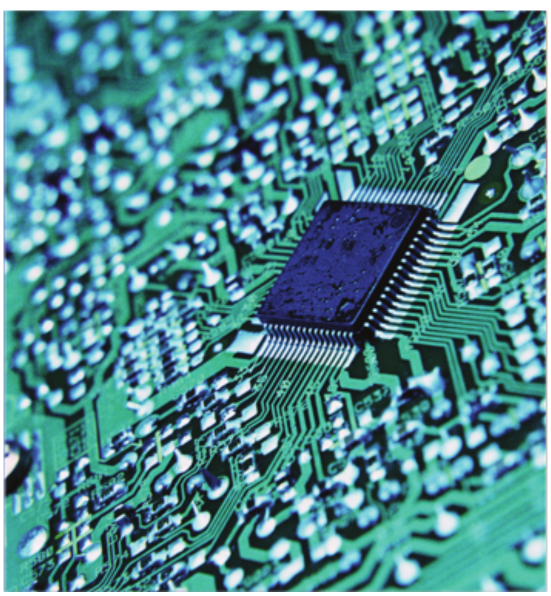
Chapter 4

Exponential Functions



We next consider another important family of functions, called **exponential functions**. These functions describe growth by a constant factor in equal time periods. Exponential functions model many familiar processes, including the growth of populations, compound interest, and radioactive decay. Here is an example.

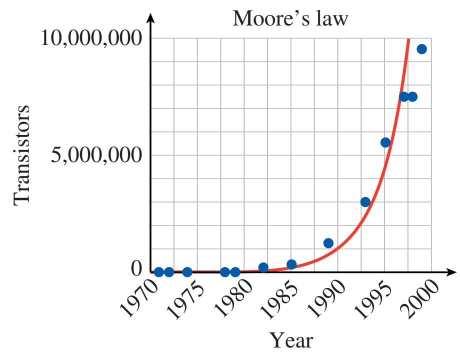
In 1965, Gordon Moore, the cofounder of Intel, observed that the number of transistors on a computer chip had doubled every year since the integrated circuit was invented. Moore predicted that the pace would slow down a bit, but the number of transistors would continue to double every 2 years. More recently, data density has doubled approximately every 18 months, and this is the current definition of Moore's law. Most experts, including Moore himself, expected Moore's law to hold for at least another two decades.

Year	Name of circuit	Transistors
1971	4004	2300
1972	8008	3300
1974	8080	6000
1978	8086	29,000
1979	8088	30,000
1982	80286	134,000
1985	80386	275,000
1989	90486	1,200,000
1993	Pentium	3,000,000
1995	Pentium Pro	5,500,000
1997	Pentium II	7,500,000
1998	Pentium II Xeon	7,500,000
1999	Pentium III	9,500,000

The data shown are modeled by the exponential function

$$N(t) = 2200(1.356)^t,$$

where t is the number of years since 1970.



Investigation 23 Population Growth.

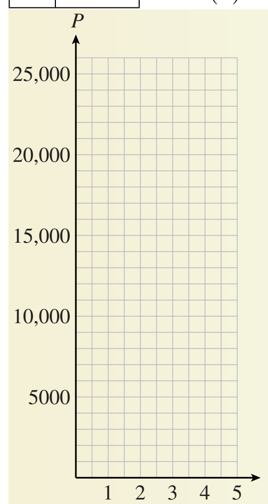
A In a laboratory experiment, researchers establish a colony of 100 bacteria and monitor its growth. The colony triples in population every day.

1 Fill in the table showing the population P(t) of bacteria t days later.

t	$P(t)$
0	100
1	
2	
3	
4	
5	

$P(0) = 100$
$P(1) = 100 \cdot 3 =$
$P(2) = [100 \cdot 3] \cdot 3 =$
$P(3) =$
$P(4) =$
$P(5) =$

2 Plot the data points from the table and connect them with a smooth curve.



3 Write a function that gives the population of the colony at any time t, in days. *Hint:* Express the values you calculated in part (1) using powers of 3. Do you see a connection between the value of t and the exponent on 3?

4 Graph your function from part (3) using a calculator. (Use the table to choose an appropriate domain and range.) The graph should resemble your hand-drawn graph from part (2).

5 Evaluate your function to find the number of bacteria present after 8 days. How many bacteria are present after 36 hours?

B Under ideal conditions, the number of rabbits in a certain area can double every 3 months. A rancher estimates that 60 rabbits live on his land.

1 Fill in the table showing the population $P(t)$ of rabbits t months later.

2 Plot the data points and connect them with a smooth curve.

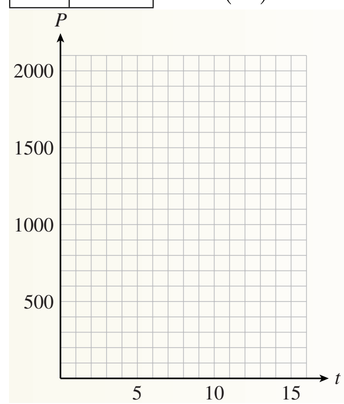
3 Write a function that gives the population of rabbits at any time t, in months. *Hint:* Express the values you calculated in part (1) using powers of 2. Note that the population of rabbits is multiplied by 2 every 3 months. If you know the value of t, how do you find the corresponding exponent in $P(t)$?

4 Graph your function from part (3) using a calculator. (Use the table to choose an appropriate domain and range.) The graph should resemble your hand-drawn graph from part (2).

5 Evaluate your function to find the number of rabbits present after 2 years. How many rabbits are present after 8 months?

t	$P(t)$
0	60
3	
6	
9	
12	
15	

$P(0) = 60$
$P(3) = 60 \cdot 2 =$
$P(6) = [60 \cdot 3] \cdot 2 =$
$P(9) =$
$P(12) =$
$P(15) =$

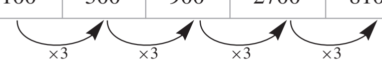


4.1 Exponential Growth and Decay

4.1.1 Exponential Growth

The functions in Investigation 23, p. 392 describe **exponential growth**. During each time interval of a fixed length, the population is multiplied by a certain constant amount. In Part A, the bacteria population grows by a factor of 3 every day.

t	0	1	2	3	4
$P(t)$	100	300	900	2700	8100



For this reason, we say that 3 is the **growth factor** for the function. Functions that describe exponential growth can be expressed in a standard form.

Exponential Growth.

$$P(t) = P_0 b^t$$

where $P_0 = P(0)$ is the **initial value**,
and b is the **growth factor**.

For the bacteria population, we have

$$P(t) = 100 \cdot 3^t$$

so $P_0 = 100$ and $b = 3$.

Example 4.1.1 A colony of bacteria starts with 300 organisms and doubles every week.

a Write a formula for the population of the bacteria colony after t weeks.

b How many bacteria will there be after 8 weeks? After 5 days?

Solution.

a The initial value of the population was $P_0 = 300$, and its weekly growth factor is $b = 2$. Thus, a formula for the population after t weeks is

$$P(t) = 300 \cdot 2^t$$

b After 8 weeks, the population will be

$$P(8) = 300 \cdot 2^8 = 76,800 \text{ bacteria}$$

Because 5 days is $\frac{5}{7}$ of a week, after 5 days the population will be

$$P\left(\frac{5}{7}\right) = 300 \cdot 2^{5/7} = 492.2$$

We cannot have a fraction of a bacterium, so we round to the nearest whole number, 492.

□

Caution 4.1.2 In Example 4.1.1, p. 395a, note that

$$300 \cdot 2^8 \neq 600^8$$

According to the order of operations, we compute the power 2^8 first, then multiply by 300.

Checkpoint 4.1.3 A population of 24 fruit flies triples every month.

a Write a formula for the population of fruit flies after t weeks.

b How many fruit flies will there be after 6 months? After 3 weeks? (Assume that a month equals 4 weeks.)

Answer.

a $P(t) = 24 \cdot 3^t$

b 17,496; 55

4.1.2 Growth Factors

In Part B of Investigation 23, p. 392, the rabbit population grew by a factor of 2 every 3 months.

t	0	3	6	9	12
P(t)	60	120	240	480	960

×2 ×2 ×2 ×2

To write the growth formula for this population, we divide the value of t by 3 to find the number of doubling periods.

$$P(t) = 60 \cdot 2^{t/3}$$

Now we need some algebra to see the growth factor for the function. We use the third law of exponents to write $2^{t/3}$ in another form. Recall that to raise a power to a power, we multiply exponents, so

$$\left(2^{1/3}\right)^t = 2^{t(1/3)} = 2^{t/3}$$

The growth law for the rabbit population is thus

$$P(t) = 60 \cdot \left(2^{1/3}\right)^t$$

The initial value of the function is $P_0 = 60$, and the growth factor is $b = 2^{1/3}$, or approximately 1.26. The rabbit population grows by a factor of about 1.26 every month.

If the units are the same, a population with a larger growth factor grows faster than one with a smaller growth factor.

Example 4.1.4 A lab technician compares the growth of 2 species of bacteria. She starts 2 colonies of 50 bacteria each. Species A doubles in population every 2 days, and species B triples every 3 days. Find the growth factor for each species.

Solution. A function describing the growth of species A is

$$P(t) = 50 \cdot 2^{t/2} = 50 \cdot \left(\mathbf{2^{1/2}}\right)^t$$

so the growth factor for species A is $\mathbf{2^{1/2}}$, or approximately 1.41. For species B,

$$P(t) = 50 \cdot 3^{t/3} = 50 \cdot \left(\mathbf{3^{1/3}}\right)^t$$

so the growth factor for species B is $\mathbf{3^{1/3}}$, or approximately 1.44. Species B grows faster than species A. □

Checkpoint 4.1.5 In 1999, analysts expected the number of Internet service providers to double in five years.

a What was the annual growth factor for the number of Internet service providers?

b If there were 5078 Internet service providers in April 1999, estimate the number of providers in April 2000 and in April 2001.

c Write a formula for $I(t)$, the number of Internet service providers t years after 1999.

Source: LA Times, Sept. 6, 1999

Answer.

a $2^{1/5}$ b 5833 and 6700 c $I(t) = 5078 \cdot 2^{t/5}$

4.1.3 Percent Increase

Exponential growth occurs in other circumstances, too. For example, if the interest on a savings account is compounded annually, the amount of money in the account grows exponentially.

Consider a principal of \$100 invested at 5% interest compounded annually. At the end of 1 year, the amount is

$$\begin{aligned}\text{Amount} &= \text{Principal} + \text{Interest}\\ A &= P + Pr\\ &= 100 + 100(0.05) = 105\end{aligned}$$

It will be more useful to write the formula for the amount after 1 year in factored form.

$$\begin{aligned}A &= P + Pr \qquad \text{Factor out P.}\\ &= P(1+r)\end{aligned}$$

With this version of the formula, the calculation for the amount at the end of 1 year looks like this:

$$\begin{aligned}A &= P(1+r)\\ &= 100(1+0.05)\\ &= 100(1.05) = \mathbf{105}\end{aligned}$$

The amount, \$105, becomes the new principal for the second year. To find the amount at the end of the second year, we apply the formula again, with $P = 105$.

$$\begin{aligned}A &= P(1+r)\\ &= \mathbf{105}(1+0.05)\\ &= 105(1.05) = 110.25\end{aligned}$$

Observe that to find the amount at the end of each year, we multiply the principal by a factor of $1 + r = 1.05$. Thus, we can express the amount at the end of the second year as

$$\begin{aligned}A &= [100(1.05)](1.05)\\ &= 100\mathbf{(1.05)^2}\end{aligned}$$

and at the end of the third year as

$$\begin{aligned}A &= \left[100(1.05)^2\right](1.05)\\ &= 100\mathbf{(1.05)^3}\end{aligned}$$

At the end of each year, we multiply the old balance by another factor of 1.05 to get the new amount. We organize our results into a table, where $A(t)$ represents the amount of money in the account after t years. For this example, a formula for the amount after t years is

t	$P(1+r)^t$	$A(t)$
0	100	100
1	100(1.05)	105
2	$100(1.05)^2$	110.25
3	$100(1.05)^3$	115.76

$$A(t) = 100(1.05)^t$$

In general, for an initial investment of P dollars at an interest rate r compounded annually, we have the following formula for the amount accumulated after t years.

Compound Interest.

The **amount** $A(t)$ accumulated (principal plus interest) in an account bearing interest compounded annually is

$$\boldsymbol{A(t) = P(1+r)^t}$$

where

P is the principal invested,
r is the interest rate,
t is the time period, in years.

This function describes exponential growth with an initial value of P and a growth factor of $b = 1 + r$.

Note 4.1.6 The notion of **percent increase** is often used to describe the growth factor for quantities that grow exponentially. Note carefully the distinction between the percent increase, r, and the growth factor, $b = 1 + r$.

Example 4.1.7 During a period of rapid inflation, prices rose by 12% over 6 months. At the beginning of the inflationary period, a pound of butter cost \$2.

a Make a table of values showing the rise in the cost of butter over the next 2 years.

b Write a function that gives the price of a pound of butter t years after inflation began.

c How much did a pound of butter cost after 3 years? After 15 months?

d Graph the function you found in part (b).

Solution.

a The percent increase in the price of butter is 12% every 6 months. Therefore, the growth factor for the price of butter is $1 + 0.12 = 1.12$ every half-year. If $P(t)$ represents the price of butter after t years, then $P(0) = 2$, and every half-year we multiply the price by 1.12, as shown in the table.

t	$P(t)$		
0	2		$P(0) = 2.00$
$\frac{1}{2}$	$2(1.12)$	$\times 1.12$	$P\left(\frac{1}{2}\right) = 2.24$
1	$2(1.12)^2$	$\times 1.12$	$P(1) = 2.51$
$\frac{3}{2}$	$2(1.12)^3$	$\times 1.12$	$P\left(\frac{3}{2}\right) = 2.81$
2	$2(1.12)^4$	$\times 1.12$	$P(2) = 3.15$

b Look closely at the second column of the table. After t years of inflation, the original price of \$2 has been multiplied by $2t$ factors of 1.12. Thus,

$$P = 2(1.12)^{2t}$$

c To find the price of butter at any time after inflation began, we evaluate the function at the appropriate value of t.

$$\begin{aligned} P(\mathbf{3}) &= 2(1.12)^{2(\mathbf{3})} \\ &= 2(1.12)^6 \approx 3.95 \end{aligned}$$

After 3 years, the price was \$3.95. Fifteen months is 1.25 years, so we evaluate $P(1.25)$.

$$\begin{aligned} P(\mathbf{1.25}) &= 2(1.12)^{2(\mathbf{1.25})} \\ &= 2(1.12)^{2.5} \approx 2.66 \end{aligned}$$

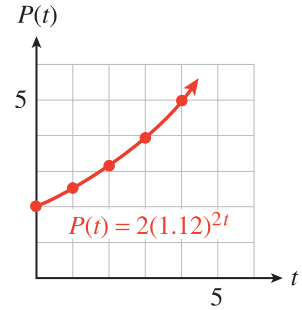
After 15 months, the price of butter was \$2.66.

d Evaluate the function

$$P(t) = 2(1.12)^{2t}$$

for several values, as shown in the table below. We plot the points and connect them with a smooth curve to obtain the graph shown in the figure at right below.

t	
0	2.00
1	2.51
2	3.15
3	3.95
4	4.95



□

In Example 4.1.7, p. 398, we can rewrite the formula for $P(t)$ as follows:

$$\begin{aligned} P(t) &= 2(1.12)^{2t} \\ &= 2\left[(1.12)^2\right]^t = 2(1.2544)^t \end{aligned}$$

Thus, the annual growth factor for the price of butter is 1.2544, and the annual percent growth rate is 25.44%.

Checkpoint 4.1.8 In 1998, the average annual cost of attending a public college was \$10,069, and costs were climbing by 6% per year.

a Write a formula for $C(t)$, the cost of one year of college t years after 1998.

b Complete the table and sketch a graph of $C(t)$.

t	0	5	10	15	20	25
$C(t)$						

c If the percent growth rate remained steady, how much did a year of college cost in 2005?

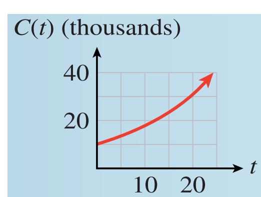
d If the percent growth rate continues to remain steady, how much will a year of college cost in 2020?

Answer.

a $C(t) = 10,069 \cdot 1.06^t$

b

t	0	5	10	15	20	25
$C(t)$	10,069	13,475	18,032	24,131	32,293	43,215



c \$15,140 per year

d \$36,284

4.1.4 Exponential Decay

In the preceding examples, exponential growth was modeled by increasing functions of the form

$$P(t) = P_0 b^t$$

where $b > 1$. The function $P(t) = P_0 b^t$ is a *decreasing* function if $0 < b < 1$. In this case, we say that the function describes **exponential decay**, and the constant b is called the **decay factor**. In Investigation 24, p. 400, we consider two examples of exponential decay.

Investigation 24 Exponential Decay.

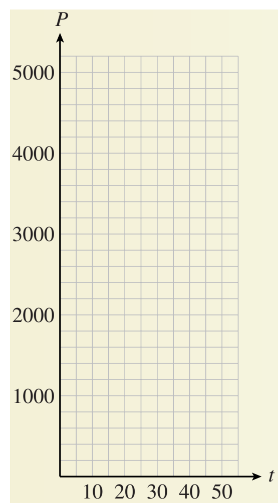
A A small coal-mining town has been losing population since 1940, when 5000 people lived there. At each census thereafter (taken at 10-year intervals), the population declined to approximately 0.90 of its earlier figure.

t	$P(t)$
0	5000
10	
20	
30	
40	
50	

$P(0) = 5000$
$P(10) = 5000 \cdot 0.90 =$
$P(20) = [5000 \cdot 0.90] \cdot 0.90 =$
$P(3) =$
$P(4) =$
$P(5) =$

1 Fill in the table showing the population $P(t)$ of the town t years after 1940.

2 Plot the data points and connect them with a smooth curve.



3 Write a function that gives the population of the town at any time t in years after 1940.

Express the values you calculated in part (1) using powers of 0.90. Do you see a connection between the value of t and the exponent on 0.90?

4 Graph your function from part (3) using a calculator. (Use the table to choose an appropriate domain and range.) The graph should resemble your hand-drawn graph from part (2).

5 Evaluate your function to find the population of the town in 1995. What was the population in 2000?

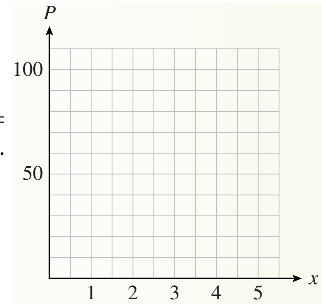
B A plastic window coating 1 millimeter thick decreases the light coming through a window by 25%. This means that 75% of the original amount of light comes through 1 millimeter of the coating. Each additional millimeter of coating reduces the light by another 25%.

