.) A galaxy in the constellation Ursa Major is 980 million light-years away and is receding at a speed of $15,000$ kilometers per second.

a Find a function that gives the speed, v, of a galaxy in terms of its distance, d, from Earth. Complete the table and graph your function in a suitable window. (Distances are given in millions of light-years.)

d	500	1000	2000	4000
m				

b How far away is a galaxy in the constellation Hydra that is receding at $61,000$ kilometers per second?

c A galaxy in Leo is 1240 million light-years away. How fast is it receding from us?

d If one constellation is twice as distant as another, how do their speeds compare?

15. The length, L, of a pendulum varies directly with the square of its period, T, the time required for the pendulum to make one complete swing back and forth. The pendulum on a grandfather clock is 3.25 feet long and has a period of 2 seconds.

a Express L as a function of T. Complete the table and graph your function in a suitable window.

T	1	5	10	20
L				

b How long is the Foucault pendulum in the Pantheon in Paris, which has a period of 17 seconds?

c A hypnotist uses a gold pendant as a pendulum to mesmerize his clients. If the chain on the pendant is 9 inches long, what is the period of its swing?

d In order to double the period of a pendulum, how must you vary its length?

Answer.

a $L = 0.8125T^2$

T	1	5	10	20
L	**0.8125**	**20.3**	**81.25**	**325**

b 234.8125 ft

c 0.96 sec

d It must be four times as long.

16. The load, L, that a beam can support varies directly with the square of its vertical thickness, h. A beam that is 4 inches thick can support a load of 2000 pounds.

a Express L as a function of h. Complete the table and graph your function in a suitable window.

h	1	2	4	8
L				

b What size load can be supported by a beam that is 6 inches thick?

c How thick a beam is needed to support a load of 100 pounds?

d If you double the thickness of a beam, how will the load it can support change?

17. Computer monitors produce a magnetic field. The effect of the field, B, on the user varies inversely with his or her distance, d, from the screen. The field from a certain color monitor was measured at 22 milligauss 4 inches from the screen.

a Express the field strength as a function of distance from the screen. Complete the table and graph your function in a suitable window.

d	1	2	12	24
B				

b What is the field strength 10 inches from the screen?

c An elevated risk of cancer can result from exposure to field strengths of 4.3 milligauss. How far from the screen should the computer user sit to keep the field level below 4.3 milligauss?

d If you double your distance from the screen, how does the field strength change?

Answer.

a $B = \dfrac{88}{d}$

d	1	2	12	24
B	**88**	**44**	**7.3**	**3.7**

b 8.8 milligauss

c More than 20.47 in

d It is one half as strong.

18. The amount of current, I, that flows through a circuit varies inversely with the resistance, R, on the circuit. An iron with a resistance of 12 ohms draws 10 amps of current.

a Express the current as a function of the resistance. Complete the table and graph your function in a suitable window.

R	1	2	10	20
I				

b How much current is drawn by a light bulb with a resistance of 533.3 ohms?

c What is the resistance of a toaster that draws 12.5 amps of current?

d If the resistance of one appliance is double the resistance of a second appliance, how does the current they draw compare?

19. The amount of power, P, generated by a windmill varies directly with the cube of the wind speed, w. A windmill on Oahu, Hawaii, produces 7300 kilowatts of power when the wind speed is 32 miles per hour.

a Express the power as a function of wind speed. Complete the table and graph your function in a suitable window.

w	10	20	40	80
P				

b How much power would the windmill produce in a light breeze of 15 miles per hour?

c What wind speed is needed to produce $10,000$ kilowatts of power?

d If the wind speed doubles, what happens to the amount of power generated?

Answer.

a $P = \dfrac{1825}{8192}w^3 \approx 0.2228w^3$

w	10	20	40	80
P	**223**	**1782**	**14,259**	**114,074**

b 752 kilowatts

c 33.54 mph

d It is multiplied by 8.

20. A crystal form of pyrite (a compound of iron and sulfur) has the shape of a regular solid with 12 faces. Each face is a regular pentagon. This compound is called pyritohedron, and its mass, M, varies directly with the cube of the length, L, of one edge. If each edge is 1.1 centimeters, then the mass is 51 grams.

a Express the mass of pyritohedron as a function of the length of one edge. Complete the table and graph your function in a suitable window.

L	0.5	1	2	4
M				

b What is the weight of a chunk of pyritohedron if each edge is 2.2 centimeters?

c How long would each edge be for a 1643-gram piece of pyritohedron?

d If one chunk has double the length of a second chunk, how do their weights compare?

For Problems 21–26,

a Use the values in the table to find the constant of variation, k, and write y as a function of x.

b Fill in the rest of the table with the correct values.

c What happens to y when you double the value of x?

21. y varies directly with x.

x	y
2	
5	1.5
	2.4
12	
	4.5

Answer.

a $y = 0.3x$

b

x	y
2	**0.6**
5	1.5
8	2.4
12	**3.6**
15	4.5

c y doubles.

22. y varies directly with x.

x	y
0.8	
1.5	54
	108
	126
6	

23. y varies directly with the square of x.

x	y
3	
6	24
	54
12	
	150

Answer.

a $y = \dfrac{2}{3}x^2$

b

x	y
3	**6**
6	24
9	
12	**96**
15	150

c y is quadrupled.

24. y varies directly with the cube of x.

x	y
2	120
3	
	1875
6	
	$15,000$

25. y varies inversely with x.

x	y
4	
	15
20	6
30	
	3

Answer.

a $y = \dfrac{120}{x}$

b

4	**30**
8	15
20	6
30	**4**
40	3

c y is halved.

26. y varies inversely with the square of x.

x	y
0.2	
	80
2	
4	1.25
	0.8

For Problems 27-30, decide whether

a y varies directly with x,

b y varies directly with x^2, or

c y does not vary directly with a power of x.

Explain why your choice is correct. If your choice is (a) or (b), find the constant of variation.

27.

x	y
2	2.0
3	4.5
5	12.5
8	32.0

Answer.
(b)
$y = 0.5x^2$

28.

x	y
2	12
4	28
6	44
9	68

29.

x	y
1.5	3.0
2.4	5.3
5.5	33
8.2	73.8

Answer.
(c) $\dfrac{y}{x^p}$ is not constant for any exponent p.

30.

x	y
1.2	7.20
2.5	31.25
6.4	204.80
12	720.00

For Problems 31-34, decide whether

a y varies inversely with x,

b y varies inversely with x^2, or

c y does not vary inversely with a power of x.

Explain why your choice is correct. If your choice is (a) or (b), find the constant of variation.

31.

x	y
0.5	288
2.0	18
3.0	8
6.0	2

Answer.
(b) $y = \dfrac{72}{x^2}$

32.

x	y
0.5	100.0
2.0	25.0
4.0	12.5
5.0	10.0

33.

x	y
1.0	4.0
1.3	3.7
3.0	2.0
4.0	1.0

Answer.
(c) $x^p y$ is not constant for any exponent p.

34.

x	y
0.5	180.00
2.0	11.25
3.0	5.00
5.0	1.80

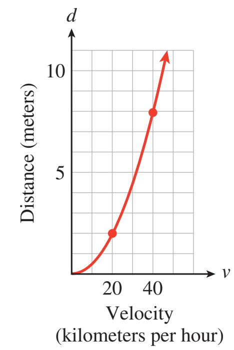
The functions described by a table of data or by a graph in Problems 35–42 are examples of direct or inverse variation.

a Find an algebraic formula for the function, including the constant of variation, k.

b Answer the question in the problem.

35.

The faster a car moves, the more difficult it is to stop. The graph shows the distance, d, required to stop a car as a function of its velocity, v, before the brakes were applied. What distance is needed to stop a car moving at 100 kilometers per hour?

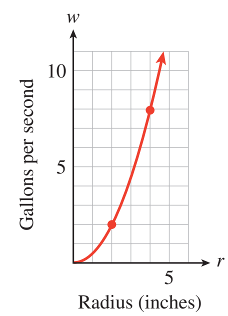


Answer.

a $d = 0.005v^2$ b 50 m

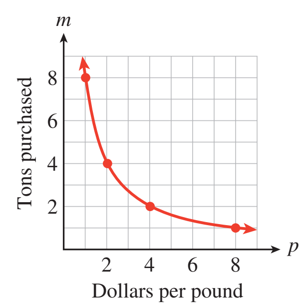
36.

A wide pipe can handle a greater water flow than a narrow pipe. The graph shows the water flow through a pipe, w, as a function of its radius, r. How great is the water flow through a pipe of radius of 10 inches?



37.

If the price of mushrooms goes up, the amount consumers are willing to buy goes down. The graph shows the number of tons of shiitake mushrooms, m, sold in California each week as a function of their price, p. If the price of shiitake mushrooms rises to \$10 per pound, how many tons will be sold?

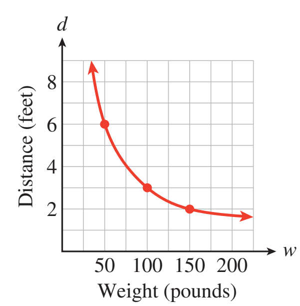


Answer.

a $m = \dfrac{8}{p}$ b 0.8 ton

38.

When an adult plays with a small child on a seesaw, the adult must sit closer to the pivot point to balance the seesaw. The graph shows this distance, d, as a function of the adult's weight, w. How far from the pivot must Kareem sit if he weighs 280 pounds?



39. Ocean temperatures are generally colder at the greater depths. The table shows the temperature of the water as a function of depth. What is the ocean temperature at a depth of 6 kilometers?

Depth (km)	Temperature (°C)
0.5	12
1	6
2	3
3	2

Answer.

a $T = \dfrac{6}{d}$

b 1°C

40. The shorter the length of a vibrating guitar string, the higher the frequency of the vibrations. The fifth string is 65 centimeters long and is tuned to A (with a frequency of 220 vibrations per second). The placement of the fret relative to the bridge changes the effective length of the guitar string. The table shows frequency as a function of effective length. How far from the bridge should the fret be placed for the note C (256 vibrations per second)?

Length (cm)	Frequency
55	260
57.2	250
65	220
71.5	200

41. The strength of a cylindrical rod depends on its diameter. The greater the diameter of the rod, the more weight it can support before collapsing. The table shows the maximum weight supported by a rod as a function of its diameter. How much weight can a 1.2-centimeter rod support before collapsing?

Diameter (cm)	Weight (newtons)
0.5	150
1.0	600
1.5	1350
2.0	2400

Answer.

a $W = 600d^2$

b 864 newtons

42. The maximum height attained by a cannonball depends on the speed at which it was shot. The table shows maximum height as a function of initial speed. What height is attained by a cannonball whose initial upward speed was 100 feet per second?

Speed (ft/sec)	Height (ft)
40	200
50	31.25
60	450
70	612.5

43. The wind resistance, W, experienced by a vehicle on the freeway varies directly with the square of its speed, v.

a If you double your speed, what happens to the wind resistance?

b If you drive one-third as fast, what happens to the wind resistance?

c If you decrease your speed by 10%, what happens to the wind resistance?

Answer.

a Wind resistance quadruples.

b It is one-ninth as great.

c It is decreased by 19% because it is 81% of the original.

44. The weight, w, of a bronze statue varies directly with the cube of its height, h.

a If you double the height of the statue, what happens to its weight?

b If you make the statue one-fourth as tall, what happens to its weight?

c If you increase the height of the statue by 50%, what happens to its weight?

45. The intensity of illumination, I, from a lamp varies inversely with the square of your distance, d, from the lamp.

a If you double your distance from a reading lamp, what happens to the illumination?

b If you triple the distance, what happens to the illumination?

c If you increase the distance by 25%, what happens to the illumination?

Answer.

a It is one-fourth the original illumination.

b It is one-ninth the illumination.

c It is 64% of the illumination.

46. The resistance, R, of an electrical wire varies inversely with the square of its diameter, d.

a If you replace an old wire with a new one whose diameter is half that of the old one, what happens to the resistance?

b If you replace an old wire with a new one whose diameter is two-thirds of the old one, what happens to the resistance?

c If you decrease the diameter of the wire by 30%, what happens to the resistance?

The quoted material in Problems 47–50 is taken from the article "Quantum Black Holes," by Bernard J. Carr and Steven B. Giddings, in the May 2005 issue of *Scientific American*. (See Algebra Skills Refresher A.1.4, p. 853 to review scientific notation.)

47. “The density to which matter must be squeezed [to create a black hole] scales as the inverse square of the mass. For a hole with the mass of the Sun, the density is about 10^{19} kilograms per cubic meter, higher than that of an atomic nucleus.”

a Recall that the density of an object is its mass per unit volume. Given that the mass of the sun is about 2×10^{30} kilograms, write a formula for the density, D, of a black hole as a function of its mass, m.

b “The known laws of physics allow for a matter density up to the so-called Planck value of 10^{97} kilograms per cubic meter." If a black hole with this density could be created, it would be the smallest possible black hole. What would its mass be?

c Assuming that a black hole is spherical, what would be the radius of the smallest possible black hole?

48. "A black hole radiates thermally, like a hot coal, with a temperature inversely proportional to its mass. For a solar-mass black hole, the temperature is around a millionth of a kelvin."

a The solar mass is given in Problem 47. Write a formula for the temperature, T, of a black hole as a function of its mass, m.

b What is the temperature of a black hole of mass 10^{12} kilograms, about the mass of a mountain?

49. "The total time for a black hole to evaporate away is proportional to the cube of its initial mass. For a solar-mass hole, the lifetime is an unobservably long 10^{64} years."

a The solar mass is given in Problem 47. Write a formula for the lifetime, L, of a black hole as a function of its mass, m.

b The present age of the universe is about 10^{10} years. What would be the mass of a black hole as old as the universe?

Answer.

a $L = (1.25 \times 10{-}27)m^3$

b 2×10^{12} kg

50. "String theory ... predicts that space has dimensions beyond the usual three. In three dimensions, the force of as strong." In three dimensions, the forcgravity quadruples as you halve the distance between two objects. But in nine dimensions, gravity would get 256 timese of gravity varies inversely with the square of distance. Write a formula for the force of gravity in nine dimensions.

Use algebra to support your answers to Problems 51–56. Begin with a formula for direct or inverse variation.

51. Suppose y varies directly with x. Show that if you multiply x by any constant c, then y will be multiplied by the same constant.

Answer. $y = kx$ implies that $k(cx) = c(kx) = cy$.

52. Suppose y varies inversely with x. Show that if you multiply x by any constant c, then y will be divided by the same constant.

53. Explain why the ratio $\dfrac{y}{x^2}$ is a constant when y varies directly with x^2.

Answer. If $y = kx^2$, then dividing both sides of the equation by x^2 gives $\dfrac{y}{x^2} = k$.

54. Explain why the product yx^2 is a constant when y varies inversely with x^2.

55. If x varies directly with y and y varies directly with z, does x vary directly with z?

Answer. Yes

56. If x varies inversely with y and y varies inversely with z, does x vary invesely with z?

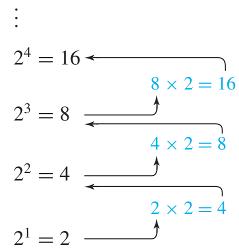
3.2 Integer Exponents

Recall that a positive integer exponent tells us how many times its base occurs as a factor in an expression. For example,

$$4a^3b^2 \quad \text{means} \quad 4aaabb$$

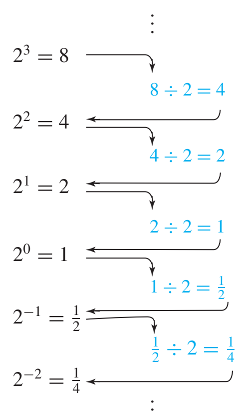
3.2.1 Negative Exponents

Study the list of powers of 2 shown in Table a and observe the pattern as we move up the list from bottom to top. Each time the exponent increases by 1 we multiply by another factor of 2. We can continue up the list as far as we like.



a.

If we move back down the list, we divide by 2 at each step, until we get to the bottom of the list, $2^1 = 2$.
What if we continue the list in the same way, dividing by 2 each time we decrease the exponent? The results are shown in Table b.
As we continue to divide by 2, we generate fractions whose denominators are powers of 2. In particular,



b.

$$2^{-1} = \frac{1}{2} = \frac{1}{2^1} \qquad \text{and} \qquad 2^{-2} = \frac{1}{4} = \frac{1}{2^2}$$

Based on these observations, we make the following definitions.

Definition of Negative and Zero Exponents.

$$a^{-n} = \frac{1}{a^n} \qquad (a \neq 0)$$
$$a^0 = 1 \qquad (a \neq 0)$$

These definitions tell us that if the base a is not zero, then any number raised to the zero power is 1, and that a negative exponent denotes a reciprocal.

Example 3.2.1

a $2^{-3} = \dfrac{1}{2^3} = \dfrac{1}{8}$

b $9x^{-2} = 9 \cdot \dfrac{1}{x^2} = \dfrac{9}{x^2}$

□

Caution 3.2.2

1. A negative exponent does *not* mean that the power is negative! For example,
$$2^{-3} \neq -2^3$$
2. In Example 3.2.1, p. 316b, note that
$$9x^{-2} \neq \frac{1}{9x^2}$$
The exponent, -2, applies only to the base x, not to 9.

Checkpoint 3.2.3 Write each expression without using negative exponents.

a 5^{-4}

b $5x^{-4}$

Answer.

a $\dfrac{1}{5^4}$

b $\dfrac{5}{x^4}$

In the next example, we see how to evaluate expressions that contain negative exponents and how to solve equations involving negative exponents.

Example 3.2.4 The body mass index, or BMI, is one measure of a person's physical fitness. Your body mass index is defined by

$$BMI = wh^{-2}$$

where w is your weight in kilograms and h is your height in meters. The World Health Organization classifies a person as obese if his or her BMI is 25 or higher.

a a. Calculate the BMI for a woman who is 1.625 meters (64 inches) tall and weighs 54 kilograms (120 pounds).

b For a fixed weight, how does BMI vary with height?

c The world's heaviest athlete is the amateur sumo wrestler Emanuel Yarbrough, who weighs 319 kg (704 pounds). What height would Yarbrough have to be to have a BMI under 25?

Solution.

a $BMI = 54(1.625^{-2}) = 54(\dfrac{1}{1.625^2}) = 20.45$

b $BMI = wh^{-2} = \dfrac{w}{h^2}$, so BMI varies inversely with the square of height. That is, for a fixed weight, BMI decreases as height increases.

c To find the height that gives a BMI of 25, we solve the equation $25 = 319h^{-2}$. Note that the variable h appears in the denominator of a fraction, so we begin by clearing the denominator -- in this case we multiply both sides of the equation by h^2.

$$\begin{aligned} 25 &= \frac{319}{h^2} && \text{Multiply both sides by } \boldsymbol{h^2}\textbf{.} \\ 25h^2 &= 319 && \text{Divide both sides by } \mathbf{25.} \\ h^2 &= 12.76 && \text{Extract square roots.} \\ h &\approx 3.57 \end{aligned}$$

To have a BMI under 25, Yarbrough would have to be over 3.57 meters, or 11 feet 8 inches tall. (In fact, he is 6 feet 8 inches tall.)

□

Checkpoint 3.2.5 Solve the equation $0.2x^{-3} = 1.5$

Hint. Rewrite without a negative exponent.
Clear the fraction.
Isolate the variable.

Answer. $x = \sqrt[3]{\dfrac{2}{15}} \approx 0.51$

3.2.2 Power Functions

The functions that describe direct and inverse variation are part of a larger family of functions called **power functions**.

Power Function.

A function of the form

$$f(x) = kx^p$$

where k and p are nonzero constants, is called a **power function**.

Examples of power functions are

$$V(r) = \frac{4}{3}\pi r^3 \quad \text{and} \quad L(T) = 0.8125T^2$$

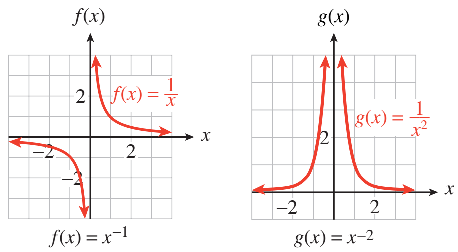
In addition, the basic functions

$$f(x) = \frac{1}{x} \quad \text{and} \quad g(x) = \frac{1}{x^2}$$

which we studied in Chapter 2, p. 149 can be written as

$$f(x) = x^{-1} \quad \text{and} \quad g(x) = x^{-2}$$

Their graphs are shown below. Note that the domains of power functions with negative exponents do not include zero.



Example 3.2.6 Which of the following are power functions?

a $f(x) = \dfrac{1}{3}x^4 + 2$ b $g(x) = \dfrac{1}{3x^4}$ c $h(x) = \dfrac{x+6}{x^3}$

Solution.

a This is not a power function, because of the addition of the constant term.

b We can write $g(x) = \frac{1}{3}x^{-4}$, so g is a power function.

c This is not a power function, but it can be treated as the sum of two power functions, because $h(x) = x^{-2} + 6x^{-3}$.