1 Fill in the table showing the percent of the light, $P(x)$, that shines through x millimeters of the window coating.

2 Plot the data points and connect them with a smooth curve.

x	$P(x)$
0	100
1	
2	
3	
4	
5	

$P(0) = 100$
$P(1) = 100 \cdot 0.75 =$
$P(2) = [100 \cdot 0.75] \cdot$
$P(3) =$
$P(4) =$
$P(5) =$



3 Write a function that gives the percent of the light that shines through x millimeters of the coating.

Express the values you calculated in part (1) using powers of 0.75. Do you see a connection between the value of x and the exponent on 0.75?

4 Graph your function from part (3) using a calculator. (Use your table of values to choose an appropriate domain and range.) The graph should resemble your hand-drawn graph from part (2).

5 Evaluate your function to find the percent of the light that comes through 6 millimeters of plastic coating. What percent comes through $\frac{1}{2}$ millimeter?

4.1.5 Decay Factors

Before Example 4.1.7, p. 398, we noted that a percent increase of r (in decimal form) corresponds to a growth factor of $b = 1 + r$. A percent *decrease* of r corresponds to a *decay* factor of $b = 1 - r$. In Part B of Investigation 24, p. 400, each millimeter of plastic reduced the amount of light by 25%, so $r = 0.25$, and the decay factor for the function $P(x)$ is

$$\begin{aligned} b &= 1 - r \\ &= 1 - 0.25 = 0.75 \end{aligned}$$

Caution 4.1.9 Note the difference in the two expressions for b:

- A percent increase of r produces a growth factor of $b = 1 + r$.
- A percent decrease of r produces a decay factor of $b = 1 - r$.

Example 4.1.10 David Reed writes in Context magazine: "Computing prices have been falling exponentially -- 50% every 18 months -- for the past 30 years and will probably stay on that curve for another couple of decades." An accounting firm invests $50,000 in new computer equipment.

a Write a formula for the value of the equipment t years from now.

b By what percent does the equipment depreciate each year?

c What will the equipment be worth in 5 years?

Solution.

a The initial value of the equipment is $V_0 = 50,000$. Every 18 months, the value of the equipment is multiplied by

$$b = 1 - r = 1 - 0.50 = 0.50$$

However, because 18 months is 1.5 years, we must divide t by 1.5 in our formula, giving us

$$V(t) = 50,000(0.50)^{t/1.5}$$

b After 1 year, we have

$$V(1) = 50,000(0.50)^{1/1.5} = 50,000(0.63)$$

The equipment is worth 63% of its original value, so it has depreciated by $1 - 0.63$, or 37%.

c After 5 years,

$$V(5) = 50,000(0.50)^{5/1.5} = 4960.628$$

To the nearest dollar, the equipment is worth $4961.

□

Checkpoint 4.1.11 The number of butterflies visiting a nature station is declining by 18% per year. In 1998, 3600 butterflies visited the nature station.

a What is the decay factor in the annual butterfly count?

b Write a formula for $B(t)$, the number of butterflies t years after 1998.

c Complete the table and sketch a graph of $B(t)$.

t	0	2	4	6	8	10
$B(t)$						

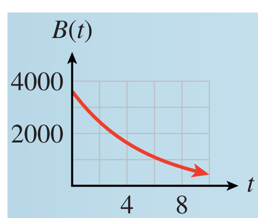
Answer.

a 0.82

b $B(t) = 3600 \cdot 0.82^t$

c

t	0	2	4	6	8	10
$B(t)$	3600	2421	1628	1094	736	495



We summarize our observations about exponential growth and decay functions as follows.

Exponential Growth and Decay.

The function

$$P(t) = P_0 b^t$$

models exponential growth and decay.

$P_0 = P(0)$ is the **initial value** of P;
b is the **growth** or **decay factor**.

1. If $b > 1$, then $P(t)$ is increasing, and $b = 1 + r$, where r represents percent increase.

2. If $0 < b < 1$, then $P(t)$ is decreasing, and $b = 1 - r$, where r represents percent decrease.

4.1.6 Comparing Linear Growth and Exponential Growth

It may be helpful to compare linear growth and exponential growth. Consider the two functions

$$L(t) = 5 + 2t \quad \text{and} \quad E(t) = 5 \cdot 2^t \quad (t \geq 0)$$

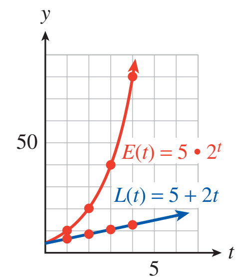
whose graphs are shown below.

t	$L(t)$
0	5
1	7
2	9
3	11
4	13

Slope $m = 2$

t	$E(t)$
0	5
1	10
2	20
3	40
4	80

Growth factor $b = 2$



L is a linear function with initial value 5 and slope 2; E is an exponential function with initial value 5 and growth factor 2. In a way, the growth factor of an exponential function is analogous to the slope of a linear function: Each measures how quickly the function is increasing (or decreasing).

However, for each unit increase in t, 2 units are *added* to the value of $L(t)$, whereas the value of $E(t)$ is *multiplied* by 2. An exponential function with growth factor 2 eventually grows much more rapidly than a linear function with slope 2, as you can see by comparing the graphs in the figure or the function values in the tables.

Example 4.1.12 A solar energy company sold \$80,000 worth of solar collectors last year, its first year of operation. This year its sales rose to \$88,000, an increase of 10%. The marketing department must estimate its projected sales for the next 3 years.

a If the marketing department predicts that sales will grow linearly, what should it expect the sales total to be next year? Graph the projected sales figures over the next 3 years, assuming that sales will grow linearly.

b If the marketing department predicts that sales will grow exponentially, what should it expect the sales total to be next year? Graph the projected sales figures over the next 3 years, assuming that sales will grow exponentially.

Solution.

a Let $L(t)$ represent the company's total sales t years after starting business, where $t = 0$ is the first year of operation. If sales grow linearly, then $L(t)$ has the form $L(t) = mt + b$. Now $L(0) = 80,000$, so the intercept b is $80,000$. The slope m of the graph is

$$\frac{\Delta S}{\Delta t} = \frac{8000 \text{ dollars}}{1 \text{ year}} = 8000 \text{ dollars/year}$$

where $\Delta S = 8000$ is the increase in sales during the first year. Thus, $L(t) = 8000t + 80,000$, and sales grow by adding \$8000 each year. The expected sales total for the next year is

$$L(2) = 8000(2) + 80,000 = 96,000$$

b Let $E(t)$ represent the company's sales assuming that sales will grow exponentially. Then $E(t)$ has the form $E(t) = E_0 b^t$. The percent increase in sales over the first year was $r = 0.10$, so the growth rate is

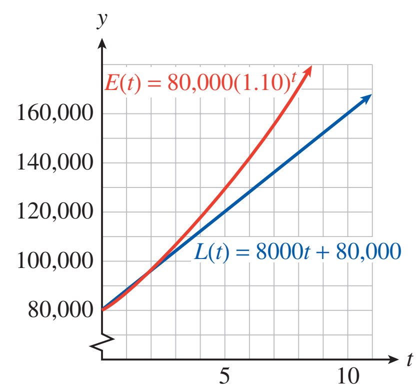
$$b = 1 + r = 1.10$$

The initial value, E_0, is 80, 000. Thus, $E(t) = 80,000(1.10)^t$, and sales grow by being multiplied each year by 1.10. The expected sales total for the next year is

$$E(2) = 80,000(1.10)^2 = 96,800$$

We evaluate each function at several points to obtain the graphs shown in the figure.

t	$L(t)$	$E(t)$
0	80, 000	80, 000
1	88, 000	88, 000
2	96, 000	96, 800
3	104, 000	106, 480
4	112, 000	117, 128



□

Checkpoint 4.1.13 A new car begins to depreciate in value as soon as you drive it off the lot. Some models depreciate linearly, and others depreciate exponentially. Suppose you buy a new car for \$20, 000, and 1 year later its value has decreased to \$17, 000.

a If the value decreased linearly, what was its annual rate of decrease?

b If the value decreased exponentially, what was its annual decay factor? What was its annual percent depreciation?

c Calculate the value of your car when it is 5 years old under each assumption, linear or exponential depreciation.

Answer.

a \$3000 per year

b 0.85; 15%

c Linear: \$5000; Exponential: \$8874

4.1.7 Section Summary

4.1.7.1 Vocabulary

Look up the definitions of new terms in the Glossary.

- Exponential growth
- Initial value
- Exponential decay
- Percent increase
- Compound interest
- Growth factor
- Amount

4.1.7.2 CONCEPTS

1 If a quantity is multiplied by a constant factor, b, in each time period, we say that it undergoes **exponential growth** or **decay**. The constant b is called the **growth factor** if $b > 1$ and the **decay factor** if $0 < b < 1$.

2 Quantities that increase or decrease by a constant percent in each time period grow or decay exponentially.

3 Exponential Growth and Decay.

The function

$$P(t) = P_0 b^t$$

models exponential growth and decay.

$P_0 = P(0)$ is the **initial value** of P;

b is the **growth** or **decay factor**.

(a) If $b > 1$, then $P(t)$ is increasing, and $b = 1 + r$, where r represents percent increase.

(b) If $0 < b < 1$, then $P(t)$ is decreasing, and $b = 1 - r$, where r represents percent decrease.

4 Compound Interest.

The amount $A(t)$ accumulated (principal plus interest) in an account bearing interest compounded annually is

$$A(t) = P(1 + r)^t$$

where

P	is the principal invested,
r	is the interest rate,
t	is the time period, in years

5 In linear growth, a constant amount is *added* to the output for each unit increase in the input. In exponential growth, the output is *multiplied* by a constant factor for each unit increase in the input.

4.1.7.3 STUDY QUESTIONS

1 Is it possible for two populations with the same initial value to grow at different percent rates?

2 If you know the percent growth rate, how can you find the growth factor? If you know the percent decay rate, how can you find the decay factor?

3 What is the growth factor for a population that grows 4% annually?

4 What is the decay factor for a population that declines by 4% annually?

5 What is the growth factor for a population that grows by 100% annually?

6 Explain the difference between the slope in linear growth and the growth factor in exponential growth.

4.1.7.4 SKILLS

Practice each skill in the Homework 4.1.8, p. 406 problems listed.

1 Calculate percent increase or decrease: #1–10–6

2 Write a formula for exponential growth or decay: #11–22

3 Evaluate an exponential growth or decay function: #11–22

4 Simplify exponential expressions: #23–32

5 Solve power equations: #33–40

6 Find the growth factor or initial value: #41–58

7 Solve for percent increase or decrease: #63–66

4.1.8 Exponential Growth and Decay (Homework 4.1)

1.

(a) A parking permit at Huron College cost $25 last year, but this year the price increased by 12%. What is the price this year?

(b) If the price of a parking permit increases by 12% again next year, what will the price be then?

Answer.

(a) $28 (b) $31.36

2.

(a) The computer you want cost $1200 when it first came on the market, but after 3 months the price was reduced by 15%. What was the price then?

(b) If the price falls by another 15% next month, what will the price be then?

3. The value of your stock portfolio fell 10% last year, but this year it increased by 10%. How does the current value of your portfolio compare to what it was two years ago?

Answer. It is 99% of what it was 2 years ago.

4. You got a 5% raise in January, but then in March everyone took a pay cut of 5%. How does your new salary compare to what it was last December?

5. The population of Summerville is currently 12 hundred people.

(a) Write a formula for the population if it grows at a constant rate of 1.5 hundred people per year. What is the population after 3 years?

(b) Write a formula for the population if it has a constant growth factor of 1.5 per year. What is the population after 3 years?

Answer.

(a) $P = 1200 + 150t$; 1650 (b) $P = 1200 \cdot 1.5^t$; 4050

6. Delbert's sports car was worth $\$45,000$ when he bought it.

(a) Write a formula for the value of the car if it depreciates at a constant rate of \$7000 per year. What is the value of the car after 4 years?

(b) Write a formula for the value of the car if it has a constant depreciation factor of 0.70 per year. What is the value of the car after 4 years?

7. Francine's truck was worth $\$18,000$ when she bought it.

(a) Write a formula for the value of the truck if it depreciates by \$2000 per year. What is the value of the truck after 5 years?

(b) Write a formula for the value of the truck if it depreciates by 20% per year. What is the value of the truck after 5 years?

Answer.

(a) $V = 18,000 - 2000t$; \$8000 (b) $V = 18,000 \cdot 0.8^t$; \$5898.24

8. The population of Lakeview is currently $150,000$ people.

(a) Write a formula for the population if it grows by 6000 people per year. What is the population after 2 years?

(b) Write a formula for the population if grows by 6% per year. What is the population after 2 years?

9. The table shows the growth factor for a number of different populations. For each population, find the percent growth rate.

Population	A	B	C	D	E
Growth factor	1.2	1.02	1.075	2.0	2.15
Percent growth rate					

Answer. A: 20%; B: 2%; C: 7.5%; D: 100%; E: 115%

10. The table shows the decay factor for a number of different populations. For each population, find the percent growth rate.

Population	A	B	C	D	E
Decay factor	0.6	0.06	0.96	0.996	0.096
Percent decay rate					

For Problems 11–16,

a Write a function that describes exponential growth.

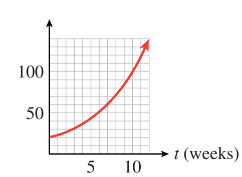
b Graph the function.

c Evaluate the function at the given values.

11. A typical beehive contains $20,000$ insects. The population can increase in size by a factor of 2.5 every 6 weeks. How many bees could there be after 4 weeks? After 20 weeks?

Answer.

a $P = 20,000 \cdot 2.5^{t/6}$

b 

c $36,840$ bees; $424,128$ bees

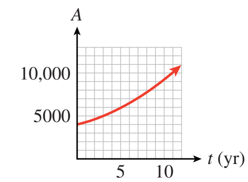
12. A rancher who started with 800 head of cattle finds that his herd increases by a factor of 1.8 every 3 years. How many head of cattle will he have after 1 year? After 10 years?

13. A sum of \$4000 is invested in an account that pays 8% interest compounded annually. How much is in the account after 2 years? After 10 years?

Answer.

a $A = 4000 \cdot 1.08^t$

b



c \$4665.60; \$8635.70

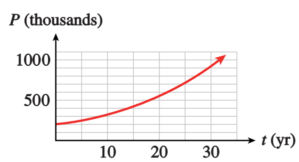
14. Otto invests \$600 in an account that pays 7.3% interest compounded annually. How much is in Otto's account after 3 years? After 6 years?

15. Paul bought a house for \$200, 000 in 2003. Since 2003, housing prices have risen an average of 5% per year. How much was the house worth in 2015? How much will it be worth in 2030?

Answer.

a $P = 200,000 \cdot 1.05^t$

b



c \$359, 171; \$746, 691

16. Sales of Windsurfers have increased 12% per year since 2010. If Sunsails sold 1500 Windsurfers in 2010, how many did it sell in 2015? How many should it expect to sell in 2022?

For Problems 17–22,

a Write a function that describes exponential decay.

b Graph the function.

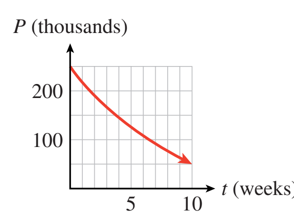
c Evaluate the function at the given values.

17. During a vigorous spraying program, the mosquito population was reduced to $\frac{3}{4}$ of its previous size every 2 weeks. If the mosquito population was originally estimated at 250, 000, how many mosquitoes remained after 3 weeks of spraying? After 8 weeks?

Answer.

a $P = 250,000 \cdot 0.75^{t/2}$

b



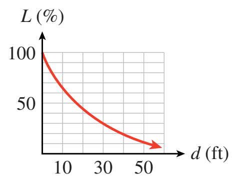
c 162, 380; 79, 102

18. The number of perch in Hidden Lake has declined to half of its previous value every 5 years since 1985, when the perch population was estimated at 8000. How many perch were there in 1995? In 2013?

19. Scuba divers find that the water in Emerald Lake filters out 15% of the sunlight for each 4 feet that they descend. How much sunlight penetrates to a depth of 20 feet? To a depth of 45 feet?

Answer.

a $L = 0.85^{d/4}$

b 

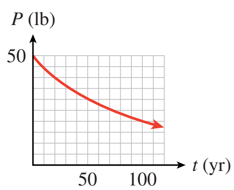
c 44%; 16%

20. Arch's motorboat cost $\$15,000$ in 2005 and has depreciated by 10% every 3 years. How much was the boat worth in 2014? In 2015?

21. Plutonium-238 is a radioactive element that decays over time into a less harmful element at a rate of 0.8% per year. A power plant has 50 pounds of plutonium-238 to dispose of. How much plutonium-238 will be left after 10 years? After 100 years?

Answer.

a $P = 50 \cdot 0.992^t$

b 

c 46.1 lb; 22.4 lb

22. Iodine-131 is a radioactive element that decays at a rate of 8.3% per day. How much of a 12-gram sample will be left after 1 week? After 15 days?

In Problems 23–26, use the laws of exponents to simplify.

23.

a $3^x\, 3^4$ b $(3^x)^4$ c $3^x\, 4^x$

Answer.

a 3^{x+4} b 3^{4x} c 12^x

24.

a $8^x\, 8^x$ b $8^{x+2}\, 8^{x-1}$ c $\dfrac{8^{2x}}{8^x}$

25.

a $b^{-4t}\, b^{2t}$ b $(b^t)^{1/2}$ c $b^{t-1}\, b^{1-t}$

Answer.

a b^{-2t} b $b^{t/2}$ c 1

26.

a $b^{t/2}\, b^{t/2}$ b $\dfrac{b^{2t}}{b}$ c $b^{1/t}\, b^t$

27. Let $P(t) = 12(3)^t$. Show that $P(t+1) = 3P(t)$.

Answer. $P(t+1) = 12(3)^{t+1} = 12(3)^t \cdot 3 = P(t) \cdot 3$

28. Let $N(t) = 8(5)^t$. Show that $\dfrac{N(t+k)}{N(t)} = 5^k$

29. Let $P(x) = P_0 a^x$. Show that $P(x+k) = a^k P(x)$.

Answer. $P(x+k) = P_0 a^{x+k} = P_0 a^x \cdot a^k = P(x) \cdot a^k$

30. Let $N(x) = N_0 b^x$. Show that $\dfrac{N(x+1)}{N(x)} = b$

31.

a Explain why $P(t) = 2 \cdot 3^t$ and $Q(t) = 6^t$ are not the same function.

b Complete the table of values for P and Q, showing that their values are not the same.

t	0	1	2
$P(t)$			
$Q(t)$			

Answer.

a In the expression $2 \cdot 3^t$, only the 3 is raised to a power t, and the result is doubled, but if both the 2 and the 3 were raised to the power t, the result would be 6^t.

b

t	0	1	2
$P(t)$	2	6	18
$Q(t)$	1	6	36

32.

a Explain why $P(t) = 4 \cdot \left(\dfrac{1}{2}\right)^t$ and $Q(t) = 2^t$ are not the same function.

b Complete the table of values for P and Q, showing that their values are not the same.

t	0	1	2
$P(t)$			
$Q(t)$			

Solve each equation. (See Section 3.3, p. 333 to review solving equations involving powers of the variable.) Round your answer to two places if necessary.