□

Checkpoint 3.2.7 Write each function as a power function in the form $y = kx^p$.

a $f(x) = \dfrac{12}{x^2}$ b $g(x) = \dfrac{1}{4x}$ c $h(x) = \dfrac{2}{5x^6}$

Answer.

a $f(x) = 12x^{-2}$ b $g(x) = \dfrac{1}{4}x^{-1}$ c $h(x) = \dfrac{2}{5}x^{-6}$

Most applications are concerned with positive variables only, so many models use only the portion of the graph in the first quadrant.

Example 3.2.8 In the Middle Ages in Europe, castles were built as defensive strongholds. An attacking force would build a huge catapult called a trebuchet to hurl rocks and scrap metal inside the castle walls. The engineers could adjust its range by varying the mass of the projectiles. The mass, m, of the projectile should be inversely proportional to the square of the distance, d, to the target.

a Use a negative exponent to write m as a function of d, $m = f(d)$.

b The engineers test the trebuchet with a 20-kilogram projectile, which lands 250 meters away. Find the constant of proportionality; then rewrite your formula for m.

c Graph $m = f(d)$.

d The trebuchet is 180 meters from the courtyard within the castle. What size projectile will hit the target?

e The attacking force would like to hurl a 100-kilogram projectile at the castle. How close must the attackers bring their trebuchet?

Solution.

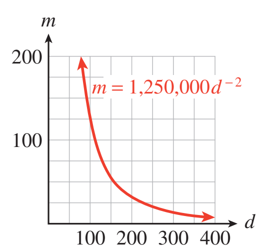
a If we use k for the constant of proportionality, then $m = \dfrac{k}{d^2}$. Rewriting this equation with a negative exponent gives $m = kd^{-2}$.

b We substitute $m = 20$ and $d = 250$ to obtain

$$\begin{aligned} 20 &= k(250)^{-2} \quad \text{Multiply both sides by } \mathbf{250^2}. \\ 1{,}250{,}000 &= k \end{aligned}$$

Thus, $m = 1{,}250{,}000d^{-2}$.

c We evaluate the function for several values of m, or use a calculator to obtain the graph below.



d We substitute $d = 180$ into the formula:

$$\begin{aligned} m &= 1{,}250{,}000(\mathbf{180})^{-2} \\ &= \frac{1{,}250{,}000}{32{,}400} \\ &\approx 38.58 \end{aligned}$$

The attackers should use a mass of approximately 38.6 kilograms.

e We substitute $m = \mathbf{100}$ into the formula and solve for d.

$$\begin{aligned} \mathbf{100} &= 1,250,000d^{-2} && \text{Multiply by } \boldsymbol{d^2}. \\ 100d^2 &= 1,250,000 && \text{Divide by } \mathbf{100}. \\ d^2 &= 12,500 && \text{Take sqwuare roots.} \\ d &= \pm\sqrt{12,500} \end{aligned}$$

They must locate the trebuchet $\sqrt{12,500} \approx 111.8$ meters from the castle.

□

The function $m = \dfrac{k}{d^2}$ is an example of an **inverse square law**, because m varies inversely with the square of d. Such laws are fairly common in physics and its applications, because gravitational and other forces behave in this way. Here is a more modern example of an inverse square law.

Checkpoint 3.2.9 Cell phone towers typically transmit signals at 10 watts of power. The signal strength varies inversely with the square of distance from the tower, and 1 kilometer away the signal strength is 0.8 picowatt. (A picowatt is 10^{-12} watt.) Cell phones can receive a signal as small as 0.01 picowatt. How far can you be from the nearest tower and still hope to have cell phone reception?

Answer. About 9 km

3.2.3 Working with Negative Exponents

A negative exponent denotes the *reciprocal* of a power. Thus, to simplify a fraction with a negative exponent, we compute the positive power of its reciprocal.

Example 3.2.10

a $\left(\dfrac{3}{5}\right)^{-2} = \dfrac{1}{\left(\dfrac{3}{5}\right)^2} = \left(\dfrac{5}{3}\right)^2 = \dfrac{25}{9}$

b $\left(\dfrac{x^3}{4}\right)^{-3} = \left(\dfrac{4}{x^3}\right)^3 = \dfrac{(4)^3}{(x^3)^3} = \dfrac{64}{x^9}$

□

Checkpoint 3.2.11 Simplify $\left(\dfrac{2}{x^2}\right)^{-4}$

Answer. $\dfrac{x^8}{16}$

Dividing by a power with a negative exponent is equivalent to multiplying by a power with a positive exponent.

Example 3.2.12

a

$$\begin{aligned} \frac{1}{5^{-3}} &= 1 \div 5^{-3} \\ &= 1 \div \frac{1}{5^3} \\ &= 1 \times 5^3 = 125 \end{aligned}$$

b

$$\begin{aligned} \frac{k^2}{m^{-4}} &= k^2 \div m^{-4} \\ &= k^2 \div \frac{1}{m^4} \\ &= k^2m^4 \end{aligned}$$

□

Checkpoint 3.2.13 Write each expression without using negative exponents.

a $\left(\dfrac{3}{b^4}\right)^{-2}$

b $\dfrac{12}{x^{-6}}$

Answer.

a $\dfrac{b^8}{9}$

b $12x^6$

3.2.4 Laws of Exponents

The laws of exponents, reviewed in Algebra Skills Refresher Section A.6, p. 895, apply to all integer exponents, positive, negative, and zero. When we allow negative exponents, we can simplify the rule for computing quotients of powers.

A Law of Exponents.

II. $\dfrac{a^m}{a^n} = a^{m-n} \qquad (a \neq 0)$

For example, by applying this new version of the law for quotients, we find

$$\frac{x^2}{x^5} = x^{2-5} = x^{-3}$$

which is consistent with our previous version of the rule,

$$\frac{x^2}{x^5} = \frac{1}{x^{5-2}} = \frac{1}{x^3}$$

For reference, we restate the laws of exponents below. The laws are valid for all integer exponents m and n, and for $a, b \neq 0$.

Laws of Exponents.

I $a^m \cdot a^n = a^{m+n}$

II $\dfrac{a^m}{a^n} = a^{m-n}$

III $(a^m)^n = a^{mn}$

IV $(ab)^n = a^n b^n$

V $\left(\dfrac{a}{b}\right)^n = \dfrac{a^n}{b^n}$

Example 3.2.14

a. $x^3 \cdot x^{-5} = x^{3-5} = x^{-2}$ Apply the first law: Add exponents.

b. $\dfrac{8x^{-2}}{4x^{-6}} = \dfrac{8}{4}x^{-2-(-6)} = 2x^4$ Apply the second law: Subtract exponents.

c. $\left(5x^{-3}\right)^{-2} = 5^{-2}(x^{-3})^{-2} = \dfrac{x^6}{25}$ Apply laws IV and III.

□

You can check that each of the calculations in Example 3.2.14, p. 321 is shorter when we use negative exponents instead of converting the expressions into algebraic fractions.

Checkpoint 3.2.15 Simplify by applying the laws of exponents.

a $\left(2a^{-4}\right)\left(-4a^{2}\right)$

b $\dfrac{(r^2)^{-3}}{3r^{-4}}$

Answer.

a $\dfrac{-8}{a^2}$

b $\dfrac{1}{3r^2}$

Caution 3.2.16 The laws of exponents do not apply to sums or differences of powers. We can add or subtract like terms, that is, powers with the same exponent. For example,

$$6x^{-2} + 3x^{-2} = 9x^{-2}$$

but *we cannot add or subtract terms with different exponents.* Thus, for example,

$$\begin{array}{ll} 4x^2 - 3x^{-2} & \text{cannot be simplified} \\ x^{-1} + x^{-3} & \text{cannot be simplified} \end{array}$$

At the start of this section, we saw that $2^0 = 1$, and in fact $a^0 = 1$ as long as $a \neq 0$. Now we can see that this definition is consistent with the laws of exponents. The quotient of any (nonzero) number divided by itself is 1. But by applying the second law of exponents, we also have

$$1 = \frac{a^m}{a^m} = a^{m-m} = a^0$$

Thus,

Zero as Exponent.

$$a^0 = 1, \quad \text{if } a \neq 0$$

For example,

$$3^0 = 1, \quad (-528)^0 = 1, \quad \text{and} \quad (0.024)^0 = 1$$

3.2.5 Section Summary

3.2.5.1 Vocabulary

Look up the definitions of new terms in the Glossary.

- Power function
- Inverse square law

3.2.5.2 CONCEPTS

1 A negative exponent denotes a reciprocal: $a^{-n} = \dfrac{1}{a^n}$, if $a \neq 0$.

2 Any number (except zero) raised to the zero power is 1: $a^0 = 1$, if $a \neq 0$.

3 A function of the form $f(x) = kx^p$, where k and p are constants, is called a **power function**.

Laws of Exponents.

I $a^m \cdot a^n = a^{m+n}$

II $\dfrac{a^m}{a^n} = a^{m-n}$

III $(a^m)^n = a^{mn}$

IV $(ab)^n = a^n b^n$

V $\left(\dfrac{a}{b}\right)^n = \dfrac{a^n}{b^n}$

3.2.5.3 STUDY QUESTIONS

1 Explain the difference between each pair of expressions.

a -2^3 and 2^{-3}
b $-x^4$ and x^{-4}
c -2^n and 2^{-n}

2 Write a power function for "y varies inversely with the cube of x."?

3 Explain why it makes sense to define $10^0 = 1$.

4 Why is zero excluded from the domain of $f(x) = 3x^{-2}$?

5 Choose a value for to show that the following statement is false:

$$2x^{-2} + 4^{x-1} = 6^{x-3} \qquad \text{False!}$$

3.2.5.4 SKILLS

Practice each skill in the Homework 3.2.6, p. 323 problems listed.

1 Simplify expressions with negative exponents: #1–12

2 Solve equations involving negative exponents: #19–24

3 Write formulas for power functions: #17 and 18, 25–34

4 Evaluate and analyze power functions: #13–16, 25–34

5 Apply the laws of exponents to simplify expressions: #35–62

3.2.6 Integer Exponents (Homework 3.2)

1. Make a table showing powers of 3 from 3^{-5} to 3^5. Illustrate why defining $3^0 = 1$ makes sense.

Answer.

n	-5	-4	-3	-2	-1	0	1	2	3	4	5
3^n	$\frac{1}{243}$	$\frac{1}{81}$	$\frac{1}{27}$	$\frac{1}{9}$	$\frac{1}{3}$	1	3	9	27	81	243

Each time n increases by 1, we multiply the power in the bottom row by 3.

2. Make a table showing powers of 5 from 5^{-4} to 5^4. Illustrate why defining $5^0 = 1$ makes sense.

For Problems 3–6, compute each power.

3.

a 2^3 b $(-2)^3$ c 2^{-3} d $(-2)^{-3}$

Answer.

a 8 b -8 c $\frac{1}{8}$ d $\frac{-1}{8}$

4.

a 4^2 b $(-4)^2$ c 4^{-2} d $(-4)^{-2}$

5.

a $\left(\frac{1}{2}\right)^3$ b $\left(-\frac{1}{2}\right)^3$ c $\left(\frac{1}{2}\right)^{-3}$ d $\left(-\frac{1}{2}\right)^{-3}$

Answer.

a $\frac{1}{8}$ b $\frac{-1}{8}$ c 8 d -8

6.

a $\left(\frac{1}{4}\right)^2$ b $\left(-\frac{1}{4}\right)^2$ c $\left(\frac{1}{4}\right)^{-2}$ d $\left(-\frac{1}{4}\right)^{-2}$

For Problems 7–12, write without negative exponents and simplify.

7.

a 2^{-1} b $(-5)^{-2}$ c $\left(\frac{1}{3}\right)^{-3}$ d $\frac{1}{(-2)^{-4}}$

Answer.

a $\frac{1}{2^1} = \frac{1}{2}$ b $\frac{1}{(-5)^2} = \frac{1}{25}$ c $3^3 = 27$ d $(-2)^4 = 16$

8.

a 3^{-2} b $(-2)^{-3}$ c $\left(\frac{3}{5}\right)^{-2}$ d $\frac{1}{(-3)^{-3}}$

9.

a $\frac{5}{4^{-3}}$ b $(2q)^{-5}$ c $-4x^{-2}$ d $\frac{8}{b^{-3}}$

Answer.

a $5 \cdot 4^3 = 320$ b $\frac{1}{(2q)^5} = \frac{1}{32q^5}$ c $\frac{-4}{x^2}$ d $8b^3$

10.

a $\frac{3}{2^{-6}}$ b $(4k)^{-3}$ c $-7x^{-4}$ d $\frac{5}{a^{-5}}$

11.

a $(m-n)^{-2}$ c $2pq^{-4}$

b $y^{-2} + y^{-3}$ d $\frac{-5y^{-2}}{x^{-5}}$

Answer.

a $\dfrac{1}{(m-n)^2}$

c $\dfrac{2p}{q^4}$

b $\dfrac{1}{y^2}+\dfrac{1}{y^3}$

d $\dfrac{-5x^5}{y^2}$

12.

a $(p+q)^{-3}$

c $8m^{-2}n^2$

b $z^{-1}-z^{-2}$

d $\dfrac{-6y^{-3}}{x^{-3}}$

Use your calculator to fill in the tables in Problems 13 and 14. Round your answers to two decimal places.

13. $f(x)=x^{-2}$

a

x	1	2	4	8	16
$f(x)$					

b What happens to the values of $f(x)$ as the values of x increase? Explain why.

c

x	1	0.5	0.25	0.125	0.0625
$f(x)$					

d What happens to the values of $f(x)$ as the values of x decrease toward 0? Explain why.

Answer.

(a)

x	1	2	4	8	16
x^{-2}	1	0.25	0.06	0.02	0.00

(b) The values of $f(x)$ decrease, because x^{-2} is the reciprocal of x^2.

(c)

x	1	0.5	0.25	0.125	0.0625
x^{-2}	1	4	16	64	256

(d) The values of $f(x)$ increase toward infinity, because x^{-2} is the reciprocal of x^2.

14. $g(x)=x^{-3}$

a

x	1	2	4.5	6.2	9.3
$g(x)$					

b What happens to the values of $g(x)$ as the values of x increase? Explain why.

c

x	1.5	0.6	0.1	0.03	0.002
$f(x)$					

d What happens to the values of $g(x)$ as the values of x decrease toward 0? Explain why.

15.

(a) Use your calculator to graph each of the following functions on the window

$$\text{Xmin} = -5 \qquad \text{Xmax} = 5$$
$$\text{Ymin} = -2 \qquad \text{Ymax} = 10$$

i. $f(x) = x^2$

ii. $f(x) = x^{-2}$

iii. $f(x) = \dfrac{1}{x^2}$

iv. $f(x) = \left(\dfrac{1}{x}\right)^2$

(b) Which functions have the same graph? Explain your results.

Answer. b. (ii), (iii), and (iv) have the same graph, because they represent the same function.

16.

(a) Use your calculator to graph each of the following functions on the window

$$\text{Xmin} = -3 \qquad \text{Xmax} = 5$$
$$\text{Ymin} = -5 \qquad \text{Ymax} = 5$$

i. $f(x) = x^3$

ii. $f(x) = x^{-3}$

iii. $f(x) = \dfrac{1}{x^3}$

iv. $f(x) = \left(\dfrac{1}{x}\right)^3$

(b) Which functions have the same graph? Explain your results.

For Problems 17–18, write each expression as a power function using negative exponents.

17.

(a) $F(r) = \dfrac{3}{r^4}$ (b) $G(w) = \dfrac{2}{5w^3}$ (c) $H(z) = \dfrac{1}{(3z)^2}$

Answer.

(a) $F(r) = 3r^{-4}$ (b) $G(w) = \dfrac{2}{5}w^{-3}$ (c) $H(z) = \dfrac{1}{9}z^{-2}$

18.

(a) $h(s) = \dfrac{9}{s^3}$ (b) $f(v) = \dfrac{3}{8v^6}$ (c) $g(t) = \dfrac{1}{(5t)^4}$

For Problems 19–24, olve.

19. $6x^{-2} = 3.84$

Answer. $x = -1.25$ or $x = 1.25$

20. $0.8w^{-2} = 1.25$

21. $12 + 0.04t^{-3} = 175.84$

Answer. $t = \dfrac{1}{16}$

22. $854 - 48z^{-3} = 104$

23. $100 - 0.15v^{-4} = 6.25$

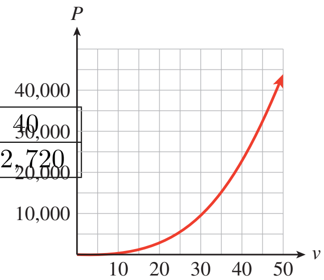
Answer. $v = \dfrac{1}{5}$ or $v = \dfrac{-1}{5}$

24. $8100p^{-4} - 250 = 3656.25$

25. When an automobile accelerates, the power, P, needed to overcome air resistance varies directly with a power of the speed, v.

(a) Use the data and the graph to find the scaling exponent and the constant of variation. Then write a formula for P as a power function of v.

v (mph)	10	20	30	40
P (watts)	355	2840	9585	22,720



(b) Find the speed that requires 50,000 watts of power.

(c) If you increase your speed by 50%, by what factor does the power requirement increase?

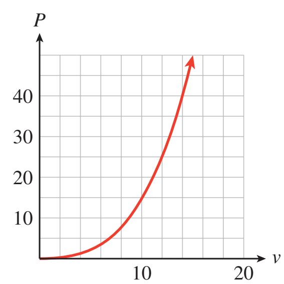
Answer.

(a) $P = 0.355v^3$ (b) $v \approx 52.03$ mph (c) 3.375

26. The power, P, generated by a windmill varies directly with a power of wind velocity, v.

(a) Use the data and the graph to find the scaling exponent and the constant of variation. Then write a formula for P as a power function of v.

v (mph)	10	20	30	40
P (watts)	15	120	405	960



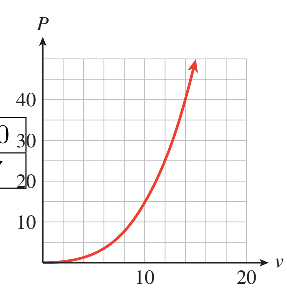
(b) Find the wind velocity needed to generate 500 watts of power.

(c) If the wind speed drops by half, what happens to the power generated?

27. The "Rule of 70" is used to estimate how long it takes an investment to double in value when interest is compounded annually. The doubling time, D, is inversely proportional to the interest rate, i. (Note that i is expressed as a percent, not as a decimal fraction. For example, if the interest rate is 8%, then $i = 8$.)

(a) Use the data and the graph to find the constant of proportionality and write D as a power function of i.

i (mph)	4	6	8	10
D (watts)	17.5	11.67	8.75	7



(b) If the interest rate increases from 5% to 6%, how will the doubling time change?

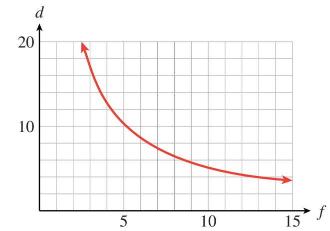
Answer.

(a) $D = \dfrac{70}{i}$ (b) It decreases by about 2.3 years.

28. The f-stop setting on a camera regulates the size of the aperture and thus the amount of light entering the camera. The f-stop f is inversely proportional to the diameter, d, of the aperture.

(a) Use the data and the graph to find the constant of proportionality and write d as a power function of f. Values of d have been rounded to one decimal place.

f	2.8	4	5.6	8	11
d	17.9	12.5	8.9	6.3	4.5



(b) Why are the f-stop settings labeled with the values given in the table?

Hint. As you stop down the aperture from one f-value to the next, by what factor does d increase?

29. The Stefan-Boltzmann law relates the total amount of radiation emitted by a star to its temperature, T, in kelvins, by the following formula:

$$sT^4 = \frac{L}{4\pi R^2}$$

where R is the radius of the star, L is its luminosity, and $s = 5.7 \times 10^{-8}$ watt/m^2 is a constant governing radiation. (See Algebra Skills Refresher A.1.4, p. 853 to review scientific notation.)

a Write a formula for luminosity as a power function of temperature for a fixed radius.

b The radius of the Sun is $R = 9.96 \times 10^8$ meters, and its luminosity is $L = 3.9 \times 10^{26}$ watts. Calculate the temperature of the Sun.

Answer.

(a) $L = \left(4\pi sR^2\right) T^4 \approx 7.2 \times 10^{-7} R^2 T^4$

(b) 4840 K

30. Poiseuille's law for the flow of liquid through a tube can be used to describe blood flow through an artery. The rate of flow, F, in liters per minute is proportional to the fourth power of the radius, r, divided by the length, L, of the artery.

a Write a formula for the rate of flow as a power function of radius.

b If the radius and length of the artery are measured in centimeters, then the constant of variation, $k = 7.8 \times 10^5$, is determined by blood pressure and viscosity. If a certain artery is 20 centimeters long, what should its radius be in order to allow a blood flow of 5 liters per minute?

31. Airplanes use radar to detect the distances to other objects. A radar unit transmits a pulse of energy, which bounces off a distant object, and the echo of the pulse returns to the sender. The power, P, of the returning echo is inversely proportional to the fourth power of the distance, d, to the object. A radar operator receives an echo of 5×10^{-10} watts from an aircraft 2 nautical miles away.

a Express the power of the echo received in picowatts. (1 picowatt $= 10^{-12}$ watts.)

b Write a function that expresses P in terms of d using negative exponents. Use picowatts for the units of power.

c Complete the table of values for the power of the echo received from objects at various distances.

d (nautical miles)	4	5	7	10
P (picowatts)				

d Radar unit scan typically detect signals as low as 10^{-13} watts. How far away is an aircraft whose echo is 10^{-13} watts?

e Sketch a graph of P as a function of d. Use units of picowatts on the vertical axis

Hint. Convert 10^{-13} watts to picowatts.