33. $768 = 12b^3$

Answer. 4

34. $75 = 3b^4$

35. $14,929.92 = 5000b^6$

Answer. 1.2

36. $151,875 = 20,000b^5$

37. $1253 = 260(1+r)^{12}$

Answer. $r \approx 0.14$

38. $116,473 = 48,600(1+r)^{15}$

39. $56.27 = 78(1-r)^8$

Answer. $r \approx 0.04$

40. $10.56 = 12.4(1-r)^{20}$

41.

a Riverside County is the fastest growing county in California. In 2000, the population was 1,545,387. Write a formula for the population of Riverside County. (You do not know the value of the growth factor, b, yet.)

b In 2004, the population had grown to 1,871,950. Find the growth factor and the percent rate of growth, rounded to the nearest tenth of a percent.

c Estimate the population of Riverside County in 2010.

Answer.

a $P(t) = 1,545,387b^t$

b Growth factor 1.049; Percent rate of growth 4.9%

c $2,493,401$

42.

a In 2006, a new Ford Focus cost \$15,574. The value of a Focus decreases exponentially over time. Write a formula for the value of a Focus. (You do not know the value of the decay factor, b, yet.)

b A 2-year old Focus cost \$11,788. Find the decay factor and the percent rate of depreciation, rounded to the nearest tenth of a percent.

c About how much would a 4-year old Focus cost?

43. In the 1940s, David Lack undertook a study of the European robin. He tagged 130 one-year-old robins and found that on average 35.6% of the birds survived each year. (Source: Burton, 1998)

a According to the data, how many robins would have originally hatched to produce 130 one-year-olds?

b Write a formula for the number of the original robins still alive after t years.

c Graph your function.

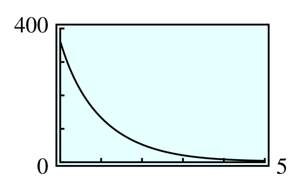
d One of the original robins actually survived for 9 years. How many robins does the model predict will survive for 9 years?

Answer.

a 365

b $N(t) = 365(0.356)^t$

c



d 0.03. (Therefore, none)

44. Many insects grow by discrete amounts each time they shed their exoskeletons. Dyar's rule says that the size of the insect increases by a constant ratio at each stage. (Source: Burton, 1998)

a Dyar measured the width of the head of a caterpillar of a swallowtail butterfly at each stage. The caterpillar's head was initially approximately 42 millimeters wide, and 63.84 millimeters wide after its first stage. Find the growth ratio.

b Write a formula for the width of the caterpillar's head at the nth stage.

c Graph your function.

d What head width does the model predict after 5 stages?

For Problems 45–54,

a Each table describes exponential growth or decay. Find the growth or decay factor.

b Complete the table. Round values to two decimal places if necessary.

45.

t	0	1	2	3	4
P	8	12	18		

Answer. The growth factor is 1.5.

t	0	1	2	3	4
P	8	12	18	27	40.5

46.

t	0	1	2	3	4
P	4	5	6.25		

47.

x	0	1	2	3	4
Q	20	24			

Answer. The growth factor is 1.2.

x	0	1	2	3	4
Q	20	24	28.8	34.56	41.47

48.

x	0	1	2	3	4
Q	100	105			

49.

w	0	1	2	3	4
N	120	96			

Answer. The decay factor is 0.8.

w	0	1	2	3	4
N	120	96	76.8	61.44	49.15

50.

w	0	1	2	3	4
N	640	480			

51.

t	0	1	2	3	4
C	10		6.4		

Answer. The decay factor is 0.8.

t	0	1	2	3	4
C	10	8	6.4	5.12	4.10

52.

t	0	1	2	3	4
C	20			2.5	

53.

n	0	1	2	3	4
B	200			266.2	

Answer. The growth factor is 1.1.

n	0	1	2	3	4
B	200	220	242	266.2	292.82

54.

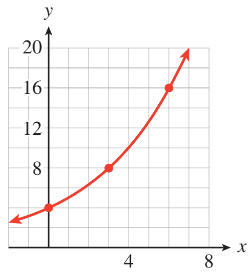
n	0	1	2	3	4
B	40		62.5		

Each graph in Problems 55–58 represents exponential growth or decay.

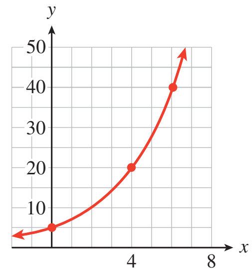
(a) Find the initial value and the growth or decay factor.

(b) Write a formula for the function.

55.



56.

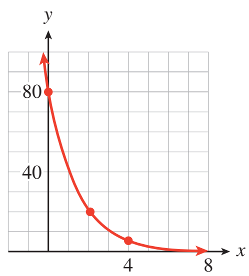


Answer.

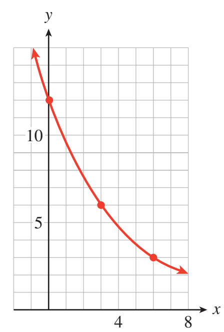
(a) Initial value 4, growth factor $2^{1/3}$

(b) $f(x) = 4 \cdot 2^{x/3}$

57.



58.



Answer.

(a) Initial value 80, decay factor $\frac{1}{2}$

(b) $f(x) = 80 \cdot \left(\dfrac{1}{2}\right)^x$

59. If 8% of the air leaks out of Brian's bicycle tire every day, what percent of the air will be left after 2 days? After a week?

Answer. 84.6%, 55.8%

60. If housing prices are increasing by 15% per year, by what percent will they increase in 2 years? In 3 years?

61. Francine says that if a population grew by 48% in 6 years, then it grew by 8% per year. Is she correct? Either justify or correct her calculation.

Answer. No, an increase of 48% in 6 years corresponds to a growth factor of $1.48^{1/6} \approx 1.0675$, or an annual growth rate of about 6.75%.

62. Delbert says that if a population decreased by 60% in 5 years, then it decreased by 12% per year. Is he correct? Either justify or correct his calculation.

In Problems 63–66, assume that each population grows exponentially with constant annual percent increase, r.

63.

a The population of the state of Texas was 16,986,335 in 1990. Write a formula in terms of r for the population in millions t years later. Round to the nearest hundredth.

b In 2000, the population was 20.85 million. Write an equation and solve for r. What was the annual percent increase to the nearest hundredth of a percent?

Answer.

a $P(t) = 16,986,335(1 + r)^t$ b 2.07%

64.

a The population of the state of Florida was 12, 937, 926 in 1990. Write a formula in terms of r for the population of Florida t years later.

b In 2000, the population was 15, 982, 378. Write an equation and solve for r. What was the annual percent increase to the nearest hundredth of a percent?

65.

a The population of Rainville was 10, 000 in 1990 and doubled in 20 years. What was the annual percent increase to the nearest hundredth percent?

b The population of Elmira was 350, 000 in 1990 and doubled in 20 years. What was the annual percent increase to the nearest hundredth of a percent?

c If a population doubles in 20 years, does the percent increase depend on the size of the original population?

d The population of Grayling doubled in 20 years. What was the annual percent increase to the nearest hundredth of a percent?

Answer.

a 3.53% b 3.53% c No d 3.53%

66.

a The population of Boomtown was 300 in 1908 and tripled in 7 years. What was the annual percent increase to the nearest hundredth of a percent?

b The population of Fairview was 15, 000 in 1962 and tripled in 7 years. What was the annual percent increase to the nearest hundredth of a percent?

c If a population triples in 7 years, does the percent increase depend on the size of the original population?

d The population of Pleasant Lake tripled in 7 years. What was the annual percent increase to the nearest hundredth of a percent?

67. A researcher starts 2 populations of fruit flies of different species, each with 30 flies. Species A increases by 30% in 6 days and species B increases by 20% in 4 days.

a What was the population of species A after 6 days? Find the daily growth factor for species A.

b What was the population of species B after 4 days? Find the daily growth factor for species B.

c Which species multiplies more rapidly?

Answer.

a 39; 1.045 b 35; 1.047 c Species B

68. A biologist isolates two strains of a particular virus and monitors the growth of each, starting with samples of 0.01 gram. Strain A increases by 10% in 8 hours and strain B increases by 12% in 9 hours.

a How much did the sample of strain A weigh after 8 hours? What was its hourly growth factor?

b How much did the sample of strain B weigh after 9 hours? What was its hourly growth factor?

c Which strain of virus grows more rapidly?

In Problems 69–72, we compare linear and exponential growth.

69. At a large university 3 students start a rumor that final exams have been canceled. After 2 hours, 6 students (including the first 3) have heard the rumor.

a Assuming that the rumor grows linearly, complete the table below for $L(t)$, the number of students who have heard the rumor after t hours. Then write a formula for the function $L(t)$. Graph the function.

t	0	2	4	6	8
$L(t)$					

b Complete the table below, assuming that the rumor grows exponentially. Write a formula for the function $E(t)$ and graph it on the same set of axes with $L(t)$.

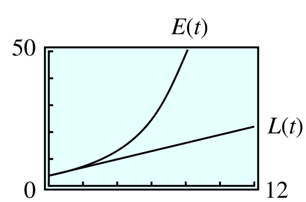
t	0	2	4	6	8
$E(t)$					

Answer.

a

t	0	2	4	6	8
$L(t)$	3	6	9	12	15

$L(t) = 3 + 1.5t$



b

t	0	2	4	6	8
$E(t)$	3	6	12	24	48

$E(t) = 3 \cdot 2^{t/2}$

70. Over the weekend the Midland Infirmary identifies four cases of 2009 H1N1 Pandemic flu. Three days later it has treated a total of ten cases.

a Assuming that the number of flu cases grows linearly, complete the table below for $L(t)$, the number of people infected after t

days. Then write a formula for the function $L(t)$. Graph the function.

t	0	3	6	9	12
$L(t)$			16		

b Complete the table below, assuming that the flu grows exponentially. Write a formula for the function $E(t)$ and graph it on the same set of axes with $L(t)$.

t	0	3	6	9	12
$E(t)$				62.5	

71. The world's population of tigers declined from 10,400 in 1980 to 6000 in 1998.

a If the population declined linearly, what was its annual rate of decrease?

b If the population declined exponentially, what was its annual decay factor? What was its annual percent decrease?

c Predict the number of tigers in 2010 under each assumption, linear or exponential decline.

Answer.

a 244 tigers per year

b 0.97; 3%

c Linear: 3067; Exponential: 4170

72. In 2003, the Center for Biological Diversity filed a lawsuit against the federal government for failing to protect Alaskan sea otters. The population of sea otters, which numbered between 150,000 and 300,000 before hunting began in 1741, declined from about 20,000 in 1992 to 6000 in 2000. (Source: Center for Biological Diversity)

a If the population declined linearly after 1992, what was its annual rate of decrease?

b If the population declined exponentially after 1992, what was its annual rate of decrease?

c Predict the number of sea otters in 2010 under each assumption, linear or exponential decline

4.2 Exponential Functions

In Section 4.1, p. 394, we studied functions that describe exponential growth or decay. More formally, we define an **exponential function** as follows.

Exponential Function.

$$f(x) = ab^x, \quad \text{where} \quad b > 0 \quad \text{and} \quad b \neq 1, \quad a \neq 0$$

Some examples of exponential functions are

$$f(x) = 5^x, \quad P(t) = 250(1.7)^t, \quad \text{and} \quad g(n) = 2.4(0.3)^n$$

The constant a is the y-intercept of the graph because

$$f(0) = a \cdot b^0 = a \cdot 1 = a$$

For the examples above, we find that the y-intercepts are

$$\begin{aligned} f(0) &= 5^0 = 1, \\ P(0) &= 250(1.7)^0 = 250, \text{ and} \\ g(0) &= 2.4(0.3)^0 = 2.4 \end{aligned}$$

The positive constant b is called the **base** of the exponential function.

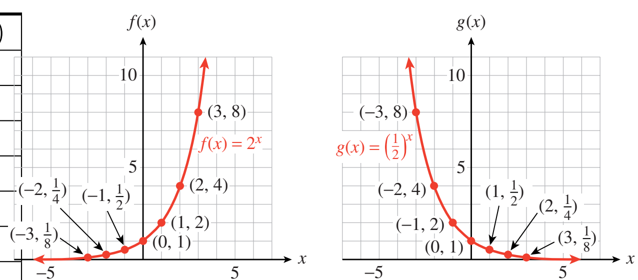
Note 4.2.1

- We do not allow b to be negative, because if $b < 0$, then b^x is not a real number for some values of x. For example, if $b = -4$ and $f(x) = (-4)^x$, then $f(1/2) = (-4)^{1/2}$ is an imaginary number.
- We also exclude $b = 1$ as a base because $1^x = 1$ for all values of x; hence the function $f(x) = 1^x$ is actually the constant function $f(x) = 1$.

4.2.1 Graphs of Exponential Functions

The graphs of exponential functions have two characteristic shapes, depending on whether the base, b, is greater than 1 or less than 1. As typical examples, consider the graphs of $f(x) = 2^x$ and $g(x) = \left(\frac{1}{2}\right)^x$ shown below. Some values for f and g are recorded in the tables.

x	$f(x)$	x	$g(x)$
-3	$\frac{1}{8}$	-3	8
-2	$\frac{1}{4}$	-2	4
-1	$\frac{1}{2}$	-1	2
0	1	0	1
1	2	1	$\frac{1}{2}$
2	4	2	$\frac{1}{4}$
3	8	3	$\frac{1}{8}$



(a) (b)

Notice that $f(x) = 2^x$ is an increasing function and $g(x) = \left(\frac{1}{2}\right)^x$ is a decreasing function. Both are concave up. In general, exponential functions have the following properties.

Properties of Exponential Functions, $f(x) = ab^x$, $a > 0$.

1. Domain: all real numbers.
2. Range: all positive numbers.
3. If $b > 1$, the function is increasing and concave up;
   if $0 < b < 1$, the function is decreasing and concave up.
4. The y-intercept is $(0, a)$. There is no x-intercept.

In the table for $f(x)$, you can see that as the x-values decrease toward negative infinity, the corresponding y-values decrease toward zero. As a result, the graph of f decreases toward the x-axis as we move to the left. Thus, the negative x-axis is a **horizontal asymptote** for exponential functions with $b > 1$, as shown in figure (a).

For exponential functions with $0 < b < 1$, the positive x-axis is an asymptote, as illustrated in figure (b). (See Section 2.2, p. 170 to review asymptotes.)

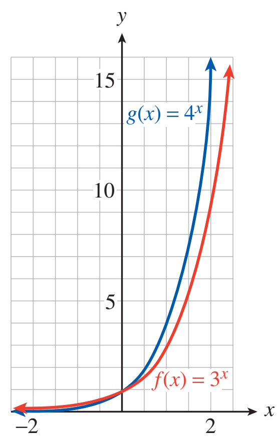
In Example 4.2.2, p. 418, we compare two increasing exponential functions. The larger the value of the base, b, the faster the function grows. In this example, both functions have $a = 1$.

Example 4.2.2 Compare the graphs of $f(x) = 3^x$ and $g(x) = 4^x$.

Solution. We evaluate each function for several convenient values, as shown in the table.

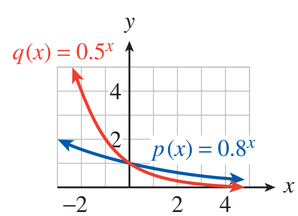
Then we plot the points for each function and connect them with smooth curves. For positive x-values, $g(x)$ is always larger than $f(x)$, and is increasing more rapidly. In the figure, we can see that $g(x) = 4^x$ climbs more rapidly than $f(x) = 3^x$. Both graphs cross the y-axis at (0, 1).

x	$f(x)$	$g(x)$
-2	$\frac{1}{9}$	$\frac{1}{16}$
-1	$\frac{1}{3}$	$\frac{1}{4}$
0	1	1
1	3	4
2	9	16



□

Note 4.2.3 For decreasing exponential functions, those with bases between 0 and 1, the smaller the base, the more steeply the graph decreases. For example, compare the graphs of $p(x) = 0.8^x$ and $q(x) = 0.5^x$ shown in the figure at right.



Checkpoint 4.2.4

a State the ranges of the functions f and g in Example 4.2.2, p. 418 on the domain $[-2, 2]$.

b State the ranges of the functions p and q shown in the Note above on the domain $[-2, 2]$. Round your answers to two decimal places.

Answer.

a $f : \left[\frac{1}{9}, 9\right]$; $g : \left[\frac{1}{16}, 16\right]$

b $p : [0.64, 1.56]$; $q : [0.25, 4]$

4.2.2 Transformations of Exponential Functions

In Chapter 2, p. 149, we considered transformations of the basic graphs. For instance, the graphs of the functions $y = x^2 - 4$ and $y = (x-4)^2$ are shifts of the basic parabola, $y = x^2$. In a similar way, we can shift or stretch the graph of an exponential function while the basic shape is preserved.

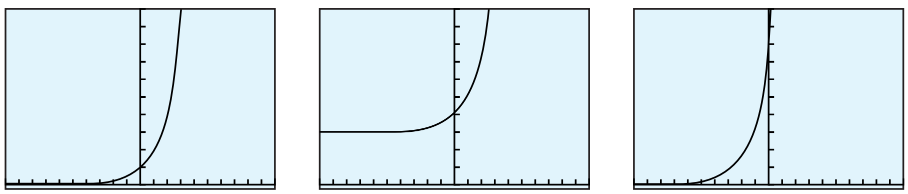
Example 4.2.5 Use your calculator to graph the following functions. Describe how these graphs compare with the graph of $h(x) = 2^x$.

a $f(x) = 2^x + 3$ b $g(x) = 2^{x+3}$

Solution. Enter the formulas for the three functions as shown below. Note the parentheses around the exponent in the keying sequence for $Y_3 = g(x)$.

$Y_1 = 2$ ^ X
$Y_2 = 2$ ^ X + 3
$Y_3 = 2$ ^ (X + 3)

The graphs of $h(x) = 2^x$, $f(x) = 2^x + 3$, and $g(x) = 2^{x+3}$ in the standard window are shown below.



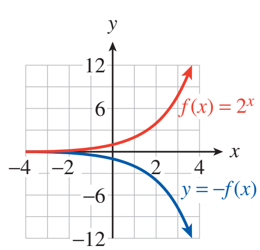
(a) $h(x) = 2^x$ **(b)** $f(x) = 2^x + 3$ **(c)** $g(x) = 2^{x+3}$

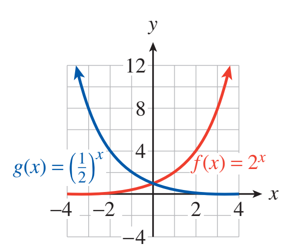
a The graph of $f(x) = 2^x + 3$, shown in figure (b), has the same basic shape as that of $h(x) = 2^x$, but it has a horizontal asymptote at $y = 3$ instead of at $y = 0$ (the x-axis). In fact, $f(x) = h(x) + 3$, so the graph of f is a vertical translation of the graph of h by 3 units. If every point on the graph of $h(x) = 2^x$ is moved 3 units upward, the result is the graph of $f(x) = 2^x + 3$.

b First note that $g(x) = 2^x + 3 = h(x+3)$. In fact, the graph of $g(x) = 2^{x+3}$ shown in figure (c) has the same basic shape as $h(x) = 2^x$ but has been translated 3 units to the left.

□

What about reflections? Recall that the graph of $y = -f(x)$ is the reflection about the x-axis of the graph of $y = f(x)$. The graphs of $y = 2^x$ and $y = -2^x$ are shown at left below.





You may have also noticed a relationship between the graphs of $f(x) = 2^x$ and $g(x) = \left(\dfrac{1}{2}\right)^x$, which are shown at right above. The graph of g is the reflection of the graph of f about the y-axis. We can see why this is true by writing the formula for $g(x)$ in another way:

$$g(x) = \left(\frac{1}{2}\right)^x = \left(2^{-1}\right)^x = 2^{-x}$$

We see that $g(x)$ is the same function as $f(-x)$. Replacing x by $-x$ in the formula for a function switches every point (p, q) on the graph with the point $(-p, q)$ and thus reflects the graph about the y-axis.

Reflections of Graphs.

1. The graph of $y = -f(x)$ is the reflection of the graph of $y = f(x)$ about the x-axis.
2. The graph of $y = f(-x)$ is the reflection of the graph of $y = f(x)$ about the y-axis.

Checkpoint 4.2.6 Which of the functions below have the same graph? Explain why.

a $f(x) = \left(\frac{1}{4}\right)^x$ b $g(x) = -4^x$ c $h(x) = 4^{-x}$

Answer. (a) and (c)

4.2.3 Comparing Exponential and Power Functions

Exponential functions are not the same as the power functions we studied in Chapter 3, p. 289. Although both involve expressions with exponents, it is the location of the variable that makes the difference.

Power Functions vs Exponential Functions.

	Power Functions	**Exponential Functions**
General formula	$h(x) = kx^p$	$f(x) = ab^x$
Description	variable base and constant exponent	constant base and variable exponent
Example	$h(x) = 2x^3$	$f(x) = 2(3^x)$

These two families of functions have very different properties, as well.

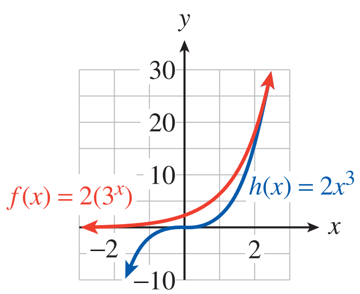
Example 4.2.7 Compare the power function $h(x) = 2x^3$ and the exponential function $f(x) = 2(3^x)$.

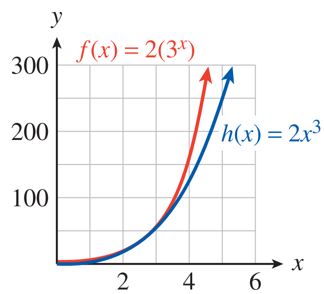
Solution. First, compare the values for these two functions shown in the table.

The scaling exponent for $h(x)$ is 3, so that when x doubles, say, from 1 to 2, the output is multiplied by 2^3, or 8.
On the other hand, we can tell that f is exponential because its values increase by a factor of 3 for each unit increase in x. (To see this, divide any function value by the previous one.)

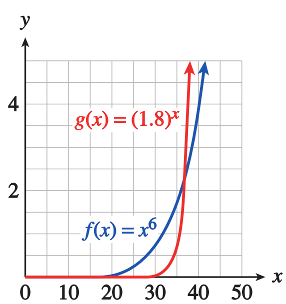
x	$h(x) = 2x^3$	$f(x) = 2(3^x)$
-3	-54	$\frac{2}{27}$
-2	-16	$\frac{1}{4}$
-1	-2	$\frac{2}{3}$
0	0	2
1	2	6
2	16	18
3	54	54

As you would expect, the graphs of the two functions are also quite different. For starters, note that the power function goes through the origin, while the exponential function has y-intercept $(0, 2)$as shown at left below.





From the table, we see that $h(3) = f(3) = 54$, so the two graphs intersect at $x = 3$. (They also intersect at approximately $x = 2.48$.) However, if you compare the values of $h(x) = 2x^3$ and $f(x) = 2(3^x)$ for larger values of x, you will see that eventually the exponential function overtakes the power function, as shown at right above. □



The relationship in Example 4.2.7, p. 420 holds true for all increasing power and exponential functions: For large enough values of x, the exponential function will always be greater than the power function, regardless of the parameters in the functions. The figure at left shows the graphs of $f(x) = x^6$ and $g(x) = 1.8^x$. At first, $f(x) > g(x)$, but at around $x = 37$, $g(x)$ overtakes $f(x)$, and $g(x) > f(x)$ for all $x > 37$.

Checkpoint 4.2.8 Which of the following functions are exponential functions, and which are power functions?

a $F(x) = 1.5^x$

b $G(x) = 3x^{1.5}$

c $H(x) = 3^{1.5x}$

d $K(x) = (3x)^{1.5}$

Answer. Exponential: (a) and (c); power: (b) and (d)

4.2.4 Exponential Equations

An **exponential equation** is one in which the variable is part of an exponent. For example, the equation

$$3^x = 81$$

is exponential.

Many exponential equations can be solved by writing both sides of the equation as powers with the same base. To solve the equation above, we write

$$3^x = 3^4$$

which is true if and only if $x = 4$.

In general, if two equivalent powers have the same base, then their exponents must be equal also, as long as the base is not 0 or ± 1.

Sometimes the laws of exponents can be used to express both sides of an equation as single powers of a common base.

Example 4.2.9 Solve the following equations.

a $3^{x-2} = 9^3$ b $27 \cdot 3^{-2x} = 9^{x+1}$

Solution.

a Using the fact that $9 = 3^2$, we write each side of the equation as a power of 3:

$$3^{x-2} = \left(3^2\right)^3$$
$$3^{x-2} = 3^6$$

Now we equate the exponents to obtain

$$x - 2 = 6$$
$$x = 8$$

b We write each factor as a power of 3.

$$3^3 \cdot 3^{-2x} = \left(3^2\right)^{x+1}$$

We use the laws of exponents to simplify each side:

$$3^{3-2x} = 3^{2x+2}$$

Now we equate the exponents to obtain

$$3 - 2x = 2x + 2$$
$$-4x = -1$$

The solution is $x = \dfrac{1}{4}$.

□

Checkpoint 4.2.10 Solve the equation $2^{x+2} = 128$.

Hint. Write each side as a power of 2.
Equate exponents.

Answer. $x = 5$

Example 4.2.11 During the summer a population of fleas doubles in number every 5 days. If a population starts with 10 fleas, how long will it be before there are 10,240 fleas?

Solution. Let P represent the number of fleas present after t days. The original population of 10 is multiplied by a factor of 2 every 5 days, or

$$P(t) = 10 \cdot 2^{t/5}$$

We set $P = \mathbf{10,240}$ and solve for t:

$$\begin{aligned} \mathbf{10,240} &= 10 \cdot 2^{t/5} && \text{Divide both sides by 10.} \\ 1024 &= 2^{t/5} && \text{Write 1024 as a power of 2.} \\ 2^{10} &= 2^{t/5} \end{aligned}$$

We equate the exponents to get $10 = \dfrac{t}{5}$, or $t = 50$. The population will grow to 10,240 fleas in 50 days. □

Checkpoint 4.2.12 During an advertising campaign in a large city, the makers of Chip-O's corn chips estimate that the number of people who have heard of Chip-O's increases by a factor of 8 every 4 days.

a If 100 people are given trial bags of Chip-O's to start the campaign, write a function, $N(t)$, for the number of people who have heard of Chip-O's after t days of advertising.

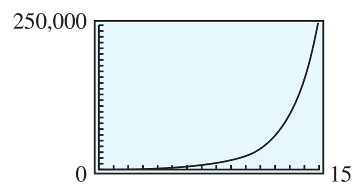
b Use your calculator to graph the function $N(t)$ on the domain $0 \leq t \leq 15$.

c How many days should the makers run the campaign in order for Chip-O's to be familiar to 51,200 people? Use algebraic methods to find your answer and verify on your graph.

Answer.

a $N(t) = 100 \cdot 8^{t/4}$

b



c 12 days

Checkpoint 4.2.13 Use the graph of $y = 5^x$ to find an approximate solution to $5^x = 285$, accurate to two decimal places.

Answer. $x \approx 3.51$

4.2.5 Section Summary

4.2.5.1 Vocabulary

Look up the definitions of new terms in the Glossary.

- Exponential function
- Base
- Exponential equation

4.2.5.2 CONCEPTS

1 An exponential function has the form

$$f(x) = ab^x, \text{ where } \ b > 0 \quad \text{and} \quad b \neq 1, \ a \neq 0$$

2 Quantities that increase or decrease by a constant percent in each time period grow or decay exponentially.

3 **Properties of Exponential Functions** $f(x) = ab^x, \quad a > 0$**.**

(a) Domain: all real numbers.

(b) Range: all positive numbers.

(c) If $b > 1$, the function is increasing and concave up; if $0 < b < 1$, the function is decreasing and concave up.

(d) The y-intercept is $(0, a)$. There is no x-intercept.

4 The graphs of exponential functions can be transformed by shifts, stretches, and reflections.

5 **Reflections of Graphs.**

1 The graph of $y = -f(x)$ is the reflection of the graph of $y = f(x)$ about the x-axis.

2 The graph of $y = f(-x)$ is the reflection of the graph of $y = f(x)$ about the y-axis.

6 Exponential functions $f(x) = ab^x$ have different properties than power functions $f(x) = kx^p$.

7 We can solve some **exponential equations** by writing both sides with the same **base** and equating the exponents.

8 We can use graphs to find approximate solutions to exponential equations.

4.2.5.3 STUDY QUESTIONS

1 Give the general form for an exponential function. What restrictions do we place on the base of the function?

2 Explain why the output of an exponential function $f(x) = b^x$ is always positive, even if x is negative.

3 How are the graphs of the functions $f(x) = b^x$ and $g(x) = \left(\dfrac{1}{b}\right)^x$ related?

4 How is an exponential function different from a power function?

5 Delbert says that $8\left(\dfrac{1}{2}\right)^x$ is equivalent to 4^x. Convince him that he is mistaken.

6 Explain the algebraic technique for solving exponential equations described in this section.

4.2.5.4 SKILLS

Practice each skill in the Homework 4.2.6, p. 425 problems listed.

1 Describe the graph of an exponential function: #1–14

2 Graph transformations of exponential functions: #15–18, 53–60

3 Evaluate exponential functions: #19–22

4 Find the equation of an exponential function from its graph: #23–26

5 Solve exponential equations: #27–44

6 Distinguish between power and exponential functions: #45–52, 65, and 66

4.2.6 Exponential Functions (Homework 4.2)

Find the y-intercept of each exponential function and decide whether the graph is increasing or decreasing.

1.

(a) $f(x) = 26(1.4)^x$

(b) $g(x) = 1.2(0.84)^x$

(c) $h(x) = 75\left(\frac{4}{5}\right)^x$

(d) $k(x) = \frac{2}{3}\left(\frac{9}{8}\right)^x$

Answer.

(a) 26; increasing

(b) 1.2; decreasing

(c) 75; decreasing

(d) $\frac{2}{3}$; increasing

2.

(a) $M(x) = 1.5(0.05)^x$

(b) $N(x) = 0.05(1.05)^x$

(c) $P(x) = \left(\frac{5}{8}\right)^x$

(d) $Q(x) = \left(\frac{4}{3}\right)^x$

Sketch the functions on the same set of axis with a domain of $[-3, 3]$. Be sure to label your functions. Describe the similarities and differences between the two graphs.

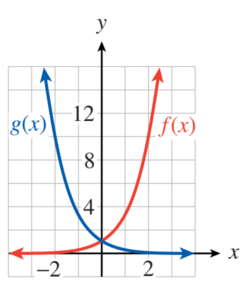
3.

a $f(x) = 3^x$

b $g(x) = \left(\frac{1}{3}\right)^x$

Answer.

x	-3	-2	-1	0	1	2	3
$f(x) = 3^x$	$\frac{1}{27}$	$\frac{1}{9}$	$\frac{1}{3}$	1	3	9	27
$g(x) = \left(\frac{1}{3}\right)^x$	27	9	3	1	$\frac{1}{3}$	$\frac{1}{9}$	$\frac{1}{27}$



The two graphs are reflections of each other across the y-axis. f is increasing, g is decreasing. f has the negative x-axis as an asymptote, and g has the positive x-axis as its asymptote.

4.

a $F(x) = \left(\dfrac{1}{10}\right)^x$

b $G(x) = 10^x$

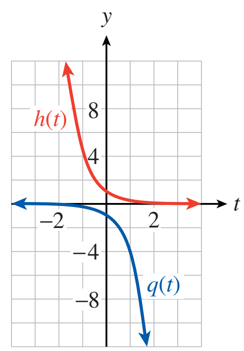
5.

a $h(t) = 4^{-t}$

b $q(t) = -4^t$

Answer.

t	-3	-2	-1	0	1	2	3
$h(t) = 4^{-t}$	64	16	4	1	$\frac{1}{4}$	$\frac{1}{16}$	$\frac{1}{64}$
$q(t) = -4^t$	$\frac{-1}{64}$	$\frac{-1}{16}$	$\frac{-1}{4}$	-1	-4	-16	-64



The graphs are reflections of each other across the origin. Both are decreasing, but h has the negative t-axis as an asymptote, and q has the positive t-axis as its asymptote.

6.

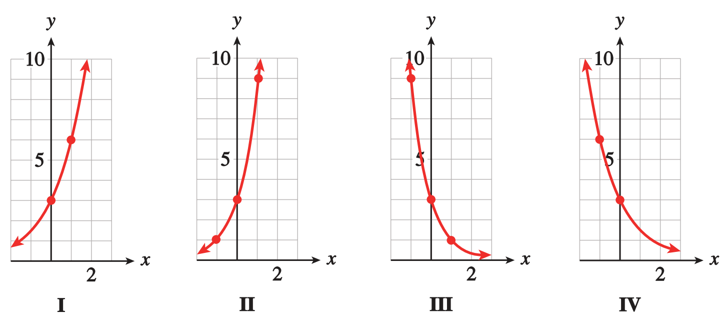
a $g(t) = 5^t$

b $P(t) = -5^t$

c $R(t) = 5^{-t}$

Match each function with its graph.

7.



a $f(x) = 3(2^x)$

b $f(x) = 3\left(\dfrac{1}{2}\right)^x$

c $f(x) = 3\left(\dfrac{1}{3}\right)^x$

d $f(x) = 3(3^x)$

Answer.

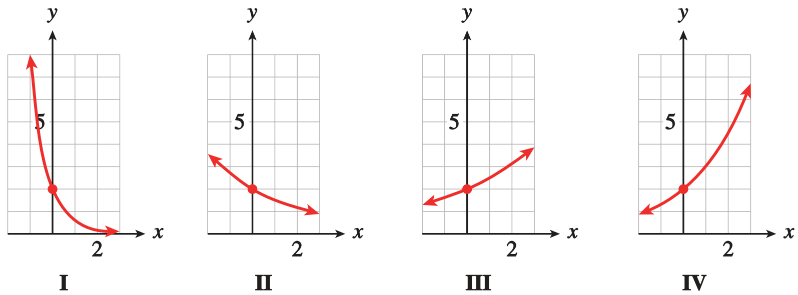
a I

b IV

c III

d II

8.



a $g(x) = 2(1.5^x)$

b $g(x) = 2\,(1.25)^x$

c $g(x) = 2\,(0.75)^x$

d $g(x) = 2(0.25)^x$

For Problems 9–12,

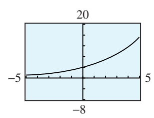
a Use a graphing calculator to graph the functions on the domain $[-5, 5]$.

b Give the range of the function on that domain, accurate to hundredths.

9. $g(t) = 4(1.3^t)$

Answer.

a



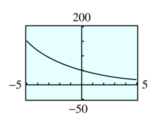
b $[1.08, 14.85]$

10. $h(t) = 3(2.4^t)$

11. $N(x) = 50(0.8^x)$

Answer.

a



b $[16.38, 152.59]$

12. $P(x) = 80(0.7^x)$

In each group of functions, which have identical graphs? Explain why using algebra and the properties of exponents.

13.

a $h(x) = 6^x$

b $k(x) = \left(\dfrac{1}{6}\right)^x$

c $m(x) = 6^{-x}$

d $n(x) = \dfrac{1}{6^x}$

Answer. Because they are defined by equivalent expressions, (b), (c), and (d) have identical graphs

14.

a $Q(t) = 5^t$

b $R(t) = \left(\dfrac{1}{5}\right)^t$

c $F(t) = \left(\dfrac{1}{5}\right)^{-t}$

d $G(t) = \dfrac{1}{5^{-t}}$

For Problems 15–18,

a Use the order of operations to explain why the two functions are different.

b Complete the table of values and graph both functions on the same set of axes.

c Describe each as a transformation of $y = 2^x$ or $y = 3^x$.