Answer.

(a) 500 picowatts

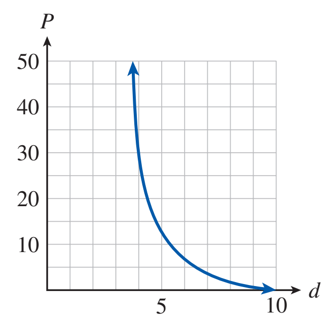
(b) $P = 8000d^{-4}$

(c)

d (nautical miles)	4	5	7	10
P (picowatts)	31.3	12.8	3.3	0.8

(d) 16.8 nautical miles

(e)



32. The lifetime of a star is roughly inversely proportional to the cube of its mass. Our Sun, which has a mass of one solar mass, will last for approximately 10 billion years.

(a) Write a power function for the lifetime, L, of a star in terms of its mass, m.

(b) Sketch a graph of the function using units of solar mass on the horizontal axis.

(c) How long will a star that is 10 times as massive as the Sun last?

(d) One solar mass is about 2×10^{30} kilograms. Rewrite your formula for L with the units of mass in kilograms.

(e) How long will a star that is half as massive as the Sun last?

33. The amount of force or thrust generated by the propeller of a ship is a function of two variables: the diameter of the propeller and its speed, in rotations per minute. The thrust, T, in pounds, is proportional to the square of the speed, r, and the fourth power of the diameter, d, in feet.

(a) Write a formula for the thrust in terms of the speed if the diameter of the propeller is 2 feet.

(b) A propeller of diameter 2 feet generates a thrust of 1000 pounds at 100 rotations per minute. Find the constant of variation in the formula for thrust.

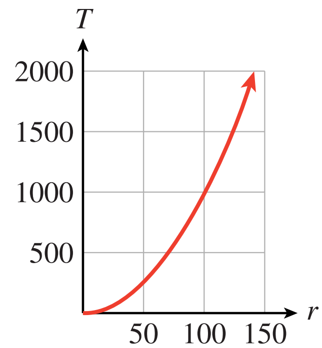
(c) Sketch a graph of the thrust as a function of the propeller speed for a propellor of diameter 4 feet. If the speed of the propeller is doubled, by what factor does the thrust increase?

Answer.

(a) $T = 16kr^2$

(b) $T = 0.1r^2$

(c)



34. Refer to Problem 33.

(a) Write a formula for the thrust, T, in terms of the diameter of the propeller if its speed is 100 rotations per minute.

(b) A propeller of diameter 4 feet generates a thrust of 32,000 pounds at 100 rotations per minute. Find the constant of variation in the formula for thrust.

(c) Sketch a graph of the thrust as a function of the diameter of the propeller at a speed of 100 rotations per minute. If the diameter of the propeller is doubled, by what factor does the thrust increase?

For Problems 35–40, use the laws of exponents to simplify and write without negative exponents.

35.

(a) $a^{-3} \cdot a^8$

(b) $5^{-4} \cdot 5^{-3}$

(c) $\dfrac{p^{-7}}{p^{-4}}$

(d) $\left(7^{-2}\right)^5$

Answer.

(a) a^5

(b) $\dfrac{1}{5^7}$

(c) $\dfrac{1}{p^3}$

(d) $\dfrac{1}{7^{10}}$

36.

(a) $b^2 \cdot b^{-6}$

(b) $4^{-2} \cdot 4^{-6}$

(c) $\dfrac{w^{-9}}{w^2}$

(d) $\left(9^{-4}\right)^3$

37.

(a) $\left(4x^{-5}\right)\left(5x^2\right)$

(b) $\dfrac{3u^{-3}}{9u^9}$

(c) $\dfrac{5^6t^0}{5^{-2}t^{-1}}$

Answer.

(a) $\dfrac{20}{x^3}$

(b) $\dfrac{1}{3u^{12}}$

(c) 5^8t

38.

(a) $\left(3y^{-8}\right)\left(2y^4\right)$

(b) $\dfrac{4c^{-4}}{8c^{-8}}$

(c) $\dfrac{3^{10}s^{-1}}{3^{-5}s^0}$

39.

(a) $\left(3x^{-2}y^3\right)^{-2}$

(b) $\left(\dfrac{6a^{-3}}{b^2}\right)^{-2}$

(c) $\dfrac{5h^{-3}(h^4)^{-2}}{6h^{-5}}$

Answer.

(a) $\dfrac{x^4}{9y^6}$

(b) $\dfrac{a^6b^4}{36}$

(c) $\dfrac{5}{6h^6}$

40.

(a) $\left(2x^3y^{-4}\right)^{-3}$

(b) $\left(\dfrac{a^4}{4b^{-5}}\right)^{-3}$

(c) $\dfrac{4v^{-5}(v^{-2})^{-4}}{3v^{-8}}$

For Problems 41–44, write each expression as a sum of terms of the form kx^p.

41.

(a) $\dfrac{x}{3} + \dfrac{3}{x}$

(b) $\dfrac{x - 6x^2}{4x^3}$

Answer.

(a) $\dfrac{1}{3}x + 3x^{-1}$

(b) $\dfrac{1}{4}x^{-2} - \dfrac{3}{2}x^{-1}$

42.

(a) $\dfrac{2}{x^2} - \dfrac{x^2}{2}$

(b) $\dfrac{5x+1}{(3x)^2}$

43.

(a) $\dfrac{2}{x^4}\left(\dfrac{x^2}{4} + \dfrac{x}{2} - \dfrac{1}{4}\right)$

(b) $\dfrac{x^2}{3}\left(\dfrac{2}{x^4} - \dfrac{1}{3x^2} + \dfrac{1}{2}\right)$

Answer.

(a) $\dfrac{1}{2}x^{-2} + x^{-3} - \dfrac{1}{2}x^{-4}$

(b) $\dfrac{2}{3}x^{-2} - \dfrac{1}{9} + \dfrac{1}{6}x^2$

44.

(a) $\dfrac{9}{x^3}\left(\dfrac{x^3}{3} - 1 - \dfrac{1}{x^3}\right)$

(b) $\dfrac{x^2}{2}\left(\dfrac{3}{x} - \dfrac{5}{x^3} + \dfrac{7}{x^5}\right)$

For Problems 45–50, use the distributive law to write each product as a sum of power functions.

45. $x^{-1}(x^2 - 3x + 2)$

Answer. $x - 3 + 2x^{-1}$

46. $3x^{-2}(2x^4 + x^2 - 4)$

47. $-3t^{-2}(t^2 - 2 + 4t^{-2})$

Answer. $-3 + 6t^{-2} + 12t^{-4}$

48. $-t^{-3}(3t^2 - 1 - t^{-2})$

49. $2u^{-3}(-2u^3 - u^2 + 3u)$

Answer. $-4 - 2u^{-1} + 6u^{-2}$

50. $2u^{-1}(-1 - u - 2u^2)$

For Problems 51–54, factor as indicated, writing the second factor with positive exponents only.

51. $4x^2 + 16x^{-2} = 4x^{-2}(\ \ ?\ \ )$

Answer. $4x^{-2}(x^4 + 4)$

52. $20y - 15y^{-1} = 5y^{-1}(\ \ ?\ \ )$

53. $3a^{-3} - 3a + a^3 = a^{-3}(\ \ ?\ \ )$

Answer. $a^{-3}(3 - 3a^4 + a^6)$

54. $2 - 4q^{-2} - 8q^{-4} = 2q^{-4}(\ \ ?\ \ )$

55.

(a) Is it true that $(x + y)^{-2} = x^{-2} + y^{-2}$? Explain why or why not.

(b) Give a numerical example to support your answer.

Answer.

(a) No, because $\frac{1}{(x+y)^2}$ is not $\frac{1}{x^2} + \frac{1}{y^2}$.

(b) Let $x = 1$, $y = 2$, then $(x + y)^{-2} = (1 + 2)^{-2} = 3^{-2} = \frac{1}{9}$, but $x^{-2} + y^{-2} = 1^{-2} + 2^{-2} = 1 + \frac{1}{4} = \frac{5}{4}$

56.

(a) Is it true that $(a - b)^{-1} = a^{-1} - b^{-1}$? Explain why or why not.

(b) Give a numerical example to support your answer.

57.

(a) Show that $x + x^{-1} = \dfrac{x^2 + 1}{x}$.

(b) Show that $x^3 + x^{-3} = \dfrac{x^6 + 1}{x^3}$.

(c) Write $x^n + x^{-n}$ as an algebraic fraction. Justify your answer.

Answer.

(a) $x + x^{-1} = x + \dfrac{1}{x} = \dfrac{x^2}{x} + \dfrac{1}{x} = \dfrac{x^2 + 1}{x}$

(b) $x^3 + x^{-3} = x^3 + \dfrac{1}{x^3} = \dfrac{x^6}{x^3} + \dfrac{1}{x^3} = \dfrac{x^6 + 1}{x^3}$

(c) $x^n + x^{-n} = x^n + \dfrac{1}{x^n} = \dfrac{x^{2n}}{x^n} + \dfrac{1}{x^n} = \dfrac{x^{2n} + 1}{x^2}$

58.

(a) Show that $x^{-m} + x^{-n} = \dfrac{x^n + x^m}{x^{n+m}}$.

(b) If $m < n$, show that $x^{-m} + x^{-n} = \dfrac{x^{n-m}+1}{x^n}$.

By rewriting the expressions in Problems 59–62 as fractions, verify that the laws of exponents hold for negative exponents . Show where you apply the corresponding law for positive exponents. Here is the fourth law as an example:

$$\begin{aligned}(ab)^{-3} &= \frac{1}{(ab)^3} = \frac{1}{a^3b^3} && \text{By the fourth law of exponents.}\\ &= \frac{1}{a^3}\cdot\frac{1}{b^3} = a^{-3}b^{-3}\end{aligned}$$

59. $a^{-2}a^{-3} = a^{-5}$

Answer.

$$\begin{aligned}a^{-2}a^{-3} &= \frac{1}{a^2}\cdot\frac{1}{a^3} = \frac{1}{a^2\cdot a^3}\\ &= \frac{1}{a^{2+3}} && \text{By the first law of exponents.}\\ &= \frac{1}{a^5} = a^{-5}\end{aligned}$$

60. $\dfrac{a^{-6}}{a^{-2}} = a^{-4}$

61. $\dfrac{a^{-2}}{a^{-6}} = a^4$

Answer.

$$\begin{aligned}\frac{a^{-2}}{a^{-6}} &= a^{-2} \div a^{-6} = \frac{1}{a^2} \div \frac{1}{a^6}\\ &= \frac{1}{a^2}\cdot\frac{a^6}{1} = \frac{a^6}{a^2}\\ &= a^{6-2} && \text{By the second law of exponents.}\\ &= a^4\end{aligned}$$

62. $\left(a^{-2}\right)^{-3} = a^6$

3.3 Roots and Radicals

In Section 3.2, p. 316 we saw that inverse variation can be expressed as a power function by using negative exponents. We can also use exponents to denote square roots and other radicals.

3.3.1 nth Roots

Recall that s is a square root of b if $s^2 = b$, and s is a cube root of b if $s^3 = b$. In a similar way, we can define the fourth, fifth, or sixth root of a number. For instance, the fourth root of b is a number s whose fourth power is b. In general, we make the following definition.

nth Roots.

s is called an **nth root of** b if $s^n = b$.

We use the symbol $\sqrt[n]{b}$ to denote the nth root of b. An expression of the form $\sqrt[n]{b}$ is called a **radical**, b is called the **radicand**, and n is called the **index of the radical**.

Example 3.3.1

a $\sqrt[4]{81} = 3$ because $3^4 = 81$

b $\sqrt[5]{32} = 2$ because $2^5 = 32$

c $\sqrt[6]{64} = 2$ because $2^6 = 64$

d $\sqrt[4]{1} = 1$ because $1^4 = 1$

e $\sqrt[5]{100,000} = 10$ because $10^5 = 100,000$

□

Checkpoint 3.3.2 Evaluate each radical.

a $\sqrt[4]{16}$

b $\sqrt[5]{243}$

Answer.

a 2

b 3

3.3.2 Exponential Notation for Radicals

A convenient notation for radicals uses fractional exponents. Consider the expression $9^{1/2}$. What meaning can we attach to an exponent that is a fraction? The third law of exponents says that when we raise a power to a power, we multiply the exponents together:

$$(x^a)^b = x^{ab}$$

Therefore, if we square the number $9^{1/2}$, we get

$$\left(9^{1/2}\right)^2 = 9^{(1/2)(2)} = 9^1 = 9$$

Thus, $9^{1/2}$ is a number whose square is 9. But this means that $9^{1/2}$ is a square root of 9, or

$$9^{1/2} = \sqrt{9} = 3$$

In general, any nonnegative number raised to the 1/2 power is equal to the positive square root of the number, or

$$a^{1/2} = \sqrt{a}$$

Example 3.3.3

a $25^{1/2} = 5$

b $-25^{1/2} = -5$

c $(-25)^{1/2}$ is not a real number.

d $0^{1/2} = 0$

□

Checkpoint 3.3.4 Evaluate each power.

a $4^{1/2}$

b 4^{-2}

c $4^{-1/2}$

d $\left(\dfrac{1}{4}\right)^{1/2}$

Answer.

a 2

b $\dfrac{1}{16}$

c $\dfrac{1}{2}$

d $\dfrac{1}{2}$

The same reasoning works for roots with any index. For instance, $8^{1/3}$ is the cube root of 8, because

$$\left(8^{1/3}\right)^3 = 8^{(1/3)(3)} = 8^1 = 8$$

In general, we make the following definition for fractional exponents.

Exponential Notation for Radicals.

For any integer $n \geq 2$ and for $a \geq 0$,

$$a^{1/n} = \sqrt[n]{a}$$

Example 3.3.5

a $81^{1/4} = \sqrt[4]{81} = 3$

b $125^{1/3} = \sqrt[3]{125} = 5$

□

Caution 3.3.6 Note that

$$25^{1/2} \neq \frac{1}{2}(25) \quad \text{and} \quad 125^{1/3} \neq \frac{1}{3}(125)$$

An exponent of $\frac{1}{2}$ denotes the square root of its base, and an exponent of $\frac{1}{3}$ denotes the cube root of its base.

Checkpoint 3.3.7 Write each power with radical notation, and then evaluate.

a $32^{1/5}$

b $625^{1/4}$

Answer.

a $\sqrt[5]{32} = 2$

b $\sqrt[4]{625} = 5$

Of course, we can use decimal fractions for exponents as well. For example,

$$\sqrt{a} = a^{1/2} = a^{0.5} \qquad \text{and} \qquad \sqrt[4]{a} = a^{1/4} = a^{0.25}$$

Example 3.3.8

a $100^{0.5} = \sqrt{100} = 10$

b $16^{0.25} = \sqrt[4]{16} = 2$

□

Checkpoint 3.3.9 Write each power with radical notation, and then evaluate.

a $100,000^{0.2}$

b $81^{0.25}$

Answer.

a $\sqrt[5]{100,000} = 10$

b $\sqrt[4]{81} = 3$

3.3.3 Irrational Numbers

What about nth roots such as $\sqrt{23}$ and $5^{1/3}$ that cannot be evaluated easily? These are examples of **irrational numbers**. We can use a calculator to obtain decimal approximations for irrational numbers. For example, you can verify that

$$\sqrt{23} \approx 4.796 \quad \text{and} \quad 5^{1/3} \approx 1.710$$

It is not possible to write down an exact decimal equivalent for an irrational number, but we can find an approximation to as many decimal places as we like.

Caution 3.3.10 The following keying sequence for evaluating the irrational number $7^{1/5}$ is incorrect:

7 ^ 1 ÷ 5 ENTER

You can check that this sequence calculates $\frac{7^1}{5}$, instead of $7^{1/5}$. Recall that according to the order of operations, powers are computed before multiplications or divisions. We must enclose the exponent 1/5 in parentheses and enter

7 ^ (1 ÷ 5) ENTER

Or, because $\frac{1}{5} = 0.2$, we can enter

7 ^ 0.2 ENTER

3.3.4 Working with Fractional Exponents

Fractional exponents simplify many calculations involving radicals. You should learn to convert easily between exponential and radical notation. Remember that a negative exponent denotes a reciprocal.

Example 3.3.11 Convert each radical to exponential notation.

a $\sqrt[3]{12} = 12^{1/3}$

b $\sqrt[4]{2y} = (2y)^{1/4}$ or $(2y)^{0.25}$

□

Checkpoint 3.3.12 Convert each radical to exponential notation.

a $\dfrac{1}{\sqrt[5]{ab}}$

b $3\sqrt[6]{w}$

Answer.

a $(ab)^{-1/5}$

b $3w^{1/6}$

Example 3.3.13 Convert each power to radical notation.

a $5^{1/2} = \sqrt{5}$

b $x^{0.2} = \sqrt[5]{x}$

c $2x^{1/3} = 2\sqrt[3]{x}$

d $8a^{-1/4} = \dfrac{8}{\sqrt[4]{a}}$

□

Note 3.3.14 In Example 3.3.13, p. 336d, note that the exponent $-1/4$ applies only to a, not to $8a$.

Checkpoint 3.3.15

a Convert $\dfrac{3}{\sqrt[4]{2x}}$ to exponential notation.

b Convert $-5b^{0.125}$ to radical notation.

Answer.

a $3(2x)^{-1/4}$

b $-5\sqrt[8]{b}$

3.3.5 Using Fractional Exponents to Solve Equations

In Chapter 2, we learned that raising to powers and taking roots are inverse operations, that is, each operation undoes the effects of the other. This relationship is especially easy to see when the root is denoted by a fractional exponent. For example, to solve the equation

$$x^4 = 250$$

we would take the fourth root of each side. But instead of using radical notation, we can raise both sides of the equation to the power $\dfrac{1}{4}$:

$$\left(x^4\right)^{1/4} = 250^{1/4}$$
$$x \approx 3.98$$

The third law of exponents tells us that $(x^a)^b = x^{ab}$, so

$$\left(x^4\right)^{1/4} = x^{(1/4)(4)} = x^1$$

In general, to solve an equation involving a power function x^n, we first isolate the power, then raise both sides to the exponent $\frac{1}{n}$.

Example 3.3.16 For astronomers, the mass of a star is its most important property, but it is also the most difficult to measure directly. For many stars, their luminosity, or brightness, varies roughly as the fourth power of the mass.

a Our Sun has luminosity 4×10^{26} watts and mass 2×10^{30} kilograms. Because the numbers involved are so large, astronomers often use these solar constants as units of measure: The luminosity of the Sun is 1 solar luminosity, and its mass is 1 solar mass. Write a power function for the luminosity, L, of a star in terms of its mass, M, using units of solar mass and solar luminosity.

b The star Sirius is 23 times brighter than the Sun, so its luminosity is 23 solar luminosities. Estimate the mass of Sirius in units of solar mass.

Solution.

a Because L varies as the fourth power of M, we have

$$L = kM^4$$

Substituting the values of L and M for the Sun (namely, $L = 1$ and $M = 1$), we find

$$1 = k(1)^4$$

so $k = 1$ and $L = M^4$.

b We substitute the luminosity of Sirius, $L = 23$, to get

$$23 = M^4$$

To solve the equation for M, we raise both sides to the $\frac{1}{4}$ power.

$$\begin{aligned}(23)^{1/4} &= \left(M^4\right)^{1/4}\\ 2.1899 &= M\end{aligned}$$

The mass of Sirius is about 2.2 solar masses, or about 2.2 times the mass of the Sun.

□

Checkpoint 3.3.17 A spherical fish tank in the lobby of the Atlantis Hotel holds about 905 cubic feet of water. What is the radius of the fish tank?

Answer. About 6 feet

3.3.6 Power Functions

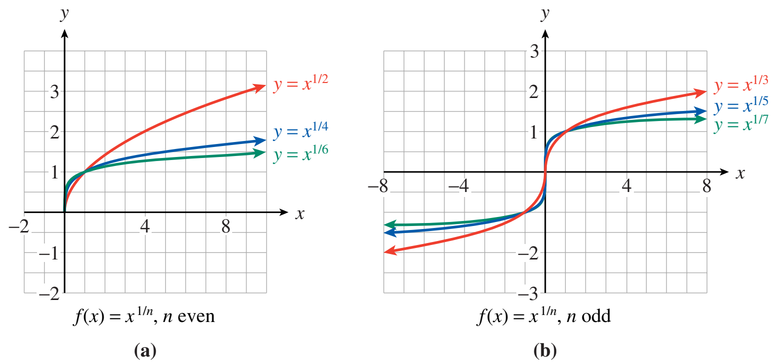
The basic functions $y = \sqrt{x}$ and $y = \sqrt[3]{x}$ are power functions of the form $f(x) = x^{1/n}$, and the graphs of all such functions have shapes similar to those two, depending on whether the index of the root is even or odd.