15. $f(x) = 2^{x-1}, \;\; g(x) = 2^x - 1$

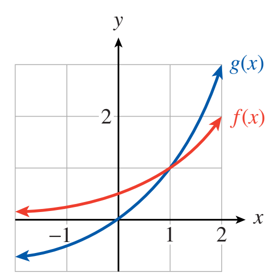
x	$y = 2^x$	$f(x)$	$g(x)$
-2			
-1			
0			
1			
2			

Answer.

a To evaluate f we subtract 1 from the input before evaluating the exponential function; to evaluate g we subtract 1 from the output of the exponential function.

b

x	$y = 2^x$	$f(x)$	$g(x)$
-2	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{-3}{4}$
-1	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{-1}{2}$
0	1	$\frac{1}{2}$	0
1	2	1	1
2	4	2	3



c The graph of f is translated 1 unit to the right; the graph of g is shifted 1 unit down.

16. $f(x) = 3^x + 2, \;\; g(x) = 3^{x+2}$

x	$y = 3^x$	$f(x)$	$g(x)$
-2			
-1			
0			
1			
2			

17. $f(x) = -3^x, \;\; g(x) = 3^{-x}$

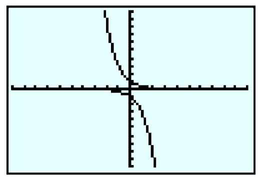
x	$y = 3^x$	$f(x)$	$g(x)$
-2			
-1			
0			
1			
2			

Answer.

a To evaluate f we take the negative of the output of the exponential function; to evaluate g we take the negative of the input.

b

x	$y = 3^x$	$f(x)$	$g(x)$
-2	$\frac{1}{9}$	$\frac{-1}{9}$	9
-1	$\frac{1}{3}$	$\frac{-1}{3}$	3
0	1	-1	1
1	3	-3	$\frac{1}{3}$
2	9	-9	$\frac{1}{9}$



c The graph of f is reflected about the x-axis; the graph of g is reflected about the y-axis.

18. $f(x) = 2^{-x}$, $g(x) = -2^x$

x	$y = 2^x$	$f(x)$	$g(x)$
-2			
-1			
0			
1			
2			

For the given function, evaluate each pair of expressions. Are they equivalent?

19. $f(x) = 3(5^x)$

a $f(a + 2)$ and $9f(a)$

b $f(2a)$ and $2f(a)$

20. $g(x) = 1.8^x$

a $g(h + 3)$ and $g(h)g(3)$

b $g(2h)$ and $[g(h)]^2$

Answer.

a $3(5^{a+2})$ is not equivalent to $9 \cdot 3(5^a)$.

b $3(5^{2a})$ is not equivalent to $2 \cdot 3(5^a)$.

21. $P(t) = 8^t$

a $P(w) - P(z)$ and $P(w - z)$

b $P(-x)$ and $\dfrac{1}{P(x)}$

Answer.

a $8^w - 8^z$ is not equivalent to 8^{w-z}.

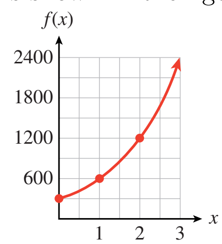
b 8^{-x} is equivalent to $\dfrac{1}{8^x}$.

22. $Q(t) = 5(0.2)^t$

a $Q(b - 1)$ and $5Q(b)$

b $Q(a)Q(b)$ and $5Q(a + b)$

23. The graph of $f(x) = P_0 b^x$ is shown in the figure.



a Read the value of P_0 from the graph.

b Make a short table of values for the function by reading values from the graph. Does your table confirm that the function is exponential?

c Use your table to calculate the growth factor, b.

d Using your answers to parts (a) and (c), write a formula for $f(x)$.

Answer.

a $P_0 = 300$

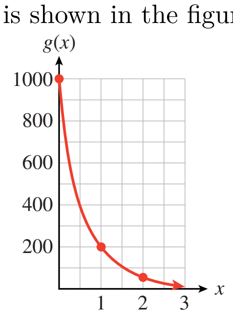
b

x	0	1	2
$f(x)$	300	600	1200

c $b = 2$

d $f(x) = 300(2)^x$

24. The graph of $g(x) = P_0 b^x$ is shown in the figure.



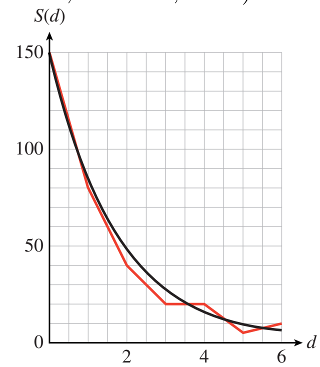
a Read the value of P_0 from the graph.

b Make a short table of values for the function by reading values from the graph. Does your table confirm that the function is exponential?

c Use your table to calculate the decay factor, b.

d Using your answers to parts (a) and (c), write a formula for $g(x)$.

25. For several days after the Northridge earthquake on January 17, 1994, the area received a number of significant aftershocks. The red graph shows that the number of aftershocks decreased exponentially over time. The graph of the function $S(d) = S_0 b^d$, shown in black, approximates the data. (Source: *Los Angeles Times*, June 27, 1995)



a Read the value of S_0 from the graph.

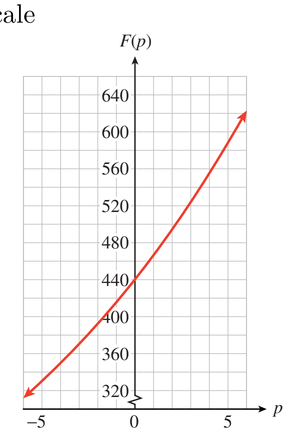
b Find an approximation for the decay factor, b, by comparing two points on the graph. (Some of the points on the graph of $S(d)$ are approximately $(1, 82)$, $(2, 45)$, $(3, 25)$, and $(4, 14)$.)

c Using your answers to (a) and (b), write a formula for $S(d)$.

Answer.

a $S_0 = 150$ b $b \approx 0.55$ c $S(d) = 150(0.55)^d$

26. The frequency of a musical note depends on its pitch. The graph shows that the frequency increases exponentially. The function $F(p) = F_0 b^p$ gives the frequency as a function of the number of half-tones, p, above the starting point on the scale



a Read the value of F_0 from the graph. (This is the frequency of the note A above middle C.)

b Find an approximation for the growth factor, b, by comparing two points on the graph. (Some of the points on the graph of $F(p)$ are approximately $(1, 466)$, $(2, 494)$, $(3, 523)$, and $(4, 554)$.)

c Using your answers to (a) and (b), write a formula for $F(p)$.

d The frequency doubles when you raise a note by one octave, which is equivalent to 12 half-tones. Use this information to find an exact value for b.

Solve each equation algebraically.

27. $5^{x+2} = 25$

Answer. $\dfrac{2}{3}$

28. $3^{x-1} = 27^{1/2}$

29. $3^{2x-1} = \dfrac{\sqrt{3}}{9}$

Answer. $\dfrac{-1}{4}$

30. $2^{3x-1} = \dfrac{\sqrt{2}}{16}$

31. $4 \cdot 2^{x-3} = 8$

Answer. $\dfrac{1}{7}$

32. $9 \cdot 3^{x+2} = 81^{-x}$

33. $27^{4x+2} = 81^{x-1}$

Answer. $\dfrac{-5}{4}$

34. $16^{2-3x} = 64^{x+5}$

35. $10^{x^2-1} = 1000$

Answer. ± 2

36. $5^{x^2-x-4} = 25$

37. Before the advent of antibiotics, an outbreak of cholera might spread through a city so that the number of cases doubled every 6 days.

a Twenty-six cases were discovered on July 5. Write a function for the number of cases of cholera t days later.

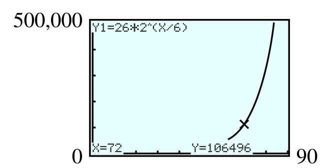
b Use your calculator to graph your function on the interval $0 \le t \le 90$.

c When should hospitals expect to be treating 106, 496 cases? Use algebraic methods to find your answer, and verify it on your graph.

Answer.

a $N(t) = 26(2)^{t/6}$

b



c 72 days later

38. An outbreak of ungulate fever can sweep through the livestock in a region so that the number of animals affected triples every 4 days.

a A rancher discovers 4 cases of ungulate fever among his herd. Write a function for the number of cases of ungulate fever t days later.

b Use your calculator to graph your function on the interval $0 \le t \le 20$.

c If the rancher does not act quickly, how long will it be until 324 head are affected? Use algebraic methods to find your answer, and verify it on your graph.

39. A color television set loses 30% of its value every 2 years.

a Write a function for the value of a television set t years after it was purchased if it cost \$700 originally.

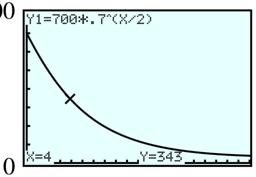
b Use your calculator to graph your function on the interval $0 \le t \le 20$.

c How long will it be before a \$700 television set depreciates to \$343? Use algebraic methods to find your answer, and verify it on your graph.

Answer.

a $V(t) = 700(0.7)^{t/2}$

b 800



0 20

c 4 yr

40. A mobile home loses 20% of its value every 3 years.

a A certain mobile home costs $20,000. Write a function for its value after t years.

b Use your calculator to graph your function on the interval $0 \le t \le 30$.

c How long will it be before a $20,000 mobile home depreciates to $12,800? Use algebraic methods to find your answer, and verify it on your graph.

Use a graph to find an approximate solution accurate to the nearest hundredth.

41. $3^{x-1} = 4$ **42.** $2^{x+3} = 5$ **43.** $4^{-x} = 7$ **44.** $6^{-x} = 3$

Answer (41).
$x = 2.26$

Answer (43).
$x = -1.40$

Decide whether each function is an exponential function, a power function, or neither.

45.

a $g(t) = 3t^{0.4}$

b $h(t) = 4(0.3)^t$

c $D(x) = 6x^{1/2}$

d $E(x) = 4x + x^4$

Answer.

a Power

b Exponential

c Power

d Neither

46.

a $R(w) = 5(5)^{w-1}$

b $Q(w) = 2^w - w^2$

c $M(z) = 0.2z^{1.3}$

d $N(z) = z^{-3}$

Decide whether the table could describe a linear function, a power function, an exponential function, or none of these.

47.

a

x	y
0	3
1	6
2	12
3	24
4	48

b

t	P
0	0
1	0.5
2	2
3	4.5
4	8

48.

a

x	N
0	0
1	2
2	16
3	54
4	128

b

p	R
0	405
1	135
2	45
3	15
4	5

Answer.

a Exponential $y = 3 \cdot 2^x$

b Power $P = 0.5t^2$

49.

a

t	y
1	100
2	50
3	$33\frac{1}{3}$
4	25
5	20

b

x	P
1	$\frac{1}{2}$
2	1
3	2
4	4
5	8

50.

a

h	a
0	70
1	7
2	0.7
3	0.07
4	0.007

b

t	Q
0	0
1	$\frac{1}{4}$
2	1
3	$\frac{9}{4}$
4	4

Answer.

a Power $y = 100x^{-1}$

b Exponential $P = \frac{1}{4} \cdot 2^x$

Fill in the tables. Graph each pair of functions in the same window. Then answer the questions below.

a Give the range of f and the range of g.

b For how many values of x does $f(x) = g(x)$?

c Estimate the value(s) of x for which $f(x) = g(x)$.

d For what values of x is $f(x) < g(x)$?

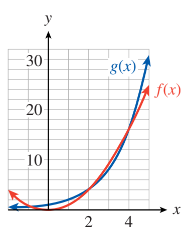
e Which function grows more rapidly for large values of x?

51.

x	$f(x) = x^2$	$g(x) = 2^x$
-2		
-1		
0		
1		
2		
3		
4		
5		

Answer.

x	$f(x) = x^2$	$g(x) = 2^x$
-2	4	$\frac{1}{4}$
-1	1	$\frac{1}{2}$
0	0	1
1	1	2
2	4	4
3	9	8
4	16	16
5	25	32



a Range of f: $[0, \infty)$;
Range of g: $(0, \infty)$

b 3

c -0.7667, 2, 4

d $(-0.7667, 2)$ and $(4, \infty)$

e g

52.

x	$f(x) = x^3$	$g(x) = 3^x$
-2		
-1		
0		
1		
2		
3		
4		
5		

For Problems 53–60, state the domain and range of each transformation, its intercept(s), and any asymptotes.

53. $f(x) = 3^x$

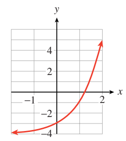
a $y = f(x) - 4$

b $y = f(x - 4)$

c $y = -4f(x)$

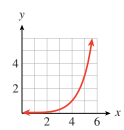
Answer.

a $y = 3^x - 4$



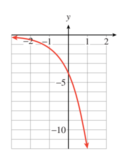
Domain: $(-\infty, \infty)$; range: $(-4, \infty)$, x-intercept $(1.26, 0)$; y-intercept $(0, -3)$; horizontal asymptote $y = -4$

b $y = 3^{x-4}$,



Domain: $(-\infty, \infty)$; range: $(0, \infty)$, no x-intercept; y-intercept $\left(0, \dfrac{1}{81}\right)$; the x-axis is the horizontal asymptote.

c $y = -4 \cdot 3^x$,



Domain: $(-\infty, \infty)$; range: $(-\infty, 0)$, no x-intercept; y-intercept $(0, -4)$; the x-axis is the horizontal asymptote.

54. $g(x) = 4^x$

a $y = g(x) + 2$

b $y = g(x + 2)$

c $y = 2g(x)$

55. $h(t) = 6^t$

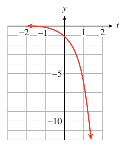
a $y = -h(t)$

b $y = h(-t)$

c $y = -h(-t)$

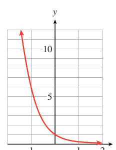
Answer.

a $y = -6^t$



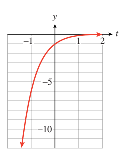
Domain: $(-\infty, \infty)$; range: $(-\infty, 0)$, no t-intercept; y-intercept $(0, -1)$; the t-axis is the horizontal asymptote.

b $y = 6^{-t}$,



Domain: $(-\infty, \infty)$; range: $(0, \infty)$, no t-intercept; y-intercept $(0, 1)$; the t-axis is the horizontal asymptote.

c $y = -6^{-t}$,



Domain: $(-\infty, \infty)$; range: $(-\infty, 0)$, no t-intercept; y-intercept $(0, -1)$; the t-axis is the horizontal asymptote.

56. $j(t) = \left(\dfrac{1}{3}\right)^t$

a $y = j(-t)$

b $y = -j(t)$

c $y = -j(-t)$

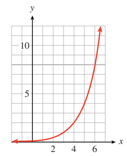
57. $g(x) = 2^x$

a $y = g(x - 3)$

b $y = g(x - 3) + 4$

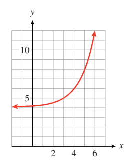
Answer.

a $y = 2^{x-3}$



Domain: $(-\infty, \infty)$; range: $(0, \infty)$, no x-intercept; y-intercept $\left(0, \frac{1}{8}\right)$; the x-axis is the horizontal asymptote.

b $y = 2^{x-3} + 4$,



Domain: $(-\infty, \infty)$; range: $(4, \infty)$, no x-intercept; y-intercept > $\left(0, \frac{33}{8}\right)$; horizontal asymptote $y = 4$

58. $f(x) = 10^x$

a $y = f(x + 5)$

b $y = f(x + 5) - 20$

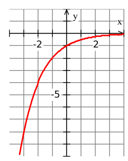
59. $N(t) = \left(\dfrac{1}{2}\right)^t$

a $y = -N(t)$

b $y = 6 - N(t)$

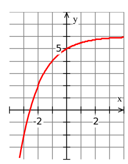
Answer.

a $y = -\left(\dfrac{1}{2}\right)^t$



Domain: $(-\infty, \infty)$; range: $(-\infty, 0)$, no t-intercept; y-intercept $(0, -1)$; the t-axis is the horizontal asymptote.

b $y = 6 - \left(\dfrac{1}{2}\right)^t$,



Domain: $(-\infty, \infty)$; range: $(-\infty, 6)$, t-intercept approximately $(-2.58, 0)$; y-intercept $(0, 5)$; horizontal asymptote is $y = 6$

60. $P(t) = 0.4^t$

a $y = -P(t)$

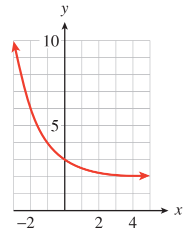
b $y = 8 - P(t)$

For Problems 61–64,

a Describe the graph as a transformation of $y = 2^x$.

b Give an equation for the function graphed.

61.

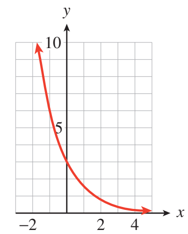


Answer.

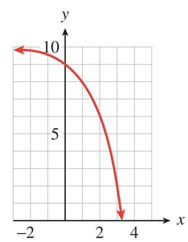
a The graph of $y = 2^x$ has been reflected about the y-axis and shifted up 2 units.

b $y = 2^{-x} + 2$

62.



63.

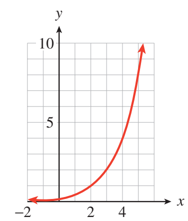


Answer.

a The graph of $y = 2^x$ has been reflected about the x-axis and shifted up 10 units.

b $y = -2^x + 10$

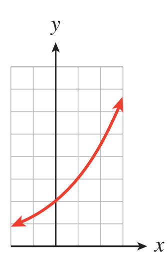
64.



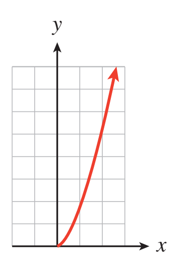
Match the graph of each function to its formula. In each formula, $a > 0$ and $b > 1$.

65.

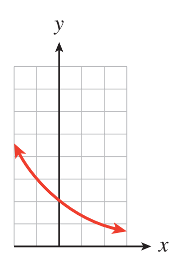
a $y = ab^x$ b $y = ab^{-x}$ c $y = ax^b$



I



II



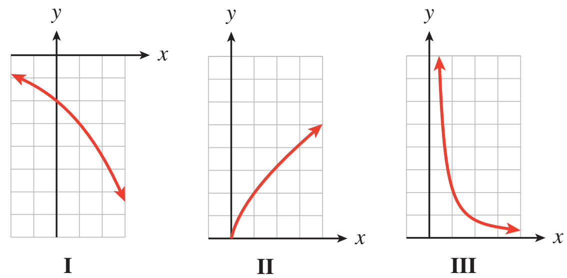
III

Answer.

a I b III c II

66.

a $y = ax^{-b}$ b $y = -ab^x$ c $y = ax^{1/b}$



67. The function $f(t)$ describes a volunteer's heart rate during a treadmill test.

$$f(t) = \begin{cases} 100 & 0 \le t < 3 \\ 56t - 68 & 3 \le t < 4 \\ 186 - 500(0.5)^t & 4 \le t < 9 \\ 100 + 6.6(0.6)^{t-14} & 9 \le t < 20 \end{cases}$$

The heart rate is given in beats per minute and t is in minutes. (See Section 2.2, p. 170 to review functions defined piecewise.) (Source: Davis, Kimmet, and Autry, 1986)

a Evaluate the function to complete the table.

t	3.5	4	8	10	15
$f(t)$					

b Sketch the graph of the function.