Figure (a) shows the graphs of

$$y = x^{1/2}, \quad y = x^{1/4}, \quad \text{and} \quad y = x^{1/6}$$

Figure (b) shows the graphs of

$$y = x^{1/3}, \quad y = x^{1/5}, \quad \text{and} \quad y = x^{1/7}$$



We cannot take an even root of a negative number. (See Subsection 3.3.8, p. 341 "A Note on Roots of Negative Numbers" at the end of this section.) Hence, if n is even, the domain of $f(x) = x^{1/n}$ is restricted to nonnegative real numbers, but if n is odd, the domain of $f(x) = x^{1/n}$ is the set of all real numbers.

We will also encounter power functions with negative exponents. For example, an animal's heart rate is related to its size or mass, with smaller animals generally having faster heart rates. The heart rates of mammals are given approximately by the power function

$$H(m) = km^{-1/4}$$

where m is the animal's mass and k is a constant.

Example 3.3.18 A typical human male weighs about 70 kilograms and has a resting heart rate of 70 beats per minute.

a Find the constant of proportionality, k, and write a formula for $H(m)$.

b Fill in the table with the heart rates of the mammals whose masses are given.

Animal	Shrew	Rabbit	Cat	Wolf	Horse	Polar bear	Elephant	Whale
Mass (kg)	0.004	2	4	80	300	600	5400	70,000
Heart rate								

c Sketch a graph of H for masses up to 6000 kilograms.

Solution.

a We substitute $H = 70$ and $m = 70$ into the equation; then solve for k.

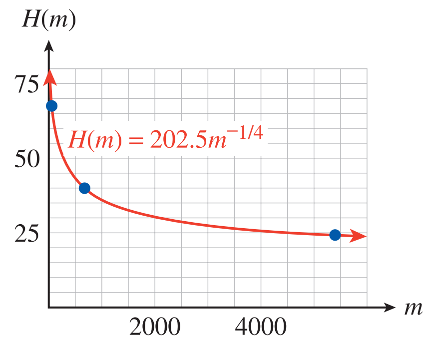
$$70 = k \cdot 70^{-1/4}$$
$$k = \frac{70}{70^{-1/4}} = 70^{5/4} \approx 202.5$$

Thus, $H(m) = 202.5m^{-1/4}$.

b We evaluate the function H for each of the masses given in the table.

Animal	Shrew	Rabbit	Cat	Wolf	Horse	Polar bear	Elephant	Whale
Mass (kg)	0.004	2	4	80	300	600	5400	70,000
Heart rate	805	170	143	68	49	41	24	12

c We plot the points in the table to obtain the graph shown below.



□

Many properties relating to the growth of plants and animals can be described by power functions of their mass. The study of the relationship between the growth rates of different parts of an organism, or of organisms of similar type, is called **allometry**. An equation of the form

$$\text{variable} = k(\text{mass})^p$$

used to describe such a relationship is called an **allometric equation**.

Of course, power functions can be expressed using any of the notations we have discussed. For example, the function in Example 3.3.18, p. 338 can be written as

$$H(m) = 202.5m^{-1/4} \quad \text{or} \quad H(m) = 202.5m^{-0.25} \quad \text{or} \quad H(m) = \frac{202.5}{\sqrt[4]{m}}$$

Checkpoint 3.3.19

a Complete the table of values for the power function $f(x) = x^{-1/2}$.

x	0.1	0.25	0.5	1	2	4	8	10	20	200
$f(x)$										

b Sketch the graph of $y = f(x)$.

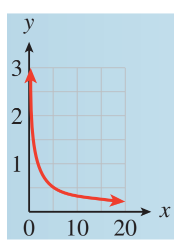
c Write the formula for $f(x)$ with a decimal exponent, and with radical notation.

Answer.

a

x	0.1	0.25	0.5	1	2	4	8	10	20	200
$f(x)$	3.2	2	1.4	1	0.71	0.5	0.35	0.32	0.22	0.1

b



c $f(x) = x^{-0.5}$, $f(x) = \dfrac{1}{\sqrt{x}}$

3.3.7 Solving Radical Equations

A **radical equation** is one in which the variable appears under a square root or other radical. The radical may be denoted by a fractional exponent. For example, the equation

$$5x^{1/3} = 32$$

is a radical equation because $x^{1/3} = \sqrt[3]{x}$. To solve the equation, we first isolate the power to get

$$x^{1/3} = 6.4$$

Then we raise both sides of the equation to the reciprocal of $\dfrac{1}{3}$, or 3.

$$\begin{aligned}\left(x^{1/3}\right)^3 &= 6.4^3\\ x &= 262.144\end{aligned}$$

Example 3.3.20 When a car brakes suddenly, its speed can be estimated from the length of the skid marks it leaves on the pavement. A formula for the car's speed, in miles per hour, is $v = f(d) = (24d)^{1/2}$, where the length of the skid marks, d, is given in feet.

a If a car leaves skid marks 80 feet long, how fast was the car traveling when the driver applied the brakes?

b How far will a car skid if its driver applies the brakes while traveling 80 miles per hour?

Solution.

a To find the velocity of the car, we evaluate the function for $d = \mathbf{80}$.

$$\begin{aligned}v &= (24\cdot \mathbf{80})^{1/2}\\ &= (1920)^{1/2} \approx 43.8178046\end{aligned}$$

The car was traveling at approximately 44 miles per hour.

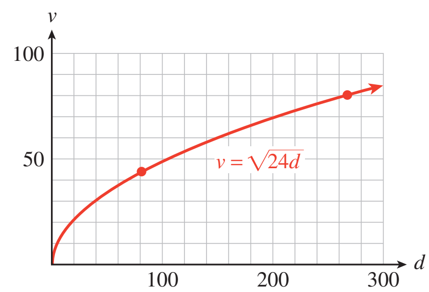
b We would like to find the value of d when the value of v is known. We substitute $v = \mathbf{80}$ into the formula and solve the equation

$$\mathbf{80} = (24d)^{1/2} \quad \text{Solve for } \boldsymbol{d}.$$

Because d appears to the power $\frac{1}{2}$, we first square both sides of the equation to get

$$\begin{aligned}80^2 &= \left((24d)^{1/2}\right)^2 && \text{Square both sides.}\\ 6400 &= 24d && \text{Divide by } \mathbf{24}.\\ 266.\overline{6} &= d\end{aligned}$$

You can check that this value for d works in the original equation. Thus, the car will skid approximately 267 feet. A graph of the function $v = (24d)^{1/2}$ is shown below, along with the points corresponding to the values in parts (a) and (b).



□

Note 3.3.21 Thus, we can solve an equation where one side is an nth root of x by raising both sides of the equation to the nth power. We must be careful when raising both sides of an equation to an even power, since extraneous solutions may be introduced. However, because most applications of power functions deal with positive domains only, they do not usually involve extraneous solutions.

Checkpoint 3.3.22 In Example 3.3.18, p. 338, we found the heart-rate function, $H(m) = 202.5m^{-1/4}$. What would be the mass of an animal whose heart rate is 120 beats per minute?

Answer. 81 kg

3.3.8 A Note on Roots of Negative Numbers

You already know that $\sqrt{-9}$ is not a real number, because there is no real number whose square is -9. Similarly, $\sqrt[4]{-16}$ is not a real number, because there is no real number r for which $r^4 = -16$. (Both of these radicals are **complex numbers**. Complex numbers are discussed in Chapter 7.) In general, we cannot find an even root (square root, fourth root, and so on) of a negative number.

On the other hand, every positive number has two even roots that are real numbers. For example, both 3 and -3 are square roots of 9. The symbol $\sqrt{9}$ refers only to the positive, or **principal root**, of 9. If we want to refer to the negative square root of 9, we must write $-\sqrt{9} = -3$. Similarly, both 2 and -2 are fourth roots of 16, because $2^4 = 16$ and $(-2)^4 = 16$. However, the symbol $\sqrt[4]{16}$ refers to the principal, or positive, fourth root only. Thus,

$$\sqrt[4]{16} = 2 \quad \text{and} \quad -\sqrt[4]{16} = -2$$

Things are simpler for odd roots (cube roots, fifth roots, and so on). Every real number, whether positive, negative, or zero, has exactly one real-valued odd root. For example,

$$\sqrt[5]{32} = 2 \quad \text{and} \quad \sqrt[5]{-32} = -2$$

Here is a summary of our discussion.

Roots of Real Numbers.

1. Every positive number has two real-valued roots, one positive and one negative, if the index is even.
2. A negative number has no real-valued root if the index is even.
3. Every real number, positive, negative, or zero, has exactly one real-valued root if the index is odd.

Example 3.3.23

a $\sqrt[4]{-625}$ is not a real number.

b $-\sqrt[4]{625} = -5$

c $\sqrt[5]{-1} = -1$

d $\sqrt[4]{-1}$ is not a real number.

□

The same principles apply to powers with fractional exponents. Thus

$$(-32)^{1/5} = -2$$

but $(-64)^{1/6}$ is not a real number. On the other hand,

$$-64^{1/6} = -2$$

because the exponent $1/6$ applies only to 64, and the negative sign is applied after the root is computed.

Checkpoint 3.3.24 Evaluate each power, if possible.

a $-81^{1/4}$

b $(-81)^{1/4}$

c $-64^{1/3}$

d $(-64)^{1/3}$

Answer.

a -3

b undefined

c -4

d -4

3.3.9 Section Summary

3.3.9.1 Vocabulary

Look up the definitions of new terms in the Glossary.

- nth root
- Radical
- Radical equation
- Radical notation
- Index
- Exponential notation
- Radicand
- Allometric equation
- Irrational number

3.3.9.2 CONCEPTS

1 nth roots: s is called an **nth root of** b if $s^n = b$.

2 Exponential notation: For any integer $n \geq 2$ and for $a \geq 0$, $a^{1/n} = \sqrt[n]{a}$.

3 We cannot write down an exact decimal equivalent for an **irrational number**, but we can approximate an irrational number to as many decimal places as we like.

4 We can solve the equation $x^n = b$ by raising both sides to the $\dfrac{1}{n}$ power.

5 An **allometric equation** is a power function of the form variable $=$ $k(\text{mass})^p$.

6 We can solve the equation $x^{1/n} = b$ by raising both sides to the nth power.

7 Roots of Real Numbers.

- Every positive number has two real-valued roots, one positive and one negative, if the index is even.
- A negative number has no real-valued root if the index is even.
- Every real number, positive, negative, or zero, has exactly one real-valued root if the index is odd.

3.3.9.3 STUDY QUESTIONS

1 Use an example to illustrate the terms radical, radicand, index, and principal root.

2 Explain why $x^{1/4}$ is a reasonable notation for $\sqrt[4]{x}$.

3 What does the notation $x^{0.2}$ mean?

4 Express each of the following algebraic notations in words; then evaluate each for = 16:

$$4x,\quad x^4,\quad \frac{x}{4},\quad \frac{1}{4}x,\quad x^{1/4},\quad x-4,\quad x^{-1/4}$$

5 How is the third law of exponents, $(xa)^b = x^{ab}$, useful in solving equations?

3.3.9.4 SKILLS

Practice each skill in the Homework 3.3.10, p. 343 problems listed.

1 Evaluate powers and roots: #1–8, 17–20

2 Convert between radical and exponential notation: #9–16, 21 and 22

3 Solve radical equations: #23–38, 59 and 60

4 Graph and analyze power functions: #39–58

5 Work with fractional exponents: #61–68

3.3.10 Roots and Radicals (Homework 3.3)

Find the indicated root without using a calculator; then check your answers.

1.

(a) $\sqrt{121}$ (b) $\sqrt[3]{27}$ (c) $\sqrt[4]{625}$

Answer.

(a) 11 (b) 3 (c) 5

2.

(a) $\sqrt{169}$ (b) $\sqrt[3]{64}$ (c) $\sqrt[4]{81}$

3.

(a) $\sqrt[5]{32}$ (b) $\sqrt[4]{16}$ (c) $\sqrt[3]{729}$

Answer.

(a) 2 (b) 2 (c) 9

4.

(a) $\sqrt[5]{100,000}$ (b) $\sqrt[4]{1296}$ (c) $\sqrt[3]{343}$

Find the indicated power without using a calculator; then check your answers.

5.

(a) $9^{1/2}$ (b) $81^{1/4}$ (c) $64^{1/6}$

Answer.

(a) 3 (b) 3 (c) 2

6.

(a) $25^{1/2}$ (b) $16^{1/4}$ (c) $27^{1/3}$

7.

(a) $32^{0.2}$ (b) $8^{-1/3}$ (c) $64^{-0.5}$

Answer.

(a) 2 (b) $\dfrac{1}{2}$ (c) $\dfrac{1}{8}$

8.

(a) $625^{0.25}$ (b) $243^{-1/5}$ (c) $49^{-0.5}$

Write each expression in radical form.

9.

(a) $3^{1/2}$ (b) $4x^{1/3}$ (c) $(4x)^{0.2}$

Answer.

(a) $\sqrt{3}$ (b) $4\sqrt[3]{x}$ (c) $\sqrt[5]{4x}$

10.

(a) $7^{1/2}$ (b) $3x^{1/4}$ (c) $(3x)^{0.25}$

11.

(a) $6^{-1/3}$ (b) $3(xy)^{-0.125}$ (c) $(x-2)^{1/4}$

Answer.

(a) $\dfrac{1}{\sqrt[3]{6}}$ (b) $\dfrac{3}{\sqrt[8]{xy}}$ (c) $\sqrt[4]{x-2}$

12.

(a) $8^{-1/4}$ (b) $y(5x)^{-0.5}$ (c) $(y+2)^{1/3}$

Write each expression in exponential form.

13.

(a) $\sqrt{7}$ (b) $\sqrt[3]{2x}$ (c) $2\sqrt[5]{z}$

Answer.

(a) $7^{1/2}$ (b) $(2x)^{1/3}$ (c) $2z^{1/5}$

14.

(a) $\sqrt{5}$ (b) $\sqrt[3]{4y}$ (c) $5\sqrt[3]{x}$

15.

(a) $\dfrac{-3}{\sqrt[4]{6}}$ (b) $\sqrt[4]{x-3y}$ (c) $\dfrac{-1}{\sqrt[5]{1+3b}}$

Answer.

(a) $-3 \cdot 6^{-1/4}$ (b) $(x-3y)^{1/4}$ (c) $-(1+3b)^{-1/5}$

16.

(a) $\dfrac{2}{\sqrt[5]{3}}$ (b) $\sqrt[3]{y+2x}$ (c) $\dfrac{-1}{\sqrt[4]{3a-2b}}$

Simplify.

17.

(a) $\left(\sqrt[3]{125}\right)^3$ (b) $\left(\sqrt[4]{2}\right)^4$ (c) $\left(3\sqrt{7}\right)^2$ (d) $\left(-x^2\sqrt[3]{2x}\right)^3$

Answer.

(a) 125 (b) 2 (c) 63 (d) $-2x^7$

18.

(a) $\left(\sqrt[4]{16}\right)^4$ (b) $\left(\sqrt[3]{6}\right)^3$ (c) $\left(2\sqrt[3]{12}\right)^3$ (d) $\left(-a^3\sqrt[4]{a^2}\right)^4$

Use a calculator to approximate each irrational number to the nearest thousandth.

19.

(a) $2^{1/2}$ (c) $\sqrt[4]{1.6}$ (e) $0.006^{-0.2}$

(b) $\sqrt[3]{75}$ (d) $365^{-1/3}$

Answer.

(a) 1.414 (b) 4.217 (c) 1.125 (d) 0.140 (e) 2.782

20.

(a) $3^{1/2}$ (c) $\sqrt[3]{1.4}$ (e) $1.05^{-0.1}$

(b) $\sqrt[4]{60}$ (d) $1058^{-1/5}$

Write each expression as a power function.

21.

(a) $g(x) = 3.7\sqrt[3]{x}$ (b) $H(x) = \sqrt[4]{85x}$ (c) $F(t) = \dfrac{25}{\sqrt[5]{t}}$

Answer.

(a) $g(x) = 3.7x^{1/3}$ (b) $H(x) = 85^{1/4}x^{1/4}$ (c) $F(t) = 25t^{-1/5}$

22.

(a) $h(v) = 12.7\sqrt{v}$ (b) $F(p) = \sqrt[3]{2.9p}$ (c) $G(w) = \dfrac{5}{8\sqrt[8]{w}}$

Solve.

23. $6.5x^{1/3} + 3.8 = 33.05$

Answer. $x = 91.125$

24. $9.8 - 76x^{1/4} + 15 = 9.6$

25. $4(x+2)^{1/5} = 12$

Answer. $x = 241$

26. $-9(x-3)^{1/5} = 18$

27. $(2x-3)^{-1/4} = \dfrac{1}{2}$

Answer. $x = \dfrac{19}{2}$

28. $(5x+2)^{-1/3} = \dfrac{1}{4}$

29. $\sqrt[3]{x^2-3} = 3$

Answer. $x = \pm\sqrt{30}$

30. $\sqrt[4]{x^3-7} = 2$

Solve each formula for the indicated variable.

31. $T = 2\pi\sqrt{\dfrac{L}{g}}$ for L. Also solve for g.

Answer. $L = \dfrac{gT^2}{4\pi^2}$

32. $T = 2\pi\sqrt{\dfrac{m}{k}}$ for m

33. $r = \sqrt{t^2-s^2}$ for s. Also solve for t.

Answer. $s = \pm\sqrt{t^2-r^2}$

34. $c = \sqrt{a^2-b^2}$ for b

35. $r = \sqrt[3]{\dfrac{3V}{4\pi}}$ for V

Answer. $v = \dfrac{4}{3}\pi r^3$

36. $d = \sqrt[3]{\dfrac{16Mr^2}{m}}$ for M

37. $R = \sqrt[4]{\dfrac{8Lvf}{\pi p}}$ for p

Answer. $p = \dfrac{8Lvf}{\pi R^4}$

36. $T = \sqrt[4]{\dfrac{E}{SA}}$ for A

39. The period of a pendulum is the time it takes for the pendulum to complete one entire swing, from left to right and back again. The greater the length, L, of the pendulum, the longer its period, T. In fact, if L is measured in feet, then the period is given in seconds by

$$T = 2\pi\sqrt{\frac{L}{32}}$$

(a) Write the formula for T as a power function in the form $f(x) = kx^p$.

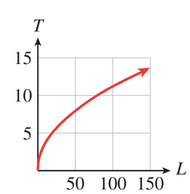
(b) Suppose you are standing in the Convention Center in Portland, Oregon, and you time the period of its Foucault pendulum (the longest in the world). Its period is approximately 10.54 seconds. How long is the pendulum?

(c) Choose a reasonable domain for the function $T = f(L)$ and graph the function.

Answer.

(a) $T = \dfrac{2\pi}{\sqrt{32}}L^{1/2}$

(b) 90 feet

(c) 

40. If you are flying in an airplane at an altitude of h miles, on a clear day you can see a distance of d miles to the horizon, where

$$d = \sqrt{7920h}.$$

(a) Write the formula for d as a power function in the form $f(x) = kx^p$.

(b) Choose a reasonable domain for the function $d = f(h)$ and graph the function.

(c) At what altitude will you be able to see for a distance of 100 miles? How high is that in feet?

41. If you walk in the normal way, your maximum speed, v, in meters per second, is limited by the length of your legs, r, according to the formula

$$v = \sqrt{gr}$$

where the constant g is approximately 10 meters per second squared. (Source: Alexander, 1992)

(a) A typical adult man has legs about 0.9 meter long. How fast can he walk?

(b) A typical four-year-old has legs 0.5 meter long. How fast can she walk?

(c) Graph maximum walking speed as a function of leg length.

(d) Race-walkers can walk as fast as 4.4 meters per second by rotating their hips so that the effective length of their legs is increased. What is that effective length?

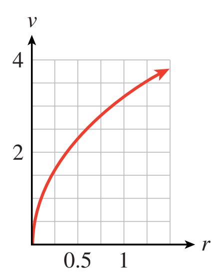
(e) On the Moon the value of g is 1.6 meters per second squared. How fast can a typical adult man walk on the Moon?

Answer.

(a) 3 meters per second

(b) $b \approx 2.2$ meters per second

(c)



(d) 1.9 meters

(e) 1.2meters per second

42. When a ship moves through the water, it creates waves that impede its own progress. Because of this resistance, there is an upper limit to the speed at which a ship can travel, given, in knots, by

$$v_{\text{max}} = 1.3\sqrt{L}$$

where L is the length of the vessel, in feet. (Source: Gilner, 1972)

(a) Graph maximum speed as a function of vessel length.

(b) The world's largest ship, the oil tanker *Jahre Viking*, is 1054 feet long. What is its top speed?

(c) As a ship approaches its maximum speed, the power required increases sharply. Therefore, most merchant ships are designed to cruise at speeds no higher than $v_c = 0.8\sqrt{L}$. Graph v_c on the same axes with $v_{\max}$.

(d) What is the cruising speed of the *Jahre Viking*? What percent of its maximum speed is that?

43. A rough estimate for the radius of the nucleus of an atom is provided by the formula

$$r = kA^{1/3}$$

where A is the mass number of the nucleus and $k \approx 1.3 \times 10^{-13}$ centimeter.

(a) Estimate the radius of the nucleus of an atom of iodine-127, which has mass number 127. If the nucleus is roughly spherical, what is its volume?

(b) The nuclear mass of iodine-127 is 2.1×10^{-22} gram. What is the density of the nucleus? (Density is mass per unit volume.)

(c) Complete the table of values for the radii of various radioisotopes.

Element	Carbon	Potassium	Cobalt	Technetium	Radium
Mass number, A	14	40	60	99	226
Radius, r					

(d) Sketch a graph of r as a function of A. (Use units of 10^{-13} centimeter on the vertical axis.)

Answer.

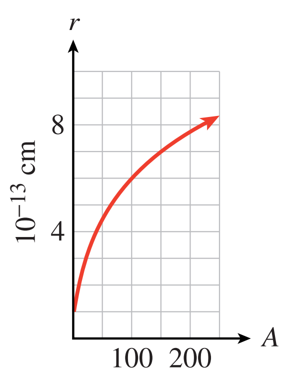
(a) 6.5×10^{-13} cm; 1.17×10^{-36} cm^3

(b) $1.8 \times 10^{14} g/\text{cm}^3$

(c)

Element	Carbon	Potassium	Cobalt	Technetium	Radium
Mass number, A	14	40	60	99	226
Radius, r (10^{-13} cm)	3.1	4.4	5.1	6	7.9

(d)



44. In the sport of crew racing, the best times vary closely with the number of men in the crew, according to the formula

$$t = kn^{-1/9}$$

where n is the number of men in the crew and t is the winning time, in minutes, for a 2000-meter race.

(a) If the winning time for the 8-man crew was 5.73 minutes, estimate the value of k.

(b) Complete the table of values of predicted winning times for the other racing classes.

Size of crew, n	1	2	4	8
Winning time, t				

(c) Sketch a graph of t as a function of n.

In Problems 45–48, one quantity varies directly with the square root of the other, that is, $y = k\sqrt{x}$.

a Find the value of k and write a power function relating the variables.

b Use your function to answer the question.

c Graph your function and verify your answer to part (b) on the graph.

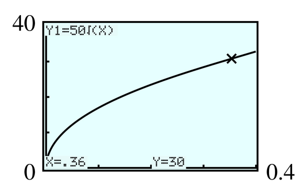
45. The stream speed necessary to move a granite particle is a function of the diameter of the particle; faster river currents can move larger particles. The table shows the stream speed necessary to move particles of different sizes. What speed is needed to carry a particle with diameter 0.36 centimeter?

Diameter, d (cm)	Speed, s (cm/sec)
0.01	5
0.04	10
0.09	15
0.16	20

Answer.

(a) $s = 50\sqrt{d}$

(c)



(b) 30 cm/sec

46. The speed at which water comes out of the spigot at the bottom of a water jug is a function of the water level in the jug; it slows down as the water level drops. The table shows different water levels and the resulting flow speeds. What is the flow speed when the water level is at 16 inches?

Level, L (in)	Speed, s (gal/min)
9	1.5
6.25	1.25
4	1
2.25	0.75

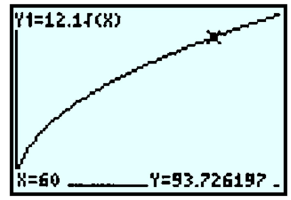
47. The rate, r, in feet per second, at which water flows from a fire hose is a function of the water pressure, P, in psi (pounds per square inch). What is the rate of water flow at a typical water pressure of 60 psi?

P (psi)	10	20	30	40
r (ft/sec)	38.3	54.1	66.3	76.5

Answer.

(a) $r = 12.1\sqrt{P}$

(b) 94 ft/sec

(c) 

48. When a layer of ice forms on a pond, the thickness of the ice, d, in centimeters, is a function of time, t, in minutes. How thick is the ice after 3 hours?

t (min)	10	30	40	60
d (cm)	0.50	0.87	1.01	1.24

49. Membership in the County Museum has been increasing since it was built in 1980. The number of members is given by the function

$$M(t) = 72 + 100t^{1/3}$$

where t is the number of years since 1980.

(a) How many members were there in 1990? In 2000?

(b) In what year will the museum have 400 members? If the membership continues to grow according to the given function, when will the museum have 500 members?

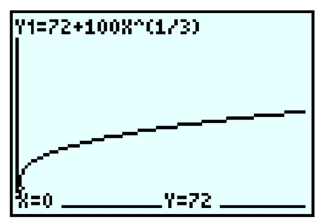
(c) Graph the function $M(t)$. How would you describe the growth of the membership over time?

Answer.

(a) 287; 343

(b) 2015; 2058

(c) The membership grows rapidly at first but is growing less rapidly with time.



50. Due to improvements in technology, the annual electricity cost of running most major appliances has decreased steadily since 1970. The average annual cost of running a refrigerator is given, in dollars, by the function

$$C(t) = 148 - 28t^{1/3}$$

where t is the number of years since 1970.

(a) How much did it cost to run a refrigerator in 1980? In 1990?

(b) When was the cost of running a refrigerator half of the cost in 1970? If the cost continues to decline according to the given function, when will it cost \$50 per year to run a refrigerator?

(c) Graph the function $C(t)$. Do you think that the cost will continue to decline indefinitely according to the given function? Why or why not?

51. Match each function with the description of its graph in the first quadrant.

I $f(x) = x^2$

II $f(x) = x^{-2}$

III $f(x) = x^{1/2}$

(a) Increasing and concave up

(b) Increasing and concave down

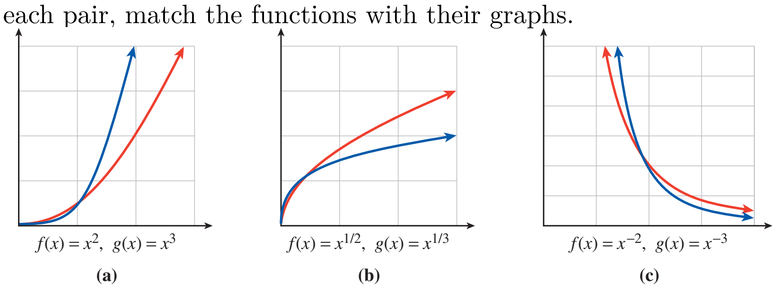
(c) Decreasing and concave up

(d) Decreasing and concave down

Answer.

(a) I (b) III (c) II (d) none

52. In each pair, match the functions with their graphs.



53.

(a) Graph the functions

$$y_1 = x^{1/2}, \quad y_2 = x^{1/3}, \quad y_3 = x^{1/4}, \quad y_4 = x^{1/5}$$

in the window

$$\begin{array}{ll} \text{Xmin} = 0 & \text{Xmax} = 100 \\ \text{Ymin} = 0 & \text{Ymax} = 10 \end{array}$$

What do you observe?

(b) Use your graphs to evaluate $100^{1/2}$, $100^{1/3}$, $100^{1/4}$, and $100^{1/5}$.

(c) Use your calculator to evaluate $100^{1/n}$ for $n = 10$, $n = 100$, and $n = 1000$. What happens when n gets large?

Answer.

(a) The graphs of $x^{1/n}$ become closer and closer to horizontal when n increases (for $x > 1$).

(b) 10, 4.64, 3.16, 2.51

(c) 1.58, 1.05, 1.005; the values decrease towards 1.

54.

(a) Graph the functions

$$y_1 = x^{1/2}, \quad y_2 = x^{1/3}, \quad y_3 = x^{1/4}, \quad y_4 = x^{1/5}$$

in the window

$$\begin{array}{ll} \text{Xmin} = 0 & \text{Xmax} = 1 \\ \text{Ymin} = 0 & \text{Ymax} = 1 \end{array}$$

What do you observe?

(b) Use your graphs to evaluate $0.5^{1/2}$, $0.5^{1/3}$, $0.5^{1/4}$, and $0.5^{1/5}$.

(c) Use your calculator to evaluate $0.5^{1/n}$ for $n = 10$, $n = 100$, and $n = 1000$. What happens when n gets large?

For Problems 55–58, graph each set of functions in the given window. What do you observe?

55. $y_1 = \sqrt{x}$, $y_2 = x^2$, $y_3 = x$

Xmin $= 0$ Xmax $= 4$
Ymin $= 0$ Ymax $= 4$

Answer. The graphs of y_1 and y_2 are symmetric about $y_3 = x$.

56. $y_1 = \sqrt[3]{x}$, $y_2 = x^3$, $y_3 = x$

Xmin $= -4$ Xmax $= 4$
Ymin $= -4$ Ymax $= 4$

57. $y_1 = \sqrt[5]{x}$, $y_2 = x^5$, $y_3 = x$

Xmin $= -2$ Xmax $= 2$
Ymin $= -2$ Ymax $= 2$

Answer. The graphs of y_1 and y_2 are symmetric about $y_3 = x$.

58. $y_1 = \sqrt[4]{x}$, $y_2 = x^4$, $y_3 = x$

Xmin $= 0$ Xmax $= 2$
Ymin $= 0$ Ymax $= 2$

59.

(a) Graph the functions $f(x) = 4\sqrt[3]{x-9}$ and $g(x) = 12$ in the window

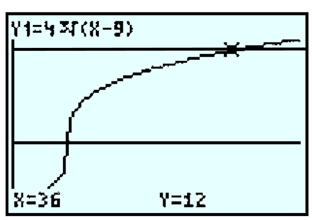
Xmin $= 0$ Xmax $= 47$
Ymin $= -8$ Ymax $= 16$

(b) Use the graph to solve the equation $4\sqrt[3]{x-9} = 12$.

(c) Solve the equation algebraically.

Answer.

(a)



(b) $x = 36$

60.

(a) Graph the functions $f(x) = 6 + 2\sqrt[4]{12-x}$ and $g(x) = 10$ in the window

Xmin $= -27$ Xmax $= 20$
Ymin $= 4$ Ymax $= 12$

(b) Use the graph to solve the equation $6 + 2\sqrt[4]{12-x} = 10$.

(c) Solve the equation algebraically.

61.

(a) Write $\sqrt{x}$ with a fractional exponent.

(b) Write $\sqrt{\sqrt{x}}$ with a fractional exponents.

(c) Use the laws of exponents to show that $\sqrt{\sqrt{x}} = \sqrt[4]{x}$.

Answer.

(a) $x^{1/2}$

(b) $\left(x^{1/2}\right)^{1/2}$

(c)

$$\begin{aligned}\sqrt{\sqrt{x}} &= \left(x^{1/2}\right)^{1/2} && \text{By definition of fractional exponents.}\\ &= x^{1/4} && \text{By the third law of exponents.}\\ &= \sqrt[4]{x} && \text{By definition of fractional exponents.}\end{aligned}$$

62.

(a) Write $\sqrt[3]{x}$ with a fractional exponent.

(b) Write $\sqrt{\sqrt[3]{x}}$ with a fractional exponents.

(c) Use the laws of exponents to show that $\sqrt{\sqrt[3]{x}} = \sqrt[6]{x}$.

Write each expression as a sum of terms of the form kx^p.

63. $\dfrac{\sqrt{x}}{4} - \dfrac{2}{\sqrt{x}} + \dfrac{x}{\sqrt{2}}$

Answer.
$\dfrac{1}{4}x^{1/2} - 2x^{-1/2} + \dfrac{1}{\sqrt{2}}x$

64. $\dfrac{\sqrt{3}}{x} + \dfrac{3}{\sqrt{x}} - \dfrac{\sqrt{x}}{3}$

65. $\dfrac{6 - \sqrt[3]{x}}{2\sqrt[3]{x}}$

Answer. $3x^{-1/3} - \dfrac{1}{2}$

66. $\dfrac{\sqrt[4]{x} + 2}{2\sqrt[4]{x}}$

67. $x^{-0.5}\left(x + x^{0.25} - x^{0.5}\right)$

Answer. $x^{0.5} + x^{-0.25} - x^0$

68. $x^{0.5}\left(x^{-1} + x^{-0.5} + x^{-0.25}\right)$

3.4 Rational Exponents

3.4.1 Powers of the Form $a^{m/n}$

In the last section, we considered powers of the form $a^{1/n}$, such as $x^{1/3}$ and $x^{-1/4}$, and saw that $a^{1/n}$ is equivalent to the root $\sqrt[n]{a}$. What about other fractional exponents? What meaning can we attach to a power of the form $a^{m/n}$?

Consider the power $x^{3/2}$. Notice that the exponent $\dfrac{3}{2} = 3(\dfrac{1}{2})$, and thus by the third law of exponents, we can write

$$(x^{1/2})^3 = x^{(1/2)^3} = x^{3/2}$$

In other words, we can compute $x^{3/2}$ by first taking the square root of x and then cubing the result. For example,

$$\begin{aligned} 100^{3/2} &= (\mathbf{100^{1/2}})^3 && \text{Take the square root of 100.} \\ &= \mathbf{10^3} = 1000 && \text{Cube the result.} \end{aligned}$$

We will define fractional powers only when the base is a positive number.

Rational Exponents.

$$a^{m/n} = (a^{1/n})^m = (a^m)^{1/n}, \qquad a > 0,\;\; n \neq 0$$

To compute $a^{m/n}$, we can compute the nth root first, or the mth power, whichever is easier. For example,

$$8^{2/3} = \left(8^{1/3}\right)^2 = 2^2 = 4$$

or

$$8^{2/3} = \left(8^2\right)^{1/3} = 64^{1/3} = 4$$

Example 3.4.1

a

$$\begin{aligned} 81^{3/4} &= \left(81^{1/4}\right)^3 \\ &= 3^3 = 27 \end{aligned}$$

c

$$\begin{aligned} 27^{-2/3} &= \frac{1}{\left(27^{1/3}\right)^2} \\ &= \frac{1}{3^2} = \frac{1}{9} \end{aligned}$$

b

$$\begin{aligned} -27^{5/3} &= -\left(27^{1/3}\right)^5 \\ &= -3^5 = -243 \end{aligned}$$

d

$$\begin{aligned} 5^{3/2} &= \left(5^{1/2}\right)^3 \\ &\approx (2.236)^3 \approx 11.180 \end{aligned}$$

□

Note 3.4.2 You can verify all the calculations in Example 3.4.1, p. 354 on your calculator. For example, to evaluate $81^{3/4}$, key in

81 ^ (3 ÷ 4) ENTER

or simply

81 ^ 0.75 ENTER

Checkpoint 3.4.3 Evaluate each power.

a $32^{-3/5}$

b $-81^{1.25}$

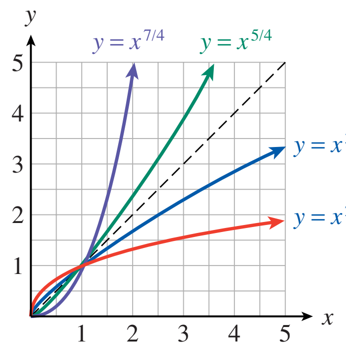
Answer.

a $\dfrac{1}{8}$

b -243

3.4.2 Power Functions

The graphs of power functions $y = x^{m/n}$, where m/n is positive, are all increasing for $x \geq 0$. If $m/n > 1$, the graph is concave up. If $0 < m/n < 1$, the graph is concave down. Some examples are shown below.



Perhaps the single most useful piece of information a scientist can have about an animal is its metabolic rate. The metabolic rate is the amount of energy the animal uses per unit of time for its usual activities, including locomotion, growth, and reproduction. The basal metabolic rate, or BMR, sometimes called the resting metabolic rate, is the minimum amount of energy the animal can expend in order to survive.

Example 3.4.4 A revised form of Kleiber's rule states that the basal metabolic rate for many groups of animals is given by

$$B(m) = 70m^{0.75}$$

where m is the mass of the animal in kilograms and the BMR is measured in kilocalories per day.

a Calculate the BMR for various animals whose masses are given in the table.

Animal	Bat	Squirrel	Raccoon	Lynx	Human	Moose	Rhinoceros
Weight (kg)	0.1	0.6	8	30	70	360	3500
BMR (kcal/day)							

b Sketch a graph of Kleiber's rule for $0 < m \leq 400$.

c Do larger species eat more or less, relative to their body mass, than smaller ones?

Solution.

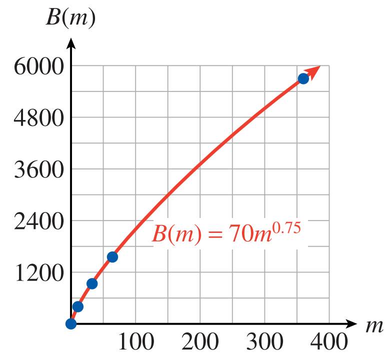
a We evaluate the function for the values of m given. For example, to calculate the BMR of a bat, we compute

$$B(0.1) = 70(0.1)^{0.75} = 12.1$$

A bat expends, and hence must consume, at least 12 kilocalories per day. We evaluate the function to complete the rest of the table.

Animal	Bat	Squirrel	Raccoon	Lynx	Human	Moose	Rhinoceros
Weight (kg)	0.1	0.6	8	30	70	360	3500
BMR (kcal/day)	12	48	333	897	1694	5785	$31,853$

b We plot the data from the table to obtain the graph below.



c If energy consumption were proportional to body weight, the graph would be a straight line. But because the exponent in Kleiber's rule, $\dfrac{3}{4}$, is less than 1, the graph is concave down, or bends downward. Therefore, larger species eat less than smaller ones, relative to their body weight.

□

Checkpoint 3.4.5

a Complete the table of values for the function $f(x) = x^{-3/4}$.

x	0.1	0.2	0.5	1	2	5	8	10
$f(x)$								

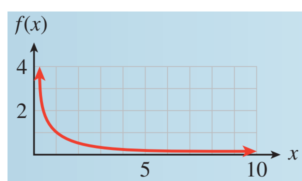
b Sketch the graph of the function.

Answer.

a

x	0.1	0.2	0.5	1	2	5	8	10
$f(x)$	5.623	3.344	1.682	1	0.595	0.299	0.210	0.178

b



3.4.3 More about Scaling

In Example 3.4.4, p. 355 we saw that large animals eat less than smaller ones, relative to their body weight. This is because the scaling exponent in Kleiber's rule is less than 1. For example, let s represent the mass of a squirrel. The mass of a moose is then $600s$, and its metabolic rate is

$$\begin{aligned} B(600s) &= 70(600s)^{0.75} \\ &= 600^{0.75} \cdot 70s^{0.75} = 121B(s) \end{aligned}$$

or 121 times the metabolic rate of the squirrel. Metabolic rate scales as $k^{0.75}$, compared to the mass of the animal.

In a famous experiment in the 1960s, an elephant was given LSD. The dose was determined from a previous experiment in which a 2.6-kg cat was given 0.26 gram of LSD. Because the elephant weighed 2970 kg, the experimenters used a direct proportion to calculate the dose for the elephant:

$$\frac{0.26 \text{ g}}{2.6 \text{ kg}} = \frac{x \text{ g}}{2970 \text{ kg}}$$

and arrived at the figure 297 g of LSD. Unfortunately, the elephant did not survive the experiment.

Example 3.4.6 Use Kleiber's rule and the dosage for a cat to estimate the corresponding dose for an elephant.

Solution. If the experimenters had taken into account the scaling exponent of 0.75 in metabolic rate, they would have used a smaller dose. Because the elephant weighs $\dfrac{2970}{2.6}$, or about 1142 times as much as the cat, the dose would be $1142^{0.75} = 196$ times the dosage for a cat, or about 51 grams. □

Checkpoint 3.4.7 A human being weighs about 70 kg, and 0.2 mg of LSD is enough to induce severe psychotic symptoms. Use these data and Kleiber's rule to predict what dosage would produce a similar effect in an elephant.

Answer. About 3.3 mg

3.4.4 Radical Notation

Because $a^{1/n} = \sqrt[n]{a}$, we can write any power with a fractional exponent in radical form as follows.

Rational Exponents and Radicals.

$$a^{m/n} = \sqrt[n]{a^m} = \left(\sqrt[n]{a}\right)^m$$

Example 3.4.8

a $125^{4/3} = \sqrt[3]{125^4}$ or $\left(\sqrt[3]{125}\right)^4$

b $x^{0.4} = x^{2/5} = \sqrt[5]{x^2}$

c $6w^{-3/4} = \dfrac{6}{\sqrt[4]{w^3}}$

□

Checkpoint 3.4.9 Write each expression in radical notation.

a $5t^{1.25}$

b $3m^{-5/3}$

Answer.

a $5\sqrt[4]{t^5}$

b $\dfrac{3}{\sqrt[3]{m^5}}$

Usually, we will want to convert from radical notation to fractional exponents, since exponential notation is easier to use.

Example 3.4.10

a $\sqrt{x^5} = x^{5/2}$

b $5\sqrt[4]{p^3} = 5p^{3/4}$

c $\dfrac{3}{\sqrt[5]{t^2}} = 3t^{-2/5}$

d $\sqrt[3]{2y^2} = \left(2y^2\right)^{1/3} = 2^{1/3}y^{2/3}$

□

Checkpoint 3.4.11 Convert to exponential notation.

a $\sqrt[3]{6w^2}$

b $\sqrt[4]{\dfrac{v^3}{s^5}}$

Answer.

a $6^{1/3}w^{2/3}$

b $v^{3/4}s^{-5/4}$

3.4.5 Operations with Rational Exponents

Powers with rational exponents -- positive, negative, or zero -- obey the laws of exponents, which we discussed in Section 3.1, p. 291. You may want to review those laws before studying the following examples.