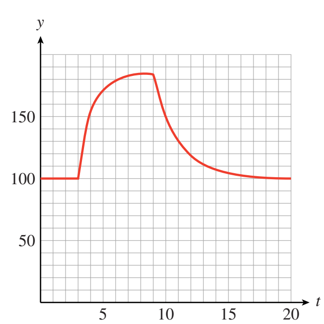
c The treadmill test began with walking at 5.5 kilometers per hour, then jogging, starting at 12 kilometers per hour and increasing to 14 kilometers per hour, and finished with a cool-down walking period. Identify each of these activities on the graph and describe the volunteer's heart rate during each phase.

Answer.

a

t	3.5	4	8	10	15
$f(t)$	128	154.75	184.05	150.93	103.96

b



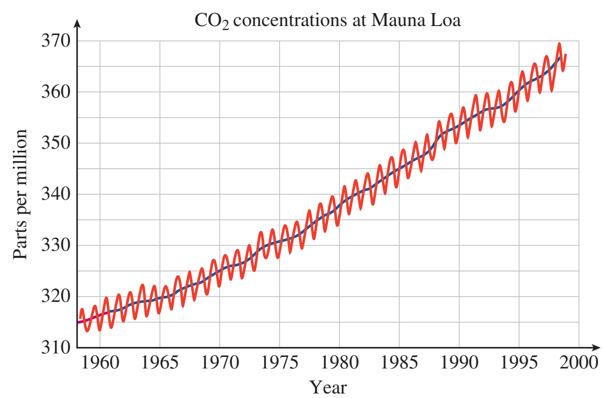
c From 0 to 3 minutes, the volunteer is walking with heart rate 100 beats per minute. The volunteer jogged at a steady pace from 3 to 4 minutes, and the heart rate increased to about 155 beats per minutes. From 4 to 9 minutes, the jogging pace increased, and the heart rate rose to about 185 beats per minute. The cooldown started at 9 minutes, and the heart rate decreased rapidly and leveled off to about 100 beats per minute.

68. Carbon dioxide (CO_2) is called a greenhouse gas because it traps part of the Earth's outgoing energy. Animals release CO_2 into the atmosphere, and plants remove CO_2 through photosynthesis. In modern times, deforestation and the burning of fossil fuels both contribute to CO_2 levels. The figure shows atmospheric concentrations of CO_2, in parts per million, measured at the Mauna Loa Observatory in Hawaii.

a The red curve shows annual oscillations in CO_2 levels. Can you explain why CO_2 levels vary throughout the year?

b The blue curve shows the average annual CO_2 readings. By approximately how much does the CO_2 level vary from its average value during the year?

c In 1960, the average CO_2 level was 316.75 parts per million, and the average level has been rising by 0.4% per year. If the level continues to rise at this rate, what CO_2 readings can we expect in the year 2100?



Hint. For part (a): Why would photosynthesis vary throughout the year?

4.3 Logarithms

4.3.1 Introduction

In this section, we introduce a new mathematical tool called a **logarithm**, which will help us solve exponential equations.

Suppose that a colony of bacteria doubles in size every day. If the colony starts with 50 bacteria, how long will it be before there are 800 bacteria? We answered questions of this type in Section 4.2, p. 416 by writing and solving an exponential equation. The function

$$P(t) = 50 \cdot 2^t$$

gives the number of bacteria present on day t, so we must solve the equation

$$800 = 50 \cdot 2^t$$

Dividing both sides by 50 yields

$$16 = 2^t$$

The solution of this equation is the answer to the following question:

To what power must we raise 2 in order to get 16?

The value of t that solves the equation is called the base 2 **logarithm** of 16. Because $2^4 = 16$, the base 2 logarithm of 16 is 4. We write this as

$$\log_2(16) = 4$$

In other words, we solve an exponential equation by computing a logarithm. You can check that $t = \mathbf{4}$ solves the problem stated above:

$$P(\mathbf{4}) = 50 \cdot 2^{\mathbf{4}} = 800$$

Thus, the unknown exponent is called a logarithm. In general, for positive values of b and x, we make the following definition.

Definition of Logarithm.

For $b > 0, b \neq 1$, the **base** b **logarithm of** x, written $\log_b(x)$, is the exponent to which b must be raised in order to yield x.

Note 4.3.1 It will help to keep in mind that a logarithm is just an exponent.

Some logarithms, like some square roots, are easy to evaluate, while others require a calculator. We will start with the easy ones.

Example 4.3.2 Compute the logarithms.

a $\log_3(9)$

b $\log_5(125)$

c $\log_4\left(\dfrac{1}{16}\right)$

d $\log_5\left(\sqrt{5}\right)$

Solution.

a To evaluate $\log_3(9)$, we ask what exponent on base 3 will produce 9. Or, 3 to what power equals 9? The exponent we need is **2**, so

$$\log_3(9) = \mathbf{2} \quad \text{because} \quad 3^{\mathbf{2}} = 9$$

We use similar reasoning to compute the other logarithms.

b $\log_5(125) = \mathbf{3} \quad \text{because} \quad 5^{\mathbf{3}} = 125$

c $\log_4\left(\dfrac{1}{16}\right) = \mathbf{-2} \quad \text{because} \quad 4^{\mathbf{-2}} = \dfrac{1}{16}$

d $\log_5\left(\sqrt{5}\right) = \dfrac{\mathbf{1}}{\mathbf{2}} \quad \text{because} \quad 5^{\mathbf{1/2}} = \sqrt{5}$

□

Checkpoint 4.3.3 Find each logarithm.

a $\log_3(81)$

b $\log\left(\dfrac{1}{1000}\right)$

Answer.

a 4

b −3

From the definition of a logarithm and the examples above, we see that the following two statements are equivalent.

Logarithms and Exponents: Conversion Equations.

If $b > 0$, $b \neq 1$, and $x > 0$,

$$\boldsymbol{y = \log_b(x)} \quad \text{if and only if} \quad \boldsymbol{x = b^y}$$

In other words, the logarithm, y, is the same as the *exponent* in $x = b^y$. We see again that *a logarithm is an exponent*; it is the exponent to which b must be raised to yield x.

These equations allow us to convert from logarithmic to exponential form, or vice versa. You should memorize the conversion equations, because we will use them frequently.

As special cases of the equivalence in (1), we can compute the following useful logarithms. For any base $b > 0, b \neq 1$,

Some Useful Logarithms.

$$\begin{aligned} \log_b(b) &= 1 \quad \text{because} \quad b^1 = b \\ \log_b(1) &= 0 \quad \text{because} \quad b^0 = 1 \\ \log_b(b^x) &= x \quad \text{because} \quad b^x = b^x \end{aligned}$$

Example 4.3.4

a $\log_2(2) = 1$ b $\log_5(1) = 0$ c $\log_3(3^4) = 4$

□

Checkpoint 4.3.5 Find each logarithm.

a $\log_n(1)$ b $\log_n(n^3)$

Answer.

a 0 b 3

4.3.2 Using the Conversion Equations

We use logarithms to solve exponential equations, just as we use square roots to solve quadratic equations. Consider the two equations

$$x^2 = 25 \quad \text{and} \quad 2^x = 8$$

We solve the first equation by taking a square root, and we solve the second equation by computing a logarithm:

$$x = \pm\sqrt{25} = \pm 5 \quad \text{and} \quad x = \log_2(8) = 3$$

The operation of taking a base b logarithm is the inverse operation for raising the base b to a power, just as extracting square roots is the inverse of squaring a number.

Every exponential equation can be rewritten in logarithmic form by using the conversion equations. Thus,

$$3 = \log_2(8) \quad \text{and} \quad 8 = 2^3$$

are equivalent statements, just as

$$5 = \sqrt{25} \quad \text{and} \quad 25 = 5^2$$

are equivalent statements. Rewriting an equation in logarithmic form is a basic strategy for finding its solution.

Example 4.3.6 Rewrite each equation in logarithmic form.

a $2^{-1} = \dfrac{1}{2}$

b $a^{1/5} = 2.8$

c $6^{1.5} = T$

d $M^v = 3K$

Solution. First identify the base b, and then the exponent or logarithm y. Use the conversion equations to rewrite $b^y = x$ in the form $\log_b(x) = y$.

a The base is 2 and the exponent is -1. Thus, $\log_2\left(\dfrac{1}{2}\right) = -1$.

b The base is a and the exponent is $\dfrac{1}{5}$. Thus, $\log_a(2.8) = \dfrac{1}{5}$.

c The base is 6 and the exponent is 1.5. Thus, $\log_6(T) = 1.5$.

d The base is M and the exponent is v. Thus, $\log_M(3K) = v$.

□

Checkpoint 4.3.7 Rewrite each equation in logarithmic form.

a $8^{-1/3} = \dfrac{1}{2}$

b $5^x = 46$

Answer.

a $\log_8\left(\dfrac{1}{2}\right) = \dfrac{-1}{3}$

b $\log_5(46) = x$

4.3.3 Base 10 Logarithms

Some logarithms are used so frequently in applications that their values are programmed into scientific and graphing calculators. These are the base 10 logarithms, such as

$$\log(1000) = 3 \quad \text{and} \quad \log(0.01) = -2$$

Base 10 logarithms are called **common logarithms**, and the subscript 10 is often omitted, so that $\log(x)$ is understood to mean $\log_{10}(x)$.

To evaluate a base 10 logarithm, we use the `LOG` key on a calculator. Many logarithms are irrational numbers, and the calculator gives as many digits as its display allows. We can then round off to the desired accuracy.

Example 4.3.8 Approximate the following logarithms to 2 decimal places.

a $\log(6.5)$

b $\log(256)$

Solution.

a The keying sequence `LOG` 6.5 `)ENTER` produces the display

```
log(6.5)
            .812913566
```

so $\log(6.5) \approx 0.81$.

b The keying sequence `LOG` 256 `)` `ENTER` yields 2.408239965, so $\log(256) \approx 2.41$.

□

Note 4.3.9 We can check the approximations found in Example 4.3.8, p. 445 with our conversion equations. Remember that a logarithm is an exponent, and in this example the base is 10. We find that

$$\begin{aligned} 10^{0.81} &\approx 6.45654229 \\ \text{and} \quad 10^{2.41} &\approx 257.0395783 \end{aligned}$$

so our approximations are reasonable, although you can see that rounding a logarithm to 2 decimal places does lose some accuracy.

For this reason, *rounding logarithms to* 4 *decimal places is customary.*

Checkpoint 4.3.10

a Evaluate $\log(250)$, and round your answer to two decimal places. Check your answer using the conversion equations.

b Evaluate $\log(250)$, and round your answer to four decimal places. Check your answer using the conversion equations.

Answer.

a 2.40

b 2.3979

4.3.4 Solving Exponential Equations

We can now solve any exponential equation with base 10. For instance, to solve the equation $16 \cdot 10^t = 360$, we first divide both sides by 16 to obtain

$$10^t = 22.5$$

Then we convert the equation to logarithmic form and evaluate:

$$t = \log(22.5) \approx 1.352182518$$

To 4 decimal places, the solution is 1.3522.

To solve exponential equations involving powers of 10, we can use the following steps.

Steps for Solving Base 10 Exponential Equations.

1. Isolate the power on one side of the equation.
2. Rewrite the equation in logarithmic form.
3. Use a calculator, if necessary, to evaluate the logarithm.
4. Solve for the variable.

Example 4.3.11 Solve the equation $38 = 95 - 15 \cdot 10^{0.4x}$

Solution. First, we isolate the power of 10: We subtract 95 from both sides of the equation and divide by −15 to obtain

$$\begin{aligned} -57 &= -15 \cdot 10^{0.4x} \quad \text{Divide by } \mathbf{-15.} \\ 3.8 &= 10^{0.4x} \end{aligned}$$

Next, we convert the equation to logarithmic form as

$$\log(3.8) = 0.4x$$

Solving for x yields

$$\frac{\log(3.8)}{0.4} = x$$

We can evaluate this expression on the calculator by entering

`LOG 3.8 ) ÷ 0.4 ENTER`

which yields 1.449458992. Thus, to four decimal places, $x \approx 1.4495$. □

Caution 4.3.12 Be careful when using a calculator to evaluate expressions involving logs. We can evaluate a single logarithm like $\log(3.8)$ by entering `LOG 3.8ENTER` without an ending parenthesis, so that the calculator shows

log(3.8)

.5795835966

But if we want to evaluate $\dfrac{\log(3.8)}{0.4}$, we must enclose 3.8 in parentheses, as shown in Example 4.3.11, p. 446. If we omit the parenthesis after 3.8, the calculator will interpret the expression as $\log\left(\dfrac{3.8}{0.4}\right)$, which is not the expression we wanted.

Checkpoint 4.3.13 Solve $12 - 30(10^{-0.2x}) = 11.25$

Answer. 8.01

4.3.5 Application to Exponential Models

We have seen that exponential functions are used to describe some applications of growth and decay, $P(t) = P_0 b^t$. There are two common questions that arise in connection with exponential models:

1. Given a value of t, what is the corresponding value of $P(t)$?
2. Given a value of $P(t)$, what is find the corresponding value of t?

To answer the first question, we evaluate the function $P(t)$ at the appropriate value. To answer the second question, we must solve an exponential equation, and this usually involves logarithms.

Example 4.3.14 The value of a large tractor originally worth \$30,000 depreciates exponentially according to the formula

$$V(t) = 30,000(10)^{-0.04t}$$

where t is in years. When will the tractor be worth half its original value?

Solution. We want to find the value of t for which $V(t) = 15,000$. That is, we want to solve the equation

$$15,000 = 30,000(10)^{-0.04t}$$

We divide both sides by 30,000 to obtain

$$0.5 = 10^{-0.04t}$$

We convert the equation to logarithmic form as

$$\log(0.5) = -0.04t$$

and divide by -0.04 to obtain

$$\frac{\log(0.5}{-0.04} = t$$

To evaluate this expression, we key in

`LOG 0.5 ) ÷ (-) 0.04 ENTER`

to find $t \approx 7.525749892$. The tractor will be worth \$15,000 in approximately $7\frac{1}{2}$ years. □

Checkpoint 4.3.15 The percentage of American homes with computers grew exponentially from 1994 to 1999. For $t = 0$ in 1994, the growth law was

$$P(t) = 25.85(10)^{0.052t}$$

[Source: Los Angeles Times, August 20, 1999]

a What percent of American homes had computers in 1994?

b If the percentage of homes with computers continued to grow at the same rate, when did 90% of American homes have a computer?

c Do you think that the function $P(t)$ will continue to model the percentage of American homes with computers? Why or why not?

Answer.

a 25.85%

b $t \approx 10.4$ (year 2004)

c No, the percent of homes with computers cannot exceed 100%.

At this stage, it seems we will only be able to solve exponential equations in which the base is 10. However, we will see in Section 4.4, p. 455 how the properties of logarithms enable us to solve exponential equations with any base.

4.3.6 Section Summary

4.3.6.1 Vocabulary

Look up the definitions of new terms in the Glossary.

- Logarithm
- Common logarithm

4.3.6.2 CONCEPTS

1 We use logarithms to help us solve exponential equations.

2 The **base b logarithm of x**, written $\log_b(x)$, is the exponent to which b must be raised in order to yield x.

3 If $b > 0$ and $x > 0$,

$$y = \log_b(x) \quad \text{if and only if} \quad x = b^y$$

4 The graphs of exponential functions can be transformed by shifts, stretches, and reflections.

5 The operation of taking a base b logarithm is the inverse operation for raising the base b to a power.

6 Base 10 logarithms are called **common logarithms**, and $\log(x)$ means $\log_{10}(x)$.

7 Steps for Solving Base 10 Exponential Equations.

(a) Isolate the power on one side of the equation.

(b) Rewrite the equation in logarithmic form.

(c) Use a calculator, if necessary, to evaluate the logarithm.

(d) Solve for the variable.

4.3.6.3 STUDY QUESTIONS

1 To find $\log_6(27)$ means to find an exponent that satisfies the equation .

2 Can a logarithm be a negative number?

3 Evaluate the following logarithms:

a $\log_8(8^{15})$ b $\log_5(5^{\sqrt{13}})$ c $\log_b(b^{2.63})$

4 Guess the solution of $10^x = 750$. Now find an approximation correct to four decimal places. Was your guess too big or too small?

4.3.6.4 SKILLS

Practice each skill in the Homework 4.3.7, p. 449 problems listed.

1 Compute logs base b using the definition: #1–10, 59–66

2 Convert from exponential to logarithmic form: #11–22

3 Approximate logarithms: #23–34

4 Solve exponential equations base 10: #35–48

5 Solve application problems: #49–58

4.3.7 Logarithms (Homework 4.3)

For Problems 1-10, find each logarithm without using a calculator.

1. (a) $\log_7(49)$ (b) $\log_2(32)$

Answer.

(a) 2 (b) 5

2. (a) $\log_4(64)$ (b) $\log_3(27)$

3. (a) $\log_3(\sqrt{3})$ (b) $\log_3\left(\dfrac{1}{3}\right)$

Answer.

(a) $\dfrac{1}{2}$ (b) -1

4. (a) $\log_5\left(\dfrac{1}{5}\right)$ (b) $\log_5(\sqrt{5})$

5.

(a) $\log_4(4)$ (b) $\log_6(1)$

Answer.

(a) 1 (b) 0

6.

(a) $\log(1)$ (b) $\log(10^{-6})$

7.

(a) $\log_8(8^5)$ (b) $\log_7(7^6)$

Answer.

(a) 5 (b) 6

8.

(a) $\log(10^{-4})$ (b) $\log(10^{-6})$

9.

(a) $\log(0.1)$ (b) $\log(0.001)$

Answer.

(a) -1 (b) -3

10.

(a) $\log(10,000)$ (b) $\log(1000)$

For Problems 11-22, rewrite the equation in logarithmic form.

11. $2^{10} = 1024$

Answer.
$\log_2(1024) = 10$

12. $11^4 = 14,641$

13. $10^{0.699} \approx 5$

Answer.
$\log(5) \approx 0.699$

14. $10^{-0.602} \approx 0.25$

15. $t^{3/2} = 16$

Answer.
$\log_t(16) = \dfrac{3}{2}$

16. $v^{5/3} = 12$

17. $0.8^{1.2} = M$

Answer.
$\log_{0.8}(M) = 1.2$

18. $3.7^{2.5} = Q$

19. $x^{5t} = W - 3$

Answer.
$\log_x(W - 3) = 5t$

20. $z^{-3t} = 2P + 5$

21. $3^{-0.2t} = 2N_0$

Answer.
$\log_3(2N_0) = -0.2t$

22. $10^{1.3t} = 3M_0$

For Problems 23–26,

a Solve each equation, writing your answer as a logarithm.

b Use trial and error to approximate the logarithm to one decimal place.