Example 3.4.12

a

$$\frac{7^{0.75}}{7^{0.5}} = 7^{0.75-0.5} = 7^{0.25} \quad \text{Apply the second law of exponents.}$$

b

$$\begin{aligned} v \cdot v^{-2/3} &= v^{1+(-2/3)} && \text{Apply the first law of exponents.} \\ &= v^{1/3} \end{aligned}$$

c

$$\left(x^8\right)^{0.5} = x^{8(0.5))} = x^4 \quad \text{Apply the third law of exponents.}$$

d

$$\begin{aligned} \frac{\left(5^{1/2}y^2\right)^2}{\left(5^{2/3}y\right)^3} &= \frac{5y^4}{5^2y^3} && \text{Apply the fourth law of exponents.} \\ &= \frac{y^{4-3}}{5^{2-1}} = \frac{y}{5} && \text{Apply the second law of exponents.} \end{aligned}$$

□

Checkpoint 3.4.13 Simplify by applying the laws of exponents.

a $x^{1/3}\left(x + x^{2/3}\right)$

b $\dfrac{n^{9/4}}{4n^{3/4}}$

Answer.

a $x^{4/3} + x$

b $\dfrac{n^{3/2}}{4}$

3.4.6 Solving Equations

According to the third law of exponents, when we raise a power to another power, we multiply the exponents together. In particular, if the two exponents are reciprocals, then their product is 1. For example,

$$\left(x^{2/3}\right)^{3/2} = x^{(2/3)(3/2)} = x^1 = x$$

This observation can help us to solve equations involving fractional exponents. For instance, to solve the equation

$$x^{2/3} = 4$$

we raise both sides of the equation to the reciprocal power, 3/2. This gives us

$$\begin{aligned} \left(x^{2/3}\right)^{3/2} &= 4^{3/2} \\ x &= 8 \end{aligned}$$

The solution is 8.

Example 3.4.14 Solve $(2x+1)^{3/4} = 27$

Solution. We raise both sides of the equation to the reciprocal power, $\dfrac{4}{3}$.

$$\begin{aligned} \left[(2x+1)^{3/4}\right]^{4/3} &= 27^{4/3} && \text{Apply the third law of exponents.} \\ 2x+1 &= 81 && \text{Solve as usual.} \\ x &= 40 \end{aligned}$$

□

Checkpoint 3.4.15 Solve the equation $3.2z^{0.6} - 9.7 = 8.7$. Round your answer to two decimal places.

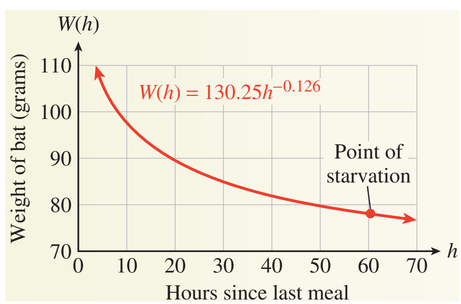
Hint. Isolate the power.
Raise both sides to the reciprocal power.

Answer. 18.45

Investigation 14 Vampire Bats. Small animals such as bats cannot survive for long without eating. The graph below shows how the weight, W, of a typical vampire bat decreases over time until its next meal, until the bat reaches the point of starvation. The curve is the graph of the function

$$W(h) = 130.25h^{-0.126}$$

where h is the number of hours since the bat's most recent meal. (Source: Wilkinson, 1984)



1. Use the graph to estimate answers to the following questions: How long can the bat survive after eating until its next meal? What is the bat's weight at the point of starvation?

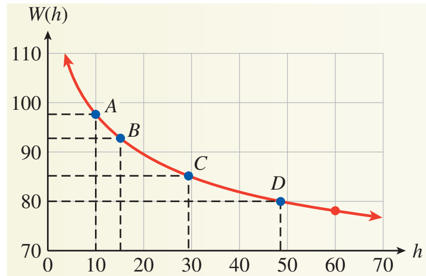
2. Use the formula for $W(h)$ to verify your answers.

3. Write and solve an equation to answer the question: When the bat's weight has dropped to 90 grams, how long can it survive before eating again?

4. Complete the table showing the number of hours since the bat last ate when its weight has dropped to the given values.

Weight (grams)	97.5	92.5	85	80
Hours since eating				
Point on graph	A	B	C	D

5. Compute the slope of the line segments from point A to point B, and from point C to point D. Include units in your answers.



6. What happens to the slope of the curve as h increases? What does this tell you about the concavity of the curve?

7. Suppose a bat that weighs 80 grams consumes 5 grams of blood. How many hours of life does it gain? Suppose a bat that weighs 97.5 grams gives up a meal of 5 grams of blood. How many hours of life does it forfeit?

8. Vampire bats sometimes donate blood (through regurgitation) to other bats that are close to starvation. Suppose a bat at point A on the curve donates 5 grams of blood to a bat at point D. Explain why this strategy is effective for the survival of the bat community.

3.4.7 Section Summary

3.4.7.1 Vocabulary

Look up the definitions of new terms in the Glossary.

- Rational exponent

3.4.7.2 CONCEPTS

1 Rational exponents $a^{m/n} = \left(a^{1/n}\right)^m = (a^m)^{1/n}, \quad a > 0, \quad n \neq 0.$

2 To compute $a^{m/n}$, we can compute the nth root first, or the mth power, whichever is easier.

3 The graphs of power functions $y = x^{m/n}$, where m/n is positive, are all increasing for $x \geq 0$. If $m/n > 1$, the graph is concave up. If $0 < m/n < 1$, the graph is concave down.

4 Radical notation: $a^{m/n} = \sqrt[n]{a^m} = \left(\sqrt[n]{a}\right)^m$.

5 Powers with rational exponents—positive, negative, or zero—obey the laws of exponents.

6 To solve the equation $x^{m/n} = k$, we raise both sides to the power n/m.

3.4.7.3 STUDY QUESTIONS

1 What does the notation $a^{0.98}$ mean?

2 Explain how to evaluate the function $f(x) = x^{-3/4}$ for $x = 625$, without using a calculator.

3 Explain why $x\sqrt{x} = x^{1.5}$.

4 What is the first step in solving the equation $(x - 2)^{-5/2} = 1.8$?

5 If the graph of $f(x) = x^{a/b}$ is concave down, and $a/b > 0$, what else can you say about a/b?

3.4.7.4 SKILLS

Practice each skill in the Homework 3.4.8, p. 361 problems listed.

1 Simplify and evaluate powers with rational exponents: #1–4, 13–18

2 Graph power functions with rational exponents: #19–22

3 Solve radical equations: #23–38, 59 and 60

4 Analyze power functions with rational exponents: #23–36

5 Simplify expressions using the laws of exponents: #37–44, 57–70

6 Solve equations involving rational exponents: #45–56

3.4.8 Rational Exponents (Homework 3.4)

For the problems in Homework 3.4, assume that all variables represent positive numbers.

Evaluate each power in Problems 1–4.

1.

(a) $81^{3/4}$ (b) $125^{2/3}$ (c) $625^{0.75}$

Answer.

(a) 27 (b) 25 (c) 125

2.

(a) $-8^{2/3}$ (b) $-64^{2/3}$ (c) $243^{0.4}$

3.

(a) $16^{-3/2}$ (b) $8^{-4/3}$ (c) $32^{-1.6}$

Answer.

(a) $\dfrac{1}{64}$ (b) $\dfrac{1}{16}$ (c) $\dfrac{1}{256}$

4.

(a) $-125^{-4/3}$ (b) $-32^{-3/5}$ (c) $100^{-2.5}$

For Problems 5–8, write each power in radical form.

5.

(a) $x^{4/5}$ (b) $b^{-5/6}$ (c) $(pq)^{-2/3}$

Answer.

(a) $\sqrt[5]{x^4}$ (b) $\dfrac{1}{\sqrt[6]{b^5}}$ (c) $\dfrac{1}{\sqrt[3]{(pq)^2}}$

6.

(a) $y^{3/4}$ (b) $a^{-2/7}$ (c) $(st)^{-3/5}$

7.

(a) $3x^{0.4}$ (b) $4z^{-4/3}$ (c) $-2x^{0.25}y^{0.75}$

Answer.

(a) $3\sqrt[5]{x^2}$ (b) $\dfrac{4}{\sqrt[3]{z^4}}$ (c) $-2\sqrt[4]{xy^3}$

8.

(a) $5y^{2/3}$ (b) $6w^{-1.5}$ (c) $-3x^{0.4}y^{0.6}$

For Problems 9–12, write each expression with fractional exponents.

9.

(a) $\sqrt[3]{x^2}$ (b) $2\sqrt[5]{ab^3}$ (c) $\dfrac{-4m}{\sqrt[6]{p^7}}$

Answer.

(a) $x^{2/3}$ (b) $2a^{1/5}b^{3/5}$ (c) $-4mp^{-7/6}$

10.

(a) $\sqrt{y^3}$ (b) $6\sqrt[5]{(ab)^3}$ (c) $\dfrac{-2n}{\sqrt[8]{q^{11}}}$

11.

(a) $\sqrt[3]{(ab)^2}$ (b) $\dfrac{8}{\sqrt[4]{x^3}}$ (c) $\dfrac{R}{3\sqrt{TK^5}}$

Answer.

(a) $(ab)^{2/3}$ (b) $8x^{-3/4}$ (c) $\dfrac{1}{3}RT^{-1/2}K^{-5/2}$

12.

(a) $\sqrt[3]{ab^2}$ (b) $\dfrac{5}{\sqrt[3]{y^2}}$ (c) $\dfrac{S}{4\sqrt{VH^3}}$

For Problems 13–16, evaluate each root without using a calculator.

13.

(a) $\sqrt[5]{32^3}$ (b) $-\sqrt[3]{27^4}$ (c) $\sqrt[4]{16y^{12}}$

Answer.

(a) 8 (b) -81 (c) $2y^3$

14.

(a) $\sqrt[4]{16^5}$ (b) $-\sqrt[3]{125^2}$ (c) $\sqrt[5]{243x^{10}}$

15.

(a) $-\sqrt{a^8b^{16}}$ (b) $\sqrt[3]{8x^9y^{27}}$ (c) $-\sqrt[4]{81a^8b^{12}}$

Answer.

(a) $-a^4b^8$ (b) $2x^3y^9$ (c) $-3a^2b^3$

16.

(a) $-\sqrt{a^{10}b^{35}}$ (b) $\sqrt[3]{64x^6y^{18}}$ (c) $\sqrt[5]{32x^{25}y^5}$

For Problems 17–18, use a calculator to approximate each power or root to the nearest thousandth.

17.

(a) $12^{5/6}$ (b) $\sqrt[3]{6^4}$ (c) $37^{-2/3}$ (d) $4.7^{2.3}$

Answer.

(a) 7.931

(b) 10.903

(c) 0.090

(d) 35.142

18.

(a) $20^{5/4}$ (b) $\sqrt[5]{8^3}$ (c) $128^{-3/4}$ (d) $16.1^{0.29}$

19. During a flu epidemic in a small town, health officials estimate that the number of people infected t days after the first case was discovered is given by

$$I(t) = 50t^{3/5}$$

(a) Make a table of values for $I(t)$ on the domain $0 \le t \le 20$. What is the range of the function on that domain?

t	5	10	15	20
$I(t)$				

(b) How long will it be before 300 people are ill?

(c) Graph the function $I(t)$ and verify your answer to part (b) on your graph.

Answer.

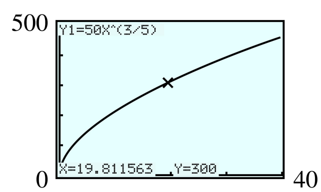
(a)

t	5	10	15	20
$I(t)$	131	199	254	302

Range: $[0, 302]$

(b) ≈ 19.812 or about 20 days

(c)



20. The research division of an advertising firm estimates that the number of people who have seen their ads t days after the campaign begins is given by the function

$$N(t) = 2000t^{5/4}$$

(a) Make a table of values for $N(t)$ on the domain $0 \le t \le 20$. What is the range of the function on that domain?

t	6	10	14	20
$N(t)$				

(b) How long will it be before 75,000 people have seen the ads?

(c) Graph the function $N(t)$ and verify your answer to part (b) on your graph.

In Problems 21–22, graph each set of power functions in the suggested window and compare the graphs.

21. $y_1 = x, \quad y_2 = x^{5/4}, \quad y_3 = x^{3/2}, \quad y_4 = x^2, \quad y_5 = x^{5/2}$
Xmin $= 0$, Xmax $= 6$, Ymin $= 0$, Ymax $= 10$

Answer. All the graphs are increasing and concave up. For $x > 1$, each graph increases more quickly than the previous one.

22. $y_1 = x^{2/5},\quad y_2 = x^{1/2},\quad y_3 = x^{2/3},\quad y_4 = x^{3/4},\quad y_5 = x$
$\text{Xmin} = 0,\quad \text{Xmax} = 6,\quad \text{Ymin} = 0,\quad \text{Ymax} = 4$

23. The *surface to volume ratio* is important in studying how organisms grow and why animals of different sizes have different characteristics.

(a) Write formulas for the volume, V, and the surface area, A, of a cube in terms of its length, L.

(b) Express the length of the cube as a function of its volume. Express the length of the cube as a function of its surface area.

(c) Express the surface area of the cube as a function of its volume.

(d) Express the surface to volume ratio of a cube in terms of its length. What happens to the surface to volume ratio as L increases?

Answer.

(a) $V = L^3$, $A = 6L^2$

(b) $L = V^{1/3}$, $L = \left(\dfrac{A}{6}\right)^{1/2}$

(c) $A = 6V^{2/3}$

(d) $\frac{A}{V} = \frac{6}{L}$. As L increases, the surface-to-volume ratio decreases.

24. Repeat Problem 23 for the volume and surface area of a sphere in terms of its radius, R.

(a) Write formulas for the volume, V, and the surface area, A, of a sphere in terms of its radius, R.

(b) Express the radius of the sphere as a function of its volume. Express the radius of the sphere as a function of its surface area.

(c) Express the surface area of the sphere as a function of its volume.

(d) Express the surface to volume ratio of a sphere in terms of its radius. What happens to the surface to volume ratio as R increases?

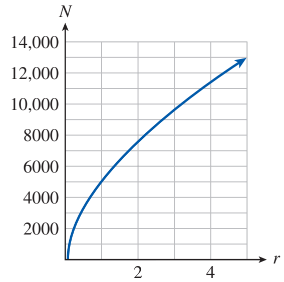
25. A brewery wants to replace its old vats with larger ones. To estimate the cost of the new equipment, the accountant uses the 0.6 rule for industrial costs, which states that the cost of a new container is approximately $N = Cr^{0.6}$, where C is the cost of the old container and r is the ratio of the capacity of the new container to the old one.

(a) If an old vat cost \$5000, graph N as a function of r.

(b) How much should the accountant budget for a new vat that holds 1.8 times as much as the old one?

Answer.

(a)



(b) \$7114.32

26. If a quantity of air expands without changing temperature, its pressure, in pounds per square inch, is given by $P = kV^{-1.4}$, where V is the volume of the air in cubic inches and $k = 2.79 \times 10^4$.

(a) Graph P as a function of V.

(b) Find the air pressure of an air sample when its volume is 50 cubic inches.

27. In the 1970s, Jared Diamond studied the number of bird species on small islands near New Guinea. He found that larger islands support a larger number of different species, according to the formula

$$S = 15.1A^{0.22}$$

where S is the number of species on an island of area A square kilometers. (Source: Chapman and Reiss, 1992)

(a) Fill in the table.

A	10	100	1000	5000	10,000
S					

(b) Graph the function on the domain $0 < A \leq 10,000$.

(c) How many species of birds would you expect to find on Manus Island, with an area of 2100 square kilometers? On Lavongai, which bird's area is 1140 square kilometers?

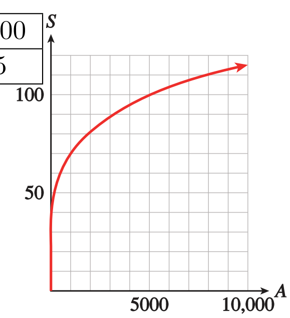
(d) How large must an island be in order to support 200 different species of bird?

Answer.

(a)

A	10	100	1000	5000	10,000
S	25	42	69	98	115

(b)



(c) 81, 71

(d) 126,000 sq km

28. The drainage basin of a river channel is the area of land that contributes water to the river. The table gives the lengths in miles of some of the world's largest rivers and the areas of their drainage basins in square miles. (Source: Leopold, Wolman, and Miller 1992)

(a) Plot the data, using units of $100,000$ on the horizontal axis and units of 500 on the vertical axis.

(b) The length, L, of the channel is related to the area, A, of its drainage basin according to the formula

$$L = 1.05A^{0.58}$$

Graph this function on top of the data points.

(c) The drainage basin for the Congo covers about $1,600,000$ square miles. Estimate the length of the Congo River.

(d) The Rio Grande is 1700 miles long. What is the area of its drainage basin?

River	Area of drainage basin	Length
Amazon	$2,700,000$	4300
Nile	$1,400,000$	4200
Mississippi	$1,300,000$	4100
Yangtze	$580,000$	2900
Volga	$480,000$	2300
St. Lawrence	$460,000$	1900
Ganges	$440,000$	1400
Orinoco	$380,000$	1400
Indus	$360,000$	2000
Danube	$350,000$	1800
Colorado	$250,000$	1700
Platte	$72,000$	800
Rhine	$63,000$	900
Seine	$48,000$	500
Delaware	$12,000$	200

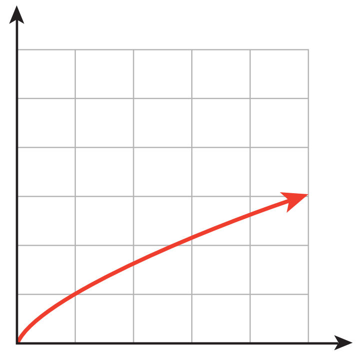
29. The table at right shows the exponent, p, in the allometric equation

$$\text{variable} = k(\text{body mass})^p$$

for some variables related to mammals. (Source: Chapman and Reiss, 1992)

Variable	Exponent, p
Home range size	1.26
Lung volume	1.02
Brain mass	0.70
Respiration rate	-0.26

a Match each equation to one of the graphs shown in the figure.



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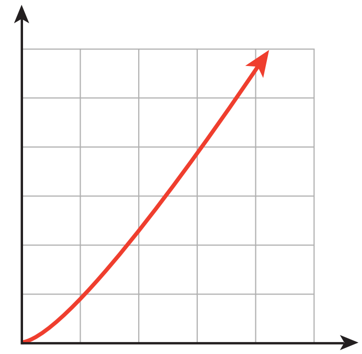
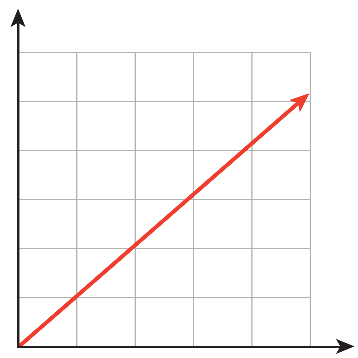
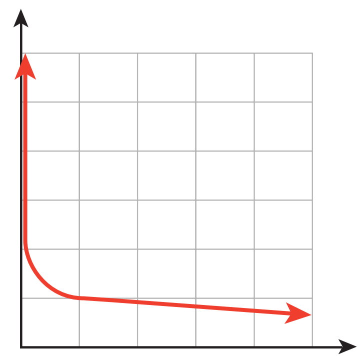


II



III



IV

b Explain how the value of p in the allometric equation determines the shape of the graph. Consider the cases $p > 1$, $0 < p < 1$, and $p < 0$.

Answer.

(a) Home range size: II, lung volume: III, brain mass: I, respiration rate: IV

(b) If $p > 1$, the graph is increasing and concave up. If $0 < p < 1$, the graph is increasing and concave down. If $p < 0$, the graph is decreasing and concave up.

30. The average body mass of a dolphin is about 140 kilograms, twice the body mass of an average human male.

(a) Using the allometric equations in Problem 29, calculate the ratio of the brain mass of a dolphin to that of a human.

(b) A good-sized brown bear weighs about 280 kilograms, twice the weight of a dolphin. Calculate the ratio of the brain mass of a brown bear to that of a dolphin.

(c) Use a ratio to compare the heartbeat frequencies of a dolphin and a human, and those of a brown bear and a dolphin. (See Example 3.3.18, p. 338 of Section 3.3, p. 333.)

31. The gourd species *Tricosanthes* grows according to the formula $L = ad^{2.2}$, where L is its length and d is its width. The species *Lagenaria* has the growth law $L = ad^{0.81}$. (Source: Burton, 1998)

(a) By comparing the exponents, predict which gourd grows into a long, thin shape, and which is relatively fatter. Which species is called the snake gourd, and which is the bottle gourd?

(b) The snake gourd reaches a length of 2 meters (200 cm), with a diameter of only 4 cm. Find the value of a in its growth law.

(c) The bottle gourd is 10 cm long and 7 cm in diameter at maturity. Find the value of a in its growth law.

(d) The giant bottle gourd grows to a length of 23 cm with a diameter of 20 cm. Does it grow according to the same law as standard bottle gourds?