23. $4^x = 2.5$

Answer.

a $\log_4(2.5)$

b 0.7

24. $2^x = 0.2$

25. $10^x = 0.003$

Answer.

a $\log(0.003)$

b -2.5

26. $10^x = 4500$

For Problems 27–30,

a By computing successive powers of the base, trap each log between two integers.

b Use a graph to approximate each logarithm to the nearest hundredth. (*Hint*: Use the conversion equations to rewrite $x = \log_b(y)$ as an appropriate exponential equation.)

27. $\log(7)$ **28.** $\log(50)$ **29.** $\log_3(67.9)$ **30.** $\log_5(86.3)$

Answer (27).

a $0 < \log(7) < 1$

b 0.85

Answer (29).

a $3 < \log_3(67.9) < 4$

b 3.84

For Problems 31-34, use a calculator to approximate each logarithm to four decimal places. Make a conjecture about logarithms based on the results of each problem.

31.

(a) $\log(5.43)$

(b) $\log(54.3)$

(c) $\log(543)$

(d) $\log(5430)$

Answer.

(a) 0.7348 (c) 2.7348

(b) 1.7348 (d) 3.7348

When the input to the common logarithm is multiplied by 10, the output is increased by 1.

32.

(a) $\log(0.625)$

(b) $\log(0.0625)$

(c) $\log(0.00625)$

(d) $\log(0.000625)$

33.

(a) $\log(2)$

(b) $\log(4)$

(c) $\log(8)$

(d) $\log(16)$

Answer.

(a) 0.3010 (c) 0.9031

(b) 0.6021 (d) 1.2041

When the input to the common logarithm is doubled, the output is increased by about 0.3010.

34.

(a) $\log(4)$

(b) $\log(0.25)$

(c) $\log(5)$

(d) $\log(0.2)$

For Problems 35-44, solve for x. Give both the exact answer and the solution rounded to the nearest hundredth.

35. $10^{-3x} = 5$

Answer. -0.23

36. $640 = 10x^3$

37. $25 \cdot 10^{0.2x} = 80$

Answer. 2.53

36. $8 \cdot 10^{1.6x} = 312$

39. $12.2 = 2\left(10^{1.4x}\right) - 11.6$

Answer. 0.77

40. $163 = 3\left(10^{0.7x}\right) - 49.3$

41. $3\left(10^{-1.5x}\right) - 14.7 = 17.1$

Answer. -0.68

42. $4\left(10^{-0.6x}\right) + 16.1 = 28.2$

43. $80\left(1 - 10^{-0.2x}\right) = 65$

Answer. 3.63

44. $250\left(1 - 10^{-0.3x}\right) = 100$

In Problems 45–48, each calculation contains an error.Identify the error and without simply correcting it, *explain* why it is a mistake.

45.

$$\begin{aligned} 2 \cdot 5^x &= 848 \\ 10^x &= 848 \\ x &= \log(848) \quad (Incorrect!) \end{aligned}$$

Answer. $2 \cdot 5^x \neq 10^x$; the first step should be to divide both sides of the equation by 2; $x = \log_5(424)$.

46.

$$\begin{aligned} 15 \cdot 10^x &= 20 \\ 10^x &= 5 \\ x &= \log(5) \quad (Incorrect!) \end{aligned}$$

47.

$$\begin{aligned} 10^{4x} &= 20 \\ 10^x &= 5 \\ x &= \log(5) \quad (Incorrect!) \end{aligned}$$

Answer. $\frac{10^{4x}}{4} \neq 10^x$; the first step should be to write $4x = \log(20)$; $x = \frac{\log(20)}{4}$.

48.

$$\begin{aligned} 12 + 6^x &= 42 \\ 6^x &= 30 \\ x &= 5 \quad (Incorrect!) \end{aligned}$$

49. The population of the state of California increased during the years 1990 to 2000 according to the formula

$$P(t) = 29{,}760{,}021(10)^{0.0056t},$$

where t is measured in years since 1990.

(a) What was the population in 2000?

(b) Assuming the same rate of growth, estimate the population of California in the year 2015.

(c) When did the population of California reach $35{,}000{,}000$?

(d) When should the population reach 40 million?

(e) Graph the function P with a suitable domain and range, then verify your answers to parts (a) through (d).

Answer.

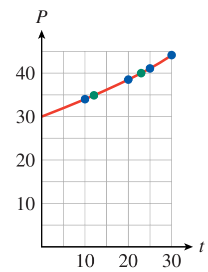
(a) $33{,}855{,}812$

(b) 38, 515, 295; 41, 080, 265; 43, 816, 051

(c) 2002

(d) 2012

(e)



50. The population of the state of New York increased during the years 1990 to 2000 according to the formula

$$P(t) = 17.9905(10)^{0.0023t},$$

where t is measured in years since 1990.

(a) What was the population in 2000? Give units in your answer.

(b) Assuming the same rate of growth, estimate the population of New York in millions in the year 2015.

(c) When did the population of New York reach 20, 000, 000?

(d) When should the population reach 30, 000, 000?

(e) Graph the function P with a suitable domain and range, then verify your answers to parts (a) through (d).

51. The absolute magnitude, M, of a star is a measurement of its brightness. For example, our Sun, not a particularly bright star, has magnitude $M = 4.83$. The magnitude in turn is a measure of the luminosity, L, or amount of light energy emitted by the star, where

$$L = L_0 10^{-0.4M}$$

(a) The luminosity of a star is measured in solar units, so that our Sun has luminosity $L = 1$. Use the values of L and M for the Sun to calculate a value of L_0 in the equation above.

(b) Is luminosity an increasing or decreasing function of magnitude? Graph the function on the domain $[-3, 3]$. What is its range on that domain?

(c) The luminosity of Sirius is 22.5 times that of the Sun, or $L = 22.5$. Calculate the magnitude of Sirius.

(d) If two stars differ in magnitude by 5, what is the ratio of their luminosities?

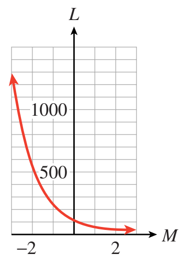
(e) A decrease in magnitude by 1 corresponds to an increase in luminosity by what factor? Give an exact value and an approximation to four decimal places.

(f) Normal stars have magnitudes between -10 and 19. What range of luminosities do stars exhibit?

Answer.

(a) 85.5

(b) Decreasing; range: $[5.4, 1355.2]$



(c) 1.45

(d) $\dfrac{1}{100}$

(e) $10^{0.4} \approx 2.5119$

(f) 2.15×10^{-6} to $855,067$

52. The loudness of a sound is a consequence of its intensity, I, or the amount of energy it generates, in watts per square meter. The intensity is related to the decibel level, D, which is another measure of loudness, by

$$I = 10^{-12+D/10}$$

(a) Is intensity an increasing or decreasing function of decibel level? The faintest sound a healthy human can hear is 0 decibels. What is the intensity of a 0 decibel sound?

(b) A whisper produces an energy intensity of 10^{-9} watts per square meter. What is the decibel level of a whisper?

(c) If two sounds differ in loudness by 10 decibels, what is the ratio of their intensities?

(d) An increase in loudness of 1 decibel produces a just noticeable difference to the human ear. By what factor does the intensity increase?

(e) Sounds of 130 decibels are at the threshold of pain for people. What is the range of the intensity function on the domain $[0, 130]$?

The atmospheric pressure decreases with altitude above the surface of the Earth. For Problems 53–58, use the function

$$P(h) = 30(10)^{-0.09h}$$

where altitude, h, is given in miles and atmospheric pressure, P, in inches of mercury. Graph this function in the window

Xmin = 0	Xmax = 9.4
Ymin = 0	Ymax = 30

Solve the problems below algebraically, and verify with your graph.

53. The elevation of Mount Everest, the highest mountain in the world, is 29,028 feet. What is the atmospheric pressure at the top?

Hint. 1 mile = 5280 feet

Answer. 9.60 in

54. The elevation of Mount McKinley, the highest mountain in the United States, is 20,320 feet. What is the atmospheric pressure at the top?

55. How high above sea level is the atmospheric pressure 20.2 inches of mercury?

Answer. 1.91 mi

56. How high above sea level is the atmospheric pressure 16.1 inches of mercury?

57. Find the height above sea level at which the atmospheric pressure is equal to one-half the pressure at sea level.

Hint. What is the altitude at sea level?

Answer. 3.34 mi

58. Find the height above sea level at which the atmospheric pressure is equal to one-fourth the pressure at sea level.

Hint. What is the altitude at sea level?

For Problems 59-66, simplify the expression.

59. $\log_2(\log_4(16))$

Answer. 1

60. $\log_5(\log_5(5))$

61. $\log[\log_3(\log_5(125))]$

Answer. 0

62. $\log\left(\log_2(\log_3(9))\right)$

63. $\log_2\left(\log_2(\log_3(81))\right)$

Answer. 1

64. $\log_4\left(\log_2(\log_3(81))\right)$

65. $\log_b(\log_b(b))$

Answer. 0

66. $\log_b(\log_a(a^b))$

4.4 Properties of Logarithms

Because logarithms are actually exponents, they have several properties that can be derived from the laws of exponents. Here are the laws we will need at present.

1. To multiply two powers with the same base, add the exponents and leave the base unchanged.

$$a^m \cdot a^n = a^{m+n}$$

2. To divide two powers with the same base, subtract the exponents and leave the base unchanged.

$$\frac{a^m}{a^n} = a^{m-n}$$

3. To raise a power to a power, keep the same base and multiply the exponents.

$$(a^m)^n = a^{mn}$$

Each of these laws corresponds to one of three properties of logarithms.

Properties of Logarithms.

If x, y, $b > 0$, and $b \neq 1$, then

1. $\log_b(xy) = \log_b(x) + \log_b(y)$

2. $\log_b\left(\dfrac{x}{y}\right) = \log_b(x) - \log_b(y)$

3. $\log_b(x^k) = k\log_b(x)$

We will consider proofs of the three properties of logarithms in the Homework problems. For now, study the examples below, keeping in mind that a logarithm is an exponent.

1. Property (1):

$$\begin{aligned} \underset{\mathbf{5}}{\log_2(32)} = \log_2(4\cdot 8) &= \log_2(4) + \log_2(8) && \text{because } 2^{\mathbf{5}} = 2^{\mathbf{2}}\cdot 2^{\mathbf{3}} \\ &= \mathbf{2} + \mathbf{3} && 32 = 4\cdot 8 \end{aligned}$$

2. Property (2):

$$\begin{aligned} \underset{\mathbf{3}}{\log_2(8)} = \log_2\left(\frac{16}{2}\right) &= \log_2(16) - \log_2(2) && \text{because } 2^{\mathbf{3}} = \frac{2^{\mathbf{4}}}{2^{\mathbf{1}}} \\ &= \mathbf{4} - \mathbf{1} && 8 = \frac{16}{2} \end{aligned}$$

3. Property (3):

$$\begin{aligned} \underset{\mathbf{6}}{\log_2(64)} = \log_2((4)^3) &= 3\log_2(4) && \text{because } \left(2^{\mathbf{2}}\right)^{\mathbf{3}} = 2^{\mathbf{6}} \\ &= \mathbf{3}\cdot\mathbf{2} && (4)^{\mathbf{3}} = 64 \end{aligned}$$

4.4.1 Using the Properties of Logarithms

Of course, these properties are useful not so much for computing logs but rather for simplifying expressions that contain variables. We will use them to solve exponential equations. But first, we will practice applying the properties. In Example 1, we rewrite one log in terms of simpler logs.

Example 4.4.1 Simplify $\log_b(\sqrt{xy})$.

Solution. First, we write $\sqrt{xy}$ using a fractional exponent:

$$\log_b(xy) = \log_b((xy)^{1/2})$$

Then we apply Property (3) to rewrite the exponent as a coefficient:

$$\log_b((xy)^{1/2}) = \frac{1}{2}\log_b(xy)$$

Finally, by Property (1) we write the log of a product as a sum of logs:

$$\frac{1}{2}(\log_b(xy)) = \frac{1}{2}(\log_b(x) + \log_b(y))$$

Thus, $\log_b(\sqrt{xy}) = \frac{1}{2}(\log_b(x) + \log_b(y))$. □

Checkpoint 4.4.2 Simplify $\log_b(xy^2)$.

Answer. $\log_b(x) - 2\log_b(y)$

Caution 4.4.3 Be careful when using the properties of logarithms! Compare the statements below:

1. $\log_b(2x) = \log_b(2) + \log_b(x)$ by Property 1,

   but

   $\log_b(2 + x) \neq \log_b(2) + \log_b(x)$

2. $\log_b\left(\dfrac{x}{5}\right) = \log_b(x) - \log_b(5)$ by Property 2,

   but

   $\log_b\left(\dfrac{x}{5}\right) \neq \dfrac{\log_b(x)}{\log_b(5)}$

We can also use the properties of logarithms to combine sums and differences of logarithms into one logarithm.

Example 4.4.4 Express $3(\log_b(x) - \log_b(y))$ as a single logarithm with a coefficient of 1.

Solution. We begin by applying Property (2) to combine the logs.

$$3(\log_b(x) - \log_b(y)) = 3\log_b\left(\frac{x}{y}\right)$$

Then, using Property (3), we replace the coefficient 3 by an exponent 3.

$$3\log_b\left(\frac{x}{y}\right) = \log_b\left(\frac{x}{y}\right)^3$$

□

Checkpoint 4.4.5 Express $2\log_b(x) + 4\log_b(x + 3)$ as a single logarithm with a coefficient of 1.

Answer. $\log_b(x^2(x + 3)^4)$

4.4.2 Solving Exponential Equations

By using Property (3), we can now solve exponential equations in which the base is not 10. For example, to solve the equation

$$5^x = 7$$

we could rewrite the equation in logarithmic form to obtain the exact solution

$$x = \log_5(7)$$

However, we cannot evaluate $\log_5(7)$; there is no log base 5 button on the calculator. If we want a decimal approximation for the solution, we begin by taking the base 10 logarithm of both sides, even though the base of the power is not 10. This gives us

$$\log(5^x) = \log 7$$

Then we use Property (3) to rewrite the left side as

$$x\log(5) = \log(7)$$

Note how using Property (3) allows us to solve the equation: The variable, x, is no longer in the exponent, and it is multiplied by a constant, $\log(5)$. To finish the solution, we divide both sides by log(5) to get

$$x = \frac{\log(7)}{\log(5)}$$

On your calculator, enter the sequence

`LOG 7 ) ÷ LOG 5 ) ENTER`

to find that $x \approx 1.2091$.

Caution 4.4.6 Do not confuse the expression $\frac{\log(7)}{\log(5)}$ with $\log\left(\frac{7}{5}\right)$; they are not the same! Property (2) allows us to simplify $\log\left(\frac{x}{y}\right)$, but not $\frac{\log(x)}{\log(y)}$. We cannot rewrite $\frac{\log(7)}{\log(5)}$, so we must evaluate it as $(\log(7))/(\log(5))$. You can check on your calculator that

$$\frac{\log(7)}{\log(5)} \neq \log\left(\frac{7}{5}\right) = \log(1.4).$$

Example 4.4.7 Solve $1640 = 80 \cdot 6^{0.03x}$

Solution. First we divide both sides by 80 to obtain

$$20.5 = 6^{0.03x}$$

Next, we take the base 10 logarithm of both sides of the equation and use Property (3) of logarithms to get

$$\log(20.5) = \log(6^{0.03x}) = 0.03x \log(6)$$

On the right side of the equation, x is multiplied by two constants, 0.03 and log(6). So, to solve for x we must divide both sides of the equation by $0.03 \log(6)$. We use a calculator to evaluate the answer:

$$x = \frac{\log(20.5)}{0.03 \log(6)} \approx 56.19$$

(On your calculator, remember to enclose the denominator, $0.03 \log(6)$, in parentheses.) □

Caution 4.4.8 In Example 4.4.7, p. 458, do not try to simplify

$$80 \cdot 6^{0.03x} \to 480^{0.03x} \quad \text{Incorrect!}$$

Remember that the order of operations tells us to compute the power $6^{0.03x}$ before multiplying by 80.

We summarize our method for solving exponential equations as follows.

Steps for Solving Exponential Equations.

1. Isolate the power on one side of the equation.
2. Take the log base 10 of both sides.
3. Simplify by applying Log Property (3).
4. Solve for the variable.

Checkpoint 4.4.9 Solve $5(1.2)^{2.5x} = 77$

Hint. Divide both sides by 5.
the log of both sides.
Apply Property (3) to simplify the left side.
Solve for $\boldsymbol{x}$.

Answer. $x = \dfrac{\log(15.4)}{2.5\log(1.2)} \approx 5.999$

4.4.3 Applications

By using the properties of logarithms, we can now solve equations that arise in exponential growth and decay models, no matter what base the exponential function uses.

Example 4.4.10 The population of Silicon City was 6500 in 1990 and has been tripling every 12 years. When will the population reach 75,000?

Solution. The population of Silicon City grows according to the formula

$$P(t) = 6500 \cdot 3^{t/12}$$

where t is the number of years after 1990. We want to find the value of t for which $P(t) = 75,000$; that is, we want to solve the equation

$$\begin{aligned} 6500 \cdot 3^{t/12} &= 75,000 && \text{Divide both sides by 6500.} \\ 3^{t/12} &= \frac{150}{13} \end{aligned}$$

Now we take the base 10 logarithm of both sides and solve for t.

$$\begin{aligned} \log\left(3^{t/12}\right) &= \log\left(\frac{150}{13}\right) && \text{Apply Property (3).} \\ \frac{t}{12}\log(3) &= \log\left(\frac{150}{13}\right) && \text{Divide by } \mathbf{\log(3)}\text{; multiply by 12.} \\ t &= \frac{12\left(\log\left(\frac{150}{13}\right)\right)}{\log(3)} \\ &\approx 26.71 \end{aligned}$$

The population of Silicon City will reach 75,000 about 27 years after 1990, or in 2017. ☐

Checkpoint 4.4.11 Traffic on U.S. highways is growing by 2.7% per year. (Source: *Time*, Jan. 25, 1999)

a Write a formula for the volume, V, of traffic as a function of time, using V_0 for the current volume.

b How long will it take the volume of traffic to double? *Hint:* Find the value of t that gives $V = 2V_0$.