Answer.

(a) Tricosanthes is the snake gourd and Lagenaria is the bottle gourd. Tricosanthes is thinner and Lagenaria is fatter.

(b) $a \approx 9.5$

(c) $a \approx 2$

(d) Yes

32. As a fiddler crab grows, one claw (called the chela) grows much faster than the rest of the body. The table shows the mass of the chela, C, versus the mass of the rest of the body, b, for a number of fiddler crabs. (Source: Burton, 1998)

b	65	110	170	205	300	360	615
C	6	15	30	40	68	110	240

(a) Plot the data.

(b) On the same axes, graph the function $C = 0.007b^{1.63}$. How well does the function fit the data?

(c) Using the function in part (b), predict the chela mass of a fiddler crab if the rest of its body weighs 400 mg.

(d) The chela from a fiddler crab weighs 250 mg. How much does the rest of its body weigh?

(e) As the body mass of a fiddler crab doubles from 100 mg to 200 mg, by what factor does the mass of its chela increase? As the body mass doubles from 200 mg to 400 mg?

33. The climate of a region has a great influence on the types of animals that can survive there. Extreme temperatures create difficult living conditions, so the diversity of wildlife decreases as the annual temperature range increases. Along the west coast of North America, the number of species of mammals, M, is approximately related to the temperature range, R, (in degrees Celsius) by the function $M = f(R) = 433.8R^{-0.742}$. (Source: Chapman and Reiss, 1992)

(a) Graph the function for temperature ranges up to 30°C.

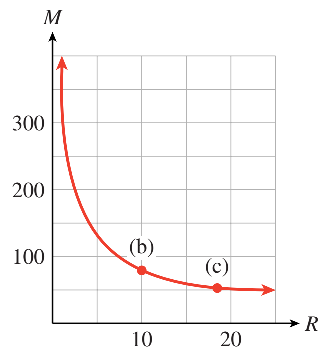
(b) How many species would you expect to find in a region where the temperature range is 10°C? Label the corresponding point on your graph.

(c) If 50 different species are found in a certain region, what temperature range would you expect the region to experience? Label the corresponding point on your graph.

(d) Evaluate the function to find $f(9)$, $f(10)$, $f(19)$, and $f(20)$. What do these values represent? Calculate the change in the number of species as the temperature range increases from 9°C to 10°C and from 19°C to 20°C. Which 1° increase results in a greater decrease in diversity? Explain your answer in terms of slopes on your graph.

Answer.

(a)

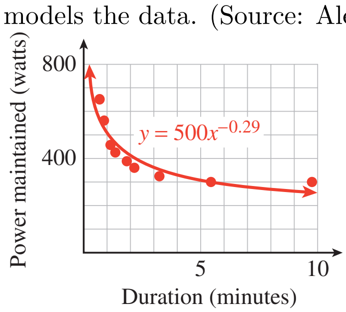


(b) ≈ 78.5 or about 79 species

(c) 18.4°C

(d) $f(9) \approx 85$, $f(10) \approx 79$, $f(19) \approx 49$, $f(20) \approx 47$; from 9°C to 10°C has the greater decrease, corresponding to the steeper slope. If the temperature range is 9°C, there will be approximately 85 species. If the temperature range is 10°C, there will be approximately 79 species. If the temperature range is 19°C, there will be approximately 49 species. If the temperature range is 20°C, there will be approximately 47 species.

34. A bicycle ergometer is used to measure the amount of power generated by a cyclist. The scatterplot shows how long an athlete was able to sustain various levels of power output. The curve is the graph of $y = 500x^{-0.29}$, which approximately models the data. (Source: Alexander, 1992)



(a) In this graph, which variable is independent and which is dependent?

(b) The athlete maintained 650 watts of power for 40 seconds. What power output does the equation predict for 40 seconds?

(c) The athlete maintained 300 watts of power for 10 minutes. How long does the equation predict that power output can be maintained?

(d) In 1979, a remarkable pedal-powered aircraft called the Gossamer

Albatross was successfully flown across the English Channel. The flight took 3 hours. According to the equation, what level of power can be maintained for 3 hours?

(e) The Gossamer Albatross needed 250 watts of power to keep it airborne. For how long can 250 watts be maintained according to the given equation?

35. Investigation 13, p. 290 at the start of this chapter gives data for the pressure inside April and Tolu's balloon as a function of its diameter. As the diameter of the balloon increases from 5 cm to 20 cm, the pressure inside decreases. Can we find a function that describes this portion of the graph?

(a) Pressure is the force per unit area exerted by the balloon on the air inside, or $P = \dfrac{F}{A}$. Because the balloon is spherical, its surface area, A, is given by $A = \pi d^2$. Because the force increases as the balloon expands, we will try a power function $F = kd^p$, where k and p are constants, and see if this fits the data. Combine the three equations, $P = \dfrac{F}{A}$, $A = \pi d^2$, and $F = kd^p$, to express P as a power function of d.

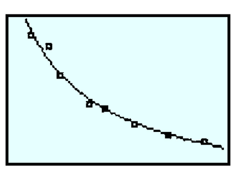
(b) Use your calculator's power regression feature to find a power function that fits the data. Graph the function $P = 211d^{-0.7}$ on top of the data. Do the data support the hypothesis that P is a power function of d?

(c) What is the value of the exponent p in $F = kd^p$?

Answer.

(a) $P = \dfrac{k}{\pi} d^{p-2}$

(b)



The power function is a good fit on this interval.

(c) 1.3

36.

The table shows the total number of frequent flyer miles redeemed by customers through the given year. (Source: www.hotelnewsresource.com)

(a) Plot the data, with $t = 0$ in 1980. What type of function might model the data?

(b) Graph the function $f(t) = 3.13t^{2.33}$ on top of the data.

(c) Evaluate $f(5)$ and $f(25)$. What do those values mean in this context?

(d) Use the regression equation to predict when the total number of miles redeemed will reach 10 trillion.

Year	Cumulative miles redeemed (billions)
1982	14.8
1984	85.3
1986	215.4
1988	387.5
1990	641.3
1992	975.2
1994	1455.9
1996	1996
1998	2670.8
2000	3379.1
2002	4123.6

Hint. (For part d): How many billions make a trillion?

For Problems 37–42, simplify by applying the laws of exponents. Write your answers with positive exponents only.

37.

(a) $4a^{6/5}a^{4/5}$ (b) $9b^{4/3}b^{1/3}$

Answer.

(a) $4a^2$ (b) $9b^{5/3}$

38.

(a) $(-2m^{2/3})^4$ (b) $(-5n^{3/4})^3$

39.

(a) $\dfrac{8w^{9/4}}{2w^{3/4}}$ (b) $\dfrac{12z^{11/3}}{4z^{5/3}}$

Answer.

(a) $4w^{3/2}$ (b) $3z^2$

40.

(a) $(-3u^{5/3})(5u^{-2/3})$ (b) $(-2v^{7/8})(-3v^{-3/8})$

41.

(a) $\dfrac{k^{3/4}}{2k}$ (b) $\dfrac{4h^{2/3}}{3h}$

Answer.

(a) $\dfrac{1}{2k^{1/4}}$ (b) $\dfrac{4}{3h^{1/3}}$

42.

(a) $c^{-2/3}\left(\dfrac{2}{3}c^2\right)$ (b) $\dfrac{r^3}{4}(r^{-5/2})$

43. The incubation time for a bird's egg is a function of the mass, m, of the egg, and has been experimentally determined as

$$I(m) = 12.0m^{0.217}$$

where m is measured in grams and I is in days. (Source: Burton, 1998)

(a) Calculate the incubation time (to the nearest day) for the wren, whose eggs weigh about 2.5 grams, and the greylag goose, whose eggs weigh 46 grams.

(b) During incubation, birds' eggs lose water vapor through their porous shells. The rate of water loss from the egg is also a function of its mass, and it appears to follow the rule

$$W(m) = 0.015m^{0.742}$$

in grams per day. Combine the functions $I(m)$ and $W(m)$ to calculate the fraction of the initial egg mass that is lost during the entire incubation period.

(c) Explain why your result shows that most eggs lose about 18% of their mass during incubation.

Answer.

(a) Wren: 15 days, greylag goose: 28 days

(b) $\dfrac{I(m) \cdot W(m)}{m} = 0.18m^{-0.041}$

(c) Because $m^{-0.041}$ is close to m^0, the fraction lost is close to 0.18.

44. The incubation time for birds' eggs is given by

$$I(m) = 12.0m^{0.217}$$

where m is the weight of the egg in grams, and I is in days. (See Problem 43.) Before hatching, the eggs take in oxygen at the rate of

$$O(m) = 22.2m^{0.77}$$

in milliliters per day. (Source: Burton, 1998)

(a) Combine the functions $I(m)$ and $O(m)$ to calculate the total amount of oxygen taken in by the egg during its incubation.

(b) Use your result from part (a) to explain why total oxygen consumption per unit mass is approximately inversely proportional to incubation time.

(c) Predict the oxygen consumption per gram of a herring gull's eggs, given that their incubation time is 26 days. (The actual value is 11 milliliters per day.)

For Problems 45–50, solve. Round your answers to the nearest thousandth if necessary.

45. $x^{2/3} - 1 = 15$

Answer.
$x = 64$

46. $x^{3/4} + 3 = 11$

47. $x^{-2/5} = 9$

Answer.
$x = \dfrac{1}{243}$

48. $x^{-3/2} = 8$

49. $2(5.2 - x^{5/3}) = 1.4$

Answer.
$x \approx 2.466$

50. $3(8.6 - x^{5/2}) = 6.5$

51. Kepler's law gives a relation between the period, p, of a planet's revolution, in years, and its average distance, a, from the sun:

$$p^2 = Ka^3$$

where $K = 1.243 \times 10^{-24}$, a is measured in miles, and p is in years.

(a) Solve Kepler's law for p as a function of a.

(b) Find the period of Mars if its average distance from the sun is 1.417×10^8 miles.

Answer.

(a) $p = 1.115 \times 10^{-12} a^{3/2}$

(b) 1.88 years

52. Refer to Kepler's law, $p^2 = Ka^3$, in Problem 51.

(a) Solve Kepler's law for a as a function of p.

(b) Find the distance from Venus to the sun if its period is 0.615 years.

53. If $f(x) = (3x - 4)^{3/2}$, find x so that $f(x) = 27$.

Answer. $\dfrac{13}{3}$

54. If $g(x) = (6x - 2)^{5/3}$, find x so that $g(x) = 32$.

55. If $S(x) = 12x^{-5/4}$, find x so that $S(x) = 20$.

Answer. 0.665

56. If $T(x) = 9x^{-6/5}$, find x so that $T(x) = 15$.

For Problems 57–64, use the distributive law to find the product.

57. $2x^{1/2}(x - x^{1/2})$

Answer. $2x^{3/2} - 2x$

58. $x^{1/3}(2x^{2/3} - x^{1/3})$

59. $\dfrac{1}{2}y^{-1/3}(y^{2/3} + 3y^{-5/6})$

Answer. $\dfrac{1}{2}y^{1/3} + \dfrac{3}{2}y^{-7/6}$

60. $3y^{-3/8}\left(\dfrac{1}{4}y^{-1/4} + y^{3/4}\right)$

61. $(2x^{1/4} + 1)(x^{1/4} - 1)$

Answer. $2x^{1/2} - x^{1/4} - 1$

62. $(2x^{1/3} - 1)(x^{1/3} + 1)$

63. $(a^{3/4} - 2)^2$

Answer. $a^{3/2} - 4a^{3/4} + 4$

64. $(a^{2/3} + 3)^2$

For Problems 65–70, factor out the smallest power from each expression. Write your answers with positive exponents only.

65. $x^{3/2} + x = x(\ \ ?\ \ )$

Answer. $x(x^{1/2} + 1)$

66. $y - y^{2/3} = y^{2/3}(\ \ ?\ \ )$

67. $y^{3/4} - y^{-1/4} = y^{-1/4}(\ \ ?\ \ )$

Answer. $\dfrac{y-1}{y^{1/4}}$

68. $x^{-3/2} + x^{-1/2} = x^{-3/2}(\ \ ?\ \ )$

69. $a^{1/3} + 3 - a^{-1/3} = a^{-1/3}(\ \ ?\ \ )$

Answer. $\dfrac{a^{2/3} + a^{1/3} - 1}{a^{1/3}}$

70. $3b - b^{3/4} + 4b^{-3/4} = b^{-3/4}(\ \ ?\ \ )$

3.5 Chapter Summary and Review

3.5.1 Key Concepts

1 **Direct and Inverse Variation.**

- y **varies directly with** x if the ratio $\dfrac{y}{x}$ is constant, that is, if $y = kx$.
- y **varies directly with a power of** x if the ratio $\dfrac{y}{x^n}$ is constant, that is, if $y = kx^n$.
- y **varies inversely with** x if the product xy is constant, that is, if $y = \dfrac{k}{x}$.
- y **varies inversely with a power of** x if the product $x^n y$ is constant, that is, if $y = \dfrac{k}{x^n}$.

2 The graph of a direct variation passes through the origin. The graph of an inverse variation has a vertical asymptote at the origin.

3 If $y = kx^n$, we say that y **scales** as x^n.

4 nth roots: s is called an nth root of b if $s^n = b$.

5 **Exponential Notation.**

The absolute value has the following properties:

$$\begin{array}{ll} a^{-n} = \dfrac{1}{x^n} & a \neq 0 \\ a^0 = 1 & a \neq 0 \\ a^{1/n} = \sqrt[n]{a} & n \text{ an integer, } n > 2 \\ a^{m/n} = (a^{1/n})^m = (a^m)^{1/n}, & a > 0,\ n \neq 0 \end{array}$$

6 In particular, a negative exponent denotes a reciprocal, and a fractional exponent denotes a root.

7 $a^{m/n} = \sqrt[n]{a^m} = \left(\sqrt[n]{a}\right)^m$

8 To compute $a^{m/n}$, we can compute the nth root first, or the mth power, whichever is easier.

9 We cannot write down an exact decimal equivalent for an irrational number, but we can approximate an irrational number to as many decimal places as we like.

10 The laws of exponents are valid for all exponents m and n, and for $b \neq 0$.

Laws of Exponents.

I $a^m \cdot a^n = a^{m+n}$	IV $(ab)^n = a^n b^n$
II $\dfrac{a^m}{a^n} = a^{m-n}$	
III $(a^m)^n = a^{mn}$	V $\left(\dfrac{a}{b}\right)^n = \dfrac{a^n}{b^n}$

11 A function of the form $f(x) = kx^p$, where k and p are constants, is called a **power function**.

12 An **allometric equation** is a power function of the form variable $= k(\text{mass})^p$.

13 We can solve the equation $x^n = b$ by raising both sides to the $\dfrac{1}{n}$ power

14 We can solve the equation $x^{1/n} = b$ by raising both sides to the nth power.

15 To solve the equation $x^{m/n} = k$, we raise both sides to the power n/m.

16 The graphs of power functions $y = x^{m/n}$, where m/n is positive are all increasing for $x \geq 0$. If $m/n > 1$, the graph is concave up. If $0 < m/n < 1$, the graph is concave down.

17 The notation $z = f(x, y)$ indicates that z is a function of two variables, x and y.

18 We can use a table with rows and columns to display the output values for a function of two variables.

19 **Joint Variation.**

- We say that z **varies jointly** with x and y if

$$z = kxy, \quad k \neq 0$$

- We say that z varies directly with x and inversely with y if

$$z = k\frac{x}{y}, \quad k \neq 0, \quad y \neq 0$$

20 We can represent a function of two variables graphically by showing a set of graphs for several fixed values of one of the variables.

21 **Roots of Real Numbers.**

- Every positive number has two real-valued roots, one positive and one negative, if the index is even.
- A negative number has no real-valued root if the index is even.
- Every real number, positive, negative, or zero, has exactly one real-valued root if the index is odd.

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3.5.2 Chapter 3 Review Problems

1. The distance s a pebble falls through a thick liquid varies directly with the square of the length of time t it falls.

a If the pebble falls 28 centimeters in 4 seconds, express the distance it will fall as a function of time.

b Find the distance the pebble will fall in 6 seconds.

Answer.

a $d = 1.75t^2$

b 63 cm

2. The volume, V, of a gas varies directly with the temperature, T, and inversely with the pressure, P, of the gas.

a If $V = 40$ when $T = 300$ and $P = 30$, express the volume of the gas as a function of the temperature and pressure of the gas.

b Find the volume when $T = 320$ and $P = 40$.

3. The demand for bottled water is inversely proportional to the price per bottle. If Droplets can sell 600 bottles at \$8 each, how many bottles can the company sell at \$10 each?

Answer. 480 bottles

4. The intensity of illumination from a light source varies inversely with the square of the distance from the source. If a reading lamp has an intensity of 100 lumens at a distance of 3 feet, what is its intensity 8 feet away?

5. A person's weight, w, varies inversely with the square of his or her distance, r, from the center of the Earth.

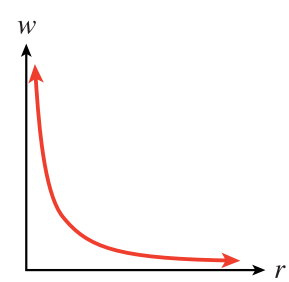
a Express w as a function of r. Let k stand for the constant of variation.

b Make a rough graph of your function.

c How far from the center of the Earth must Neil be in order to weigh one-third of his weight on the surface? The radius of the Earth is about 3960 miles.

Answer.

a $w = \dfrac{k}{r^2}$

b 

c $3960\sqrt{3} \approx 6860$ miles

6. The period, T, of a pendulum varies directly with the square root of its length, L.

a Express T as a function of L. Let k stand for the constant of variation.

b Make a rough graph of your function.

c If a certain pendulum is replaced by a new one four-fifths as long as the old one, what happens to the period?

In Problems 7–10, y varies directly or inversely with a power of x. Find the power of x and the constant of variation, k. Write a formula for each function of the form $y = kx^n$ or $y = \dfrac{k}{x^n}$.

7.

x	y
2	4.8
5	30.0
8	76.8
11	145.2

Answer.
$y = 1.2x^2$

8.

x	y
1.4	75.6
2.3	124.2
5.9	318.6
8.3	448.2

9.

x	y
0.5	40.0
2.0	10.0
4.0	5.0
8.0	2.5

Answer.
$y = \dfrac{20}{x}$

10.

x	y
1.5	320.0
2.5	115.2
4.0	45.0
6.0	20.0

For Problems 11–16, write without negative exponents and simplify.

11.

a $(-3)^{-4}$ b 4^{-3}

Answer.

a $\dfrac{1}{81}$ b $\dfrac{1}{64}$

12.

a $\left(\dfrac{1}{3}\right)^{-2}$ b $\dfrac{3}{5^{-2}}$

13.

a $(3m)^{-5}$ b $-7y^{-8}$

Answer.

a $\dfrac{1}{243m^5}$ b $\dfrac{-7}{y^8}$

14.

a $a^{-1} + a^{-2}$ b $\dfrac{3q^{-9}}{r^{-2}}$

15.

a $6c^{-7} \cdot (3)^{-1}c^4$ b $\dfrac{11z^{-7}}{3^{-2}z^{-5}}$

Answer.

a $\dfrac{2}{c^3}$ b $\dfrac{99}{z^2}$

16.

a $\left(2d^{-2}k^3\right)^{-4}$ b $\dfrac{2w^3(w^{-2})^{-3}}{5w^{-5}}$

For Problems 17–20, write each power in radical form.

17.

a $25m^{1/2}$ b $8n^{-1/3}$

Answer.

a $25\sqrt{m}$ b $\dfrac{8}{\sqrt[3]{n}}$

18.

a $(13d)^{2/3}$ b $6x^{2/5}y^{3/5}$

19.

a $(3q)^{-3/4}$ b $7(uv)^{3/2}$

Answer.

a $\dfrac{1}{\sqrt[4]{27q^3}}$ b $7\sqrt{u^3v^3}$

20.

a $(a^2+b^2)^{0.5}$ b $(16-x^2)^{0.25}$

For Problems 21–24, write each radical as a power with a fractional exponent.

21.

a $2\sqrt[3]{x^2}$ b $\dfrac{1}{4}\sqrt[4]{x}$

Answer.

a $2x^{2/3}$ b $\dfrac{1}{4}x^{1/4}$

22.

a $z^2\sqrt{z}$ b $z\sqrt[3]{z}$

23.

a $\dfrac{6}{\sqrt[4]{b^3}}$ b $\dfrac{-1}{3\sqrt[3]{b}}$

Answer.

a $6b^{-3/4}$ b $\dfrac{-1}{3}b^{-1/3}$

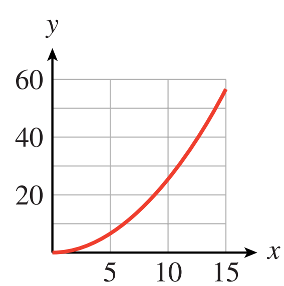
24.

a $\dfrac{-4}{(\sqrt[4]{a})^2}$ b $\dfrac{2}{(\sqrt{a})^3}$

For Problems 25–28, sketch graphs by hand for each function on the domain $(0,\infty)$.

25. y varies directly with x^2. The constant of variation is $k=0.25$.

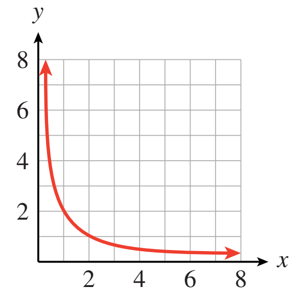
Answer.



26. y varies directly with x. The constant of variation is $k=1.5$.

27. y varies inversely with x. The constant of variation is $k=2$.

Answer.



28. y varies inversely with x^2. The constant of variation is $k = 4$.

For Problems 29–30, write each function in the form $y = kx^p$.

29. $f(x) = \dfrac{2}{3x^4}$

Answer. $f(x) = \dfrac{2}{3}x^{-4}$

30. $g(x) = \dfrac{8x^7}{29}$

For Problems 31–34,

(a) Evaluate each function for the given values.

(b) Graph the function.

31. $Q(x) = 4x^{5/2}$

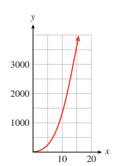
x	16	$\frac{1}{4}$	3	100
$Q(x)$				

Answer.

(a)

x	16	$\frac{1}{4}$	3	100
$Q(x)$	4096	$\frac{1}{8}$	$4\sqrt{3^5} \approx 62.35$	$400,000$

(b)



32. $T(w) = -3w^{2/3}$

w	27	$\frac{1}{8}$	20	1000
$T(w)$				

33. $f(x) = x^{0.3}$

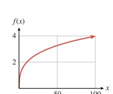
x	0	1	5	10	20	50	70	100
$f(x)$								

Answer.

(a)

x	0	1	5	10	20	50	70	100
$f(x)$	0	1	1.62	2.00	2.46	3.23	3.58	3.98

(b)



34. $g(x) = -x^{-0.7}$

x	0.1	0.2	0.5	1	2	5	8	10
$g(x)$								

35. According to the theory of relativity, the mass of an object traveling at velocity v is given by the function

$$m = \frac{M}{\sqrt{1 - \dfrac{v^2}{c^2}}}$$

where M is the mass of the object at rest and c is the speed of light. Find the mass of a man traveling at a velocity of $0.7c$ if his rest mass is 80 kilograms.

Answer. 112 kg

36. The cylinder of smallest surface area for a given volume has a radius and height both equal to $\sqrt[3]{\dfrac{V}{\pi}}$. Find the dimensions of the tin can of smallest surface area with volume 60 cubic inches.

37. Membership in the Wildlife Society has grown according to the function

$$M(t) = 30t^{3/4}$$

where t is the number of years since its founding in 1970.

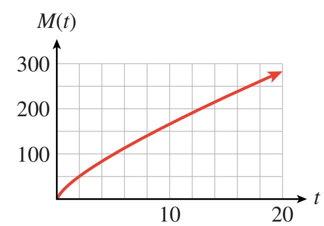
a Sketch a graph of the function $M(t)$.

b What was the society's membership in 1990?

c In what year will the membership be 810 people?

Answer.

a



b 283.7 or ≈ 284

c 2051

38. The heron population in Saltmarsh Refuge is estimated by conservationists at

$$P(t) = 360t^{-2/3}$$

where t is the number of years since the refuge was established in 1990.

a Sketch a graph of the function $P(t)$.

b How many heron were there in 1995?

c In what year will there be only 40 heron left?

39. Manufacturers of ships (and other complex products) find that the average cost of producing a ship decreases as more of those ships are produced. This relationship is called the **experience curve**, given by the equation

$$C = ax^{-b}$$

where C is the average cost per ship in millions of dollars and x is the number of ships produced. The value of the constant b depends on the complexity of the ship. (Source: Storch, Hammon, and Bunch, 1988)

a What is the significance of the constant of proportionality a?

b For one kind of ship, $b = \dfrac{1}{8}$, and the cost of producing the first ship is \$12 million. Write the equation for C as a function of x using radical notation.

c Compute the cost per ship when 2 ships have been built. By what percent does the cost per ship decrease? By what percent does the cost per ship decrease from building 2 ships to building 4 ships?

d By what percent does the average cost decrease from building n ships to building $2n$ ships? (In the shipbuilding industry, the average cost per ship usually decreases by 5 to 10% each time the number of ships doubles.)

Hint. What is the value of C if only one ship is built?

Answer.

a It is the cost of producing the first ship.

b $C = \dfrac{12}{\sqrt[8]{x}}$ million

c About \$11 million; about 8.3%

d About 8.3%

40. A population is in a period of **supergrowth** if its rate of growth, R, at any time is proportional to P^k, where P is the population at that time and k is a constant greater than 1. Suppose R is given by

$$R = 0.015P^{1.2}$$

where P is measured in thousands and R is measured in thousands per year.

a Find R when $P = 20$, when $P = 40$, and when $P = 60$.

b What will the population be when its rate of growth is 5000 per year?

c Graph R and use your graph to verify your answers to parts (a) and (b).

For Problems 41–50, solve

41. $6t^{-3} = \dfrac{3}{500}$

Answer. $t = 10$

42. $3.5 - 2.4p^{-2} = -6.1$

43. $\sqrt[3]{x+1} = 2$

Answer. $x = 7$

44. $x^{2/3} + 2 = 6$

45. $(x-1)^{-3/2} = \dfrac{1}{8}$

Answer. $x = 5$

46. $(2x+1)^{-1/2} = \dfrac{1}{3}$

47. $8\sqrt[4]{x+6} = 24$

Answer. $x = 75$

48. $9.8 = 7\sqrt[3]{z-4}$

49. $\dfrac{2}{3}(2y+1)^{0.2} = 6$

Answer. $y = 29,524$

50. $1.3w^{0.3} + 4.7 = 5.2$

For Problems 51–54, solve each formula for the indicated variable.

51. $t = \sqrt{\dfrac{2v}{g}}$, for g

Answer. $g = \dfrac{2v}{t^2}$

52. $q - 1 = 2\sqrt{\dfrac{r^2-1}{3}}$, for r

53. $R = \dfrac{1+\sqrt{p^2+1}}{2}$, for p

Answer. $p = \pm 2\sqrt{R^2 - R}$

54. $q = \sqrt[3]{\dfrac{1+r^2}{2}}$, for r

For Problems 55–60, simplify by applying the laws of exponents.

55. $(7t)^3(7t)^{-1}$

Answer. $49t^2$

56. $\dfrac{36r^{-2}s}{9r^{-3}s^4}$

57. $\dfrac{(2k^{-1})^{-4}}{4k^{-3}}$

Answer. $\dfrac{k^7}{64}$

58. $(2w^{-3})(2w^{-3})^5(-5w^2)$

59. $\dfrac{8a^{-3/4}}{a^{-11/4}}$

Answer. $8a^2$

60. $b^{2/3}(4b^{-2/3} - b^{1/3})$

61. When the Concorde landed at Heathrow Airport in London, the width, w, of the sonic boom felt on the ground is given in kilometers by the following formula:

$$w = 4\left(\frac{Th}{m}\right)^{1/2}$$

where T stands for the temperature on the ground in kelvins, h is the altitude of the Concorde when it breaks the sound barrier, and m is the drop in temperature for each gain in altitude of one kilometer.

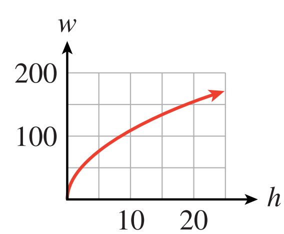
a Find the width of the sonic boom if the ground temperature was 293 K, the altitude of the Concorde was 15 kilometers, and the temperature drop was 4 K per kilometer of altitude.

b Graph w as a function of h if $T = 293$ and $m = 4$.

Answer.

(a) 132.6 km

(b)



62. The manager of an office supply store must decide how many of each item in stock she should order. The Wilson lot size formula gives the most cost-efficient quantity, Q, as a function of the cost, C, of placing an order, the number of items, N, sold per week, and the weekly inventory cost, I, per item (cost of storage, maintenance, and so on).

$$Q = \left(\frac{2CN}{I}\right)^{1/2}$$

a How many reams of computer paper should she order if she sells on average 80 reams per week, the weekly inventory cost for a ream is \$0.20, and the cost of ordering, including delivery charges, is \$25?

b Graph Q as a function of N if $C = 25$ and $I = 0.2$.

63. Two businesswomen start a small company to produce saddle bags for bicycles. The number of saddle bags, q, they can produce depends on the amount of money, m, they invest and the number of hours of labor, w, they employ, according to the Cobb-Douglas formula

$$q = 0.6m^{1/4}w^{3/4}$$

where m is measured in thousands of dollars.

a If the businesswomen invest $\$100,000$ and employ 1600 hours of labor in their first month of production, how many saddle bags can they expect to produce?

b With the same initial investment, how many hours of labor would they need in order to produce 200 saddle bags?

Answer.

(a) 480

(b) 498

64. A child who weighs w pounds and is h inches tall has a surface area (in square inches) given approximately by

$$S = 8.5h^{0.35}w^{0.55}$$

a What is the surface area of a child who weighs 60 pounds and is 40 inches tall?

b What is the weight of a child who is 50 inches tall and whose surface area is 397 square inches?

65. The cost, C, of insulating the ceiling in a building depends on the thickness of the insulation and the area of the ceiling. The table shows values of $C = f(t, A)$, where t is the thickness of the insulation and A is the area of the ceiling.

Cost of Insulation (dollars)						
	Area (sq m)					
Thickness (cm)	100	200	300	400	500	600
4	72	144	216	288	300	432
5	90	180	270	360	450	540
6	108	216	324	432	540	648
7	126	252	378	504	630	756
8	144	288	432	576	720	864
9	162	324	486	648	810	972

a What does it cost to insulate a ceiling with an area of 500 square meters with 5 cm of insulation? Write your answer in function notation.

b Solve the equation $864 = f(t, 600)$ and interpret your answer.

c Consider the row corresponding to a thickness of 4 cm. How does the cost of insulating the ceiling depend on the area of the ceiling?

d Consider the column corresponding to an area of 100 square meters. How does the cost depend on the thickness of the insulation?

e Given that the cost varies jointly with the thickness of the insulation and the area of the ceiling, write an equation for cost as a function of area and thickness of insulation.

f Use your formula from part (e) to determine the cost of insulating a building with 10 centimeters of insulation if the area of the ceiling is 800 square meters.

Answer.

a \$450

b $t = 8$: It costs \$864 to insulate a ceiling with 8 cm of insulation over an area of 600 square meters.

c $C = 0.72A$

d $C = 18T$

e $C = 0.18AT$

f \$1440

66. The volume, V, of a quantity of helium depends on both the temperature and the pressure of the gas. The table shows values of $V = f(P, T)$ for temperature in kelvins and pressure in atmospheres.

Volume (cubic meters)						
	Temperature (K)					
Pressure (atmospheres)	100	150	200	250	300	350
1	18	27	36	45	54	63
2	9	13.5	18	22.5	27	31.5
3	6	9	12	15	18	21
4	4.5	6.75	9	11.25	13.5	15.75

a What is the volume of helium when the pressure is 4 atmospheres and the temperature is 350 K? Write your answer in function notation.

b Solve the equation $15 = f(3, T)$ and interpret your answer.

c Consider the row corresponding to 2 atmospheres. How is the volume related to the absolute temperature?

d Consider the column corresponding to 300 K. How is the volume related to the pressure?

e Given that the volume of the gas varies directly with temperature and inversely with pressure, write an equation for volume as a function of temperature and pressure.

f Use your formula from part (e) to determine the volume of the helium at 50 K and pressure of 0.4 atmospheres.

67. In his hiking guidebook, *Afoot and Afield in Los Angeles County*, Jerry Schad notes that the number of people on a wilderness trail is inversely proportional to "the square of the distance and the cube of the elevation gain from the nearest road."

a Choose variables and write a formula for this relationship.

b On a sunny Saturday afternoon, you count 42 people enjoying the Rock Pool at Malibu Creek State Park. The Rock Pool is 1.5 miles from the main parking lot, and the trail includes an elevation gain of 250 feet. Calculate the constant of variation in your formula from part (a).

Hint: Convert the elevation gain to miles.

c Lookout Trail leads 1.9 miles from the parking lot and involves an elevation gain of 500 feet. How many people would you expect to encounter at the end of the trail?

Answer.

a $N = \dfrac{k}{d^2E^3}$, where N is number of people, d is distance in miles from the road, E is the elevation gain, and k is the constant of variation.

b $k \approx 0.01$

c 3

68. A company's monthly production, P, depends on the capital, C, the company has invested and the amount of labor, L, available each month. The Cobb-Douglas model for production assumes that P varies jointly with C^a and L^b, where a and b are positive constants less than 1. The Aztech Chip Company invested 625 units of capital and hired 256 workers and produces 8000 computer chips each month.

a Suppose that $a = 0.25$, $b = 0.75$. Find the constant of variation and a formula giving P in terms of C and L.

b If Aztech increases its labor force to 300 workers, what production level can they expect?

c If Aztech maintains its labor force at 256 workers, what amount of capital outlay would be required for monthly production to reach $16,000$ computer chips?

3.6 Projects for Chapter 3

Project 15 Wien's Law. A hot object such as a light bulb or a star radiates energy over a range of wavelengths, but the wavelength with maximum energy is inversely proportional to the temperature of the object. If temperature is measured in kelvins, and wavelength in micrometers, the constant of proportionality is 2898. (One micrometer is one thousandth of a millimeter, or $1\mu m = 10^{-6}$ meter.)

a Write a formula for the wavelength of maximum energy, λ_{max}, as a function of temperature, T. This formula, called Wien's law, was discovered in 1894.

b Our sun's temperature is about 5765 K. At what wavelength is most of its energy radiated?

c The color of light depends on its wavelength, as shown in the table. Can you explain why the sun does not appear to be green? Use Wien's law to describe how the color of a star depends on its temperature.

Color	Wavelength (μm)
Red	0.64 – 0.74
Orange	0.59 – 0.64
Yellow	0.56 – 0.59
Green	0.50 – 0.56
Blue	0.44 – 0.50
Violet	0.39 – 0.44

d Astronomers cannot measure the temperature of a star directly, but they can determine the color or wavelength of its light. Write a formula for T as a function of λ_{max}.

e Estimate the temperatures of the following stars, given the approximate value of λ_{max} for each.

Star	λ_{max}	Temperature
R Cygni	1.115	
Betelgeuse	0.966	
Arcturus	0.725	
Polaris	0.414	
Sirius	0.322	
Rigel	0.223	

f Sketch a graph of T as a function of λ_{max} and locate each star on the graph.

Project 16 Halley's Comet. Halley's comet which orbits the sun every 76 years, was first observed in 240 B.C. Its orbit is highly elliptical, so that its closest approach to the Sun (**perihelion**) is only 0.587 AU, while at its greatest distance (**aphelion**) the comet is 34.39 AU from the Sun. (An AU, or astronomical unit, is the distance from the Earth to the Sun, 1.5×10^8 kilometers.)

a Calculate the distances in meters from the Sun to Halley's comet at perihelion and aphelion.

b Halley's comet has a volume of 700 cubic kilometers, and its density is about 0.1 gram per cubic centimeter. Calculate the mass of the comet in kilograms.

c The gravitational force (in newtons) exerted by the Sun on its satellites is inversely proportional to the square of the distance to the satellite in meters. The constant of variation is Gm_1m_2, where $m_1 = 1.99 \times 10^{30}$ kilograms is the mass of the Sun, m_2 is the mass of the satellite, and $G = 6.67 \times 10^{-11}$ is the gravitational constant. Write a formula for the force, F, exerted by the sun on Halley's comet at a distance of d meters.

d Calculate the force exerted by the sun on Halley's comet at perihelion and at aphelion.

Project 17 World Records. Are world record times for track events proportional to the length of the race? The table gives the men's and women's world records in 2005 for races from 1 kilometer to 100 kilometers in length.

Distance (km)	Men's record (min)	Women's record (min)
1	2.199	2.483
1.5	3.433	3.841
2	4.747	5.423
3	7.345	8.102
5	12.656	14.468
10	26.379	29.530
20	56.927	65.443
25	73.93	87.098
30	89.313	105.833

a On separate graphs, plot the men's and women's times against distance. Does time appear to be proportional to distance?

b Use slopes to decide whether the graphs of time versus distance are in fact linear.

c Both sets of data can be modeled by power functions of the form $t = kx^b$, where b is called the **fatigue index**. Graph the function $M(x) = 2.21x^{1.086}$ over the men's data points, and $W(x) = 2.46x^{1.099}$ over the women's data. Describe how the graphs of the two functions differ. Explain why b is called the fatigue index.

Project 18 Naismith's Number. Fell running is a popular sport in the hills, or fells, of the British Isles. Fell running records depend on the altitude gain over the course of the race as well as its length. The equivalent horizontal distance for a race of length x kilometers with an ascent of y kilometers is given by $x + Ny$, where N is Naismith's number (see Project 6, p. 146). The record times for women's races are approximated in minutes by $t = 2.43(x + 9.5y)^{1.15}$, and men's times by $t = 2.18(x + 8.0y)^{1.14}$. (Source: Scarf, 1998)

a Whose times show a greater fatigue index, men or women? (See Project 17, p. 386.)

b Whose times are more strongly affected by ascents?

c Predict the winning times for both men and women in a 56-kilometer race with an ascent of 2750 meters.

Project 19 Elasticity. Elasticity is the property of an object that causes it to regain its original shape after being compressed or deformed. One measure of elasticity considers how high the object bounces when dropped onto a hard surface,

$$e = \sqrt{\frac{\text{height bounced}}{\text{height dropped}}}$$

(Source: Davis, Kimmet, and Autry, 1986)

a The table gives the value of e for various types of balls. Calculate the bounce height for each ball when it is dropped from a height of 6 feet onto a wooden floor.

Type of ball	Bounce height	e
Baseball		0.50
Basketball		0.75
Golfball		0.60
Handball		0.80
Softball		0.55
Superball		0.90
Tennisball		0.74
Volleyball		0.75

b Write a formula for e in terms of H, the bounce height, for the data in part (a).

c Graph the function from part (b).

d If Ball A has twice the elasticity of Ball B, how much higher will Ball A bounce than Ball B?

Project 20 Mersenne's Laws. The tone produced by a vibrating string depends on the frequency of the vibration. The frequency in turn depends on the length of the string, its weight, and its tension. In 1636, Marin Mersenne quantified these relationships as follows. The frequency, f, of the vibration is

i inversely proportional to the string's length, L,

ii directly proportional to the square root of the string's tension, T, and

iii inversely proportional to the square root of the string's weight per unit length, w. (Source: Berg and Stork, 1982)

a Write a formula for f that summarizes Mersenne's laws.

b Sketch a graph of f as a function of L, assuming that T and w are constant. (You do not have enough information to put scales on the axes, but you can show the shape of the graph.)

c On a piano, the frequency of the highest note is about 4200 hertz. This frequency is 150 times the frequency of the lowest note, at about 28 hertz. Ideally, only the lengths of the strings should change, so that all the notes have the same tonal quality. If the string for the highest note is 5 centimeters long, how long should the string for the lowest note be?

d Sketch a graph of f as a function of T, assuming that L and w are constant

e Sketch a graph of f as a function of w, assuming that L and T are constant.

f The tension of all the strings in a piano should be about the same to avoid warping the frame. Suggest another way to produce a lower note.

Look at a piano's strings.

g The longest string on the piano in part (c) is 133.5 cm long. How much heavier (per unit length) is the longest string than the shortest string?

Project 21 Damuth's Formula. In 1981, John Damuth collected data on the average body mass, m, and the average population density, D, for 307 species of herbivores. He found that, very roughly,

$$D = km^{-0.75}$$

(Source: Burton, 1998)

a Explain why you might expect an animal's rate of food consumption to be proportional to its metabolic rate. (See Example 3.4.4, p. 355 in Section 3.4, p. 353 for an explanation of metabolic rate.)

b Explain why you might expect the population density of a species to be inversely proportional to the rate of food consumption of an individual animal.

c Use Kleiber's rule and your answers to parts (a) and (b) to explain why Damuth's proposed formula for population density is reasonable.

d Sketch a graph of the function D. You do not have enough information to put scales on the axes, but you can show the shape of the graph.

Graph the function for $k = 1$.

Project 22 Self-thinning Law. Studies on pine plantations in the 1930s showed that as the trees grow and compete for space, some of the die, so that the density of trees per unit area decreases. The average mass of an individual tree is a power function of the density, d, of the trees per unit area, given by

$$M(d) = kd^{-1.5}$$

This formula is known as the $\dfrac{-3}{2}$ **self-thinning law**. (Source: Chapman and Reiss, 1992)

a To simplify the calculations, suppose that a pine tree is shaped like a tall circular cone and that as it grows, its height is always a constant multiple of its base radius, r. Explain why the base radius of the tree is proportional to the square root of the area the tree covers. Write r as a power function of d.

b Write a formula for the volume of the tree in terms of its base radius, r. Use part (b) to write the volume as a power function of d.

c The mass (or weight) of a pine tree is roughly proportional to its volume, and the area taken up by a single tree is inversely proportional to the plant density, d. Use these facts to justify the self-thinning law.

d Sketch a graph of the function M. You do not have enough information to put scales on the axes, but you can show the shape of the graph.

Graph the function for $k = 1$.