Answer.

a $V(t) = V_0(1.027)^t$

b about 26 years

4.4.4 Compound Interest

The amount of money in an account that earns interest compounded annually grows exponentially according to the formula

$$A(t) = P(1+r)^t$$

(See Section 4.1, p. 394 to review compound interest.) Many accounts compound interest more frequently than once a year. If the interest is compounded n times per year, then in t years there will be nt compounding periods, and in each period the account earns interest at a rate of $\frac{r}{n}$. The amount accumulated is given by a generalization of our earlier formula.

Compound Interest.

The amount $A(t)$ accumulated (principal plus interest) in an account bearing interest compounded n times annually is

$$A(t) = P\left(1+\frac{r}{n}\right)^{nt}$$

where

P	is the principal invested,
r	is the interest rate,
t	is the time period, in years.

Example 4.4.12 Rashad deposited \$1000 in an account that pays 4% interest. Calculate the amount in his account after 5 years if the interest is compounded

a semiannually

b quarterly

c monthly

Solution.

a *Semiannually* means "twice a year,"" so we use the formula for compound interest with $P = 1000$, $r = 0.04$, $n = 2$, and $t = 5$.

$$\begin{aligned} A(5) &= 1000\left(1+\frac{0.04}{2}\right)^{2(5)} \\ &= 1000(1.02)^{10} = 1218.99 \end{aligned}$$

If interest is compounded semiannually, the balance in the account after 5 years is \$1218.99.

b *Quarterly* means "4 times a year,"" so we use the formula for compound interest with $P = 1000$, $r = 0.04$, $n = 4$, and $t = 5$.

$$\begin{aligned} A(5) &= 1000\left(1+\frac{0.04}{4}\right)^{4(5)} \\ &= 1000(1.01)^{20} = 1220.19 \end{aligned}$$

If interest is compounded quarterly, the balance in the account after 5 years is \$1220.19.

c There are 12 months in a year, so we use the formula for compound interest with $P = 1000$, $r = 0.04$, $n = 12$, and $t = 5$.

$$A(5) = 1000\left(1 + \frac{0.04}{12}\right)^{12(5)}$$
$$= 1000(1.003)^{60} = 1221.00$$

If interest is compounded monthly, the balance in the account after 5 years is \$1221.

□

Note 4.4.13 In Example 4.4.12, p. 460, you can see that the larger the value of n, the greater the value of A, keeping the other parameters fixed. More frequent compounding periods result in a higher account balance.

Checkpoint 4.4.14 Calculate the amount in Rashad's account after 5 years if the interest is compounded daily. (See Example 4.4.12, p. 460. There are 365 days in a year.)

Answer. \$1221.39

4.4.5 Solving Formulas

The techniques for solving exponential equations can also be used to solve formulas involving exponential expressions for one variable in terms of the others.

Example 4.4.15 Solve $2C = Cb^{kt}$ for t. (Assume that C and $k \neq 0$.)

Solution. First, we divide both sides by C to isolate the power.

$$b^{kt} = 2$$

Next, we take the log base 10 of both sides.

$$\log(b^{kt}) = \log(2)$$
$$kt\log(b) = \log(2) \quad \text{Apply Log Property (3).}$$

Finally, we divide both sides by $k\log(b)$ to solve for t.

$$t = \frac{\log(2)}{k\log(b)}$$

□

Checkpoint 4.4.16 Solve $A = P(1+r)^t$ for t.

Answer. $t = \dfrac{\log(A/P)}{\log(1+r)}$

4.4.6 Section Summary

4.4.6.1 Vocabulary

Look up the definitions of new terms in the Glossary.

- Compounding period

4.4.6.2 CONCEPTS

1 Properties of Logarithms.

If x, y, $b > 0$, and $b \neq 1$, then

(a) $\log_b(xy) = \log_b(x) + \log_b(y)$

(b) $\log_b\left(\dfrac{x}{y}\right) = \log_b(x) - \log_b(y)$

(c) $\log_b(x^k) = k\log_b(x)$

2 We can use the properties of logarithms to solve exponential equations with any base.

Steps for Solving Exponential Equations.

(a) Isolate the power on one side of the equation.

(b) Take the log base 10 of both sides.

(c) Simplify by applying Log Property (3).

(d) Solve for the variable.

3 The amount in an account earning interest compounded n times per year is an exponential function of time.

Compound Interest.

The amount $A(t)$ accumulated (principal plus interest) in an account bearing interest compounded n times annually is

$$A(t) = P\left(1 + \frac{r}{n}\right)^{nt}$$

where

P is the principal invested,
r is the interest rate,
t is the time period, in years.

4.4.6.3 STUDY QUESTIONS

1 The properties of logs are really another form of which familiar laws?

2 Which log property allows us to solve an exponential equation whose base is not 10?

3 Explain why $12 \cdot 10^{3x}$ is not the same as 120^{3x}.

4 Which of the following expressions are equivalent?

$$\log\left(\frac{x}{4}\right) \qquad \frac{\log(x)}{\log(4)} \qquad \log(x-4) \qquad \log(x) - \log(4)$$

5 Which of the following expressions are equivalent?

$$\log(x+2) \qquad \log(x) + \log(2) \qquad \log(2x) \qquad (\log(2))(\log(x))$$

6 Which of the following expressions are equivalent?

$$\log(x^3) \qquad (\log(3))(\log(x)) \qquad 3\log(x) \qquad \log(3^x)$$

4.4.6.4 SKILLS

Practice each skill in the Homework 4.4.7, p. 463 problems listed.

1 Use the properties of logarithms to simplify expressions: #1–24, #45–52

2 Solve exponential equations using logs base 10: #25–36

3 Solve problems about exponential models: #37–44

4 Solve problems about compound interest: #53–58

5 Solve formulas involving exponential expressions: #59–64

4.4.7 Properties of Logarithms (Homework 4.4)

1.

(a) Simplify $10^2 \cdot 10^6$.

(b) Compute $\log(10^2)$, $\log(10^6)$, and $\log(10^2 \cdot 10^6)$. How are they related?

Answer.

(a) 10^8 (b) 2; 6; 8; $2 + 6 = 8$

2.

(a) Simplify $\dfrac{10^9}{10^6}$.

(b) Compute $\log(10^9)$, $\log(10^6)$, and $\log\left(\dfrac{10^9}{10^6}\right)$. How are they related?

3.

(a) Simplify $\dfrac{b^8}{b^5}$.

(b) Compute $\log_b(b^8)$, $\log_b(b^5)$, and $\log_b\left(\dfrac{b^8}{b^5}\right)$. How are they related?

Answer.

(a) b^3 (b) 8; 5; 3; $8 - 5 = 3$

4.

(a) Simplify $b^4 \cdot b^3$.

(b) Compute $\log_b(b^4)$, $\log_b(b^3)$, and $\log_b(b^4 \cdot b^3)$. How are they related?

5.

(a) Simplify $\left(10^3\right)^5$.

(b) Compute $\log\left(10^3\right)^5$ and $\log\left(10^3\right)$. How are they related?

Answer.

(a) 10^{15} (b) 15; 3; $15 = 3 \cdot 5$

6.

(a) Simplify $\left(b^2\right)^6$.

(b) Compute $\log_b \left(b^2\right)^6$ and $\log_b \left(b^2\right)$. How are they related?

For Problems 7-14, use the properties of logarithms to expand each expression in terms of simpler logarithms. Assume that all variable expressions denote positive numbers.

7.

(a) $\log_b(2x)$

(b) $\log_b\left(\dfrac{x}{2}\right)$

Answer.

(a) $\log_b(2)+\log_b(x)$

(b) $\log_b(2)-\log_b(x)$

8.

(a) $\log_b\left(\dfrac{2x}{x-2}\right)$

(b) $\log_b(x(2x+3))$

9.

(a) $\log_3(3x^4)$

(b) $\log_5(1.1^{1/t})$

Answer.

(a) $1+4\log_3(x)$

(b) $\dfrac{1}{t}\log_5(1.1)$

10.

(a) $\log_b\left((4b)^t\right)$

(b) $\log_2\left(5(2^x)\right)$

11.

(a) $\log_b(\sqrt{bx})$

(b) $\log_3(\sqrt[3]{x^2+1})$

Answer.

(a) $\dfrac{1}{2}+\dfrac{1}{2}\log_b(x)$

(b) $\dfrac{1}{3}\log_3((x^2+1))$

12.

(a) $\log\left(\sqrt{\dfrac{2L}{R^2}}\right)$

(b) $\log\left(2\pi\sqrt{\dfrac{l}{g}}\right)$

13.

(a) $\log\left(P_0(1-m)^t\right)$

(b) $\log_4\left(\left(1+\dfrac{r}{4}\right)^{4t}\right)$

Answer.

(a) $\log(P_0)+t\log(1-m)$

(b) $4t[\log_4(4+r)-1]$

14.

(a) $\log_3\left(\dfrac{a^2-2}{a^5}\right)$

(b) $\log\left(\dfrac{a^3b^2}{(a+b)^{3/2}}\right)$

For Problems 15-20, combine into one logarithm and simplify. Assume all expressions are defined.

15.

(a) $\log_b(8) - \log_b(2)$

(b) $2\log_4(x) + 3\log_4(y)$

Answer.

(a) $\log_b(4)$ (b) $\log_4(x^2y^3)$

16.

(a) $\log_b(5) + \log_b(2)$

(b) $\dfrac{1}{4}\log_5(x) - \dfrac{3}{4}\log_5(y)$

17.

(a) $\log(2x) + 2\log(x) - \log\left(\sqrt{x}\right)$

(b) $\log(t^2 - 16) - \log(t + 4)$

Answer.

(a) $\log(2x^{5/2})$ (b) $\log(t - 4)$

18.

(a) $\log(x^2) + \log(x^3) - 5\log(x)$

(b) $\log(x^2 - x) - \log\left(\sqrt{x^3}\right)$

19.

(a) $3 - 3\log(30)$

(b) $\dfrac{1}{3}\log_6(8w^6)$

Answer.

(a) $\log\left(\dfrac{1}{27}\right)$ (b) $\log_6(2w^2)$

20.

(a) $2 - \log_4(16z^2)$

(b) $1 - 2\log_3(x)$

For Problems 21-24, use the three logs below to find the value of each expression.

$$\log_b(2) = 1.6931, \quad \log_b(3) = 2.0986, \quad \log_b(5) = 3.6094$$

(*Hint*: For example, $\log_b(15) = \log_b(3) + \log_b(5)$.)

21.

(a) $\log_b(6)$

(b) $\log_b\left(\dfrac{2}{5}\right)$

Answer.

(a) 1.7917 (b) −0.9163

22.

(a) $\log_b(10)$ (b) $\log_b(\dfrac{3}{2})$

23.

(a) $\log_b(9)$ (b) $\log_b(\sqrt{50})$

Answer.

(a) 2.1972 (b) 1.9560

24.

(a) $\log_b(25)$ (b) $\log_b(75)$

For Problems 25-36, solve the equation by using logarithms base 10. ``Give both the exact answer and the solution rounded to the nearest four decimal places.

25. $2^x = 7$

Answer.
2.8074

26. $3^x = 4$

27. $3^{x+1} = 8$

Answer.
0.8928

28. $2^{x-1} = 9$

29. $4^{x^2} = 15$

Answer.
± 1.3977

30. $3^{x^2} = 21$

31. $4.26^{-x} = 10.3$

Answer.
-1.6092

32. $2.13^{-x} = 8.1$

33. $25 \cdot 3^{2.1x} = 47$

Answer.
0.2736

34. $12 \cdot 5^{1.5x} = 85$

35. $3600 = 20 \cdot 8^{-0.2x}$

Answer.
-12.4864

36. $0.06 = 50 \cdot 4^{-0.6x}$

37. If raw meat is allowed to thaw at 50°F, Salmonella grows at a rate of 9% per hour.

(a) Write a formula for the amount of Salmonella present after t hours, if the initial amount is S_0.

(b) Health officials advise that the amount of Salmonella initially present in meat should not be allowed to increase by more than 50%. How long can meat be left to thaw at 50°F?

Answer.

(a) $S(t) = S_0(1.09)^t$

(b) 4.7 hours

38. Starting in 1998, the demand for electricity in Ireland grew at a rate of 5.8% per year. In 1998, 20,500 gigawatts were used. (Source: Electricity Supply Board of Ireland)

(a) Write a formula for electricity demand in Ireland as a function of time.

(b) If demand continues to grow at the same rate, when would it reach 30,000 gigawatts?

39. The concentration of a certain drug injected into the bloodstream decreases by 20% each hour as the drug is eliminated from the body. The initial dose creates a concentration of 0.7 milligrams per milliliter.

(a) Write a function for the concentration of the drug as a function of time.

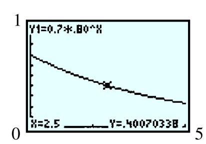
(b) The minimum effective concentration of the drug is 0.4 milligrams per milliliter. When should the second dose be administered?

(c) Verify your answer with a graph.

Answer.

(a) $C(t) = 0.7(0.80)^t$

(b) After 2.5 hours

(c) 

40. A small pond is tested for pollution and the concentration of toxic chemicals is found to be 80 parts per million. Clean water enters the pond from a stream, mixes with the polluted water, then leaves the pond so that the pollution level is reduced by 10% each month.

(a) Write a function for the concentration of toxic chemicals as a function of time.

(b) How long will it be before the concentration of toxic chemicals reaches a safe level of 25 parts per million?

(c) Verify your answer with a graph.

41. According to the National Council of Churches, the fastest growing denomination in the United States in 2004 was the Jehovah's Witnesses, with an annual growth rate of 1.82%.

(a) The Jehovah's Witnesses had $1,041,000$ members in 2004. Write a formula for the membership in the Jehovah's Witnesses in millions as a function of time, assuming that the church continues to grow at the same rate.

(b) According to this model, when will the Jehovah's Witnesses have $2,000,000$ members?

Answer.

(a) $J(t) = 1,041,000 \cdot 1.0182^t$ (b) In 2040

42. In 2004, the Presbyterian Church had $3,241,000$ members, but membership was declining by 4.87% annually.

(a) Write a formula for the membership in the Presbyterian Church as a function of time, assuming that the membership continues to decline at the same rate.

(b) When will the Presbyterian Church have $2,000,000$ members?

43. Sodium-24 is a radioactive isotope that is used in diagnosing circulatory disease. It decays into stable isotopes of sodium at a rate of 4.73% per hour.

(a) Technicians inject a quantity of sodium-24 into a patient's bloodstream. Write a formula for the amount of sodium-24 present in the bloodstream as a function of time.

(b) How long will it take for 75% of the isotope to decay?

Answer.

(a) $S(t) = S_0 \cdot 0.9527^t$ (b) 28.61 hours

44. The population of Afghanistan is growing at 2.6% per year.

(a) Write a formula for the population of Afghanistan as a function of time.

(b) In 2005, the population of Afghanistan was 29.9 million. At the given rate of growth, how long would it take the population to reach 40 million?

For Problems 46-52, evaluate each expression. Which (if any) are equal?

45.

(a) $\log_2(4 \cdot 8)$ (b) $(\log_2(4))(\log_2(8))$ (c) $\log_2(4) + \log_2(8)$

Answer.

(a) 5 (b) 6 (c) 5

(a) and (c) are equal.

46.

(a) $\log_2(16+16)$ (b) $\log_2(16) + \log_2(16)$ (c) $\log_2(2) + \log_2(16)$

47.

(a) $\log_3(27^2)$ (b) $(\log_3(27)^2)$ (c) $\log_3(27) + \log_3(27)$

Answer.

(a) 6 (b) 9 (c) 6

(a) and (c) are equal.

48.

(a) $\log_3(3 \cdot 27)$ (b) $\log_3(3) + \log_3(27)$ (c) $\log_3(3) \cdot \log_3(27)$

49.

(a) $\log\left(\dfrac{240}{10}\right)$ (b) $\dfrac{\log(240)}{\log(10)}$ (c) $\log(240) - \log(10)$

Answer.

(a) $\log(24) \approx 1.38$ (b) $\log(240) \approx 2.38$ (c) $\log(230) \approx 2.36$

None are equal.

50.

(a) $\log\left(\dfrac{1}{2} \cdot 80\right)$ (b) $\dfrac{1}{2}\log(80)$ (c) $\log\left(\sqrt{80}\right)$

51.

(a) $\log(75-15)$ (b) $\log(75) - \log(15)$ (c) $\dfrac{\log(75)}{\log(15)}$

Answer.

(a) $\log(60) \approx 1.78$ (b) $\log(5) \approx 0.70$ (c) $\dfrac{\log(75)}{\log(15)} \approx 1.59$

None are equal.

52.

(a) $\log(8 \cdot 25)$ (b) $\log(25^8)$ (c) $\log(8+25)$

For Problems 53–58, use the formula for compound interest,

$$A = P\left(1+\frac{r}{n}\right)^{nt}$$

53. What rate of interest is required so that \$1000 will yield \$1900 after 5 years if the interest rate is compounded monthly?

Answer. 12.9%

54. What rate of interest is required so that \$400 will yield \$600 after 3 years if the interest rate is compounded quarterly?

55. How long will it take a sum of money to triple if it is invested at 10% compounded daily?

Answer. About 11 years

56. How long will it take a sum of money to increase by a factor of 5 if it is invested at 10% compounded quarterly?

57.

(a) Suppose you invest \$1000 at 12% annual interest for 5 years. In this problem, we will investigate how the number of compounding periods,n, affects the amount, A. Write A as a function of n, with $P = 1000$, $r = 0.12$, and $t = 5$.

(b) Use your calculator to make a table of values for A as a function of n. What happens to A as n increases?

(c) What value of n is necessary to produce an amount $A > 1818$? To produce $A > 1820$? To produce $A > 1822$?

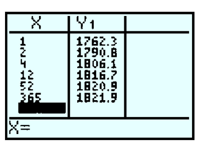
(d) Graph the function $A(n)$ in the window

Xmin = 0	Xmax = 52
Ymin = 1750	Ymax = 1850

Describe the graph: Is it increasing or decreasing? Concave up or down? Does it appear to have an asymptote? Give your best estimate for the asymptote.

Answer.

(a) $A = 1000\left(1 + \dfrac{0.12}{n}\right)^{5n}$

(b) 

A increases.

(c) 16; 31; 553

(d) Increasing, concave down, asymptotically approaching $A \approx 1822.12$

58.

(a) In this problem we will repeat Problem 49 for 4% interest. Write A as a function of n, with $P = 1000$, $r = 0.04$, and $t = 5$.

(b) Use your calculator to make a table of values for A as a function of n. What happens to A as n increases?

(c) What value of n is necessary to produce an amount $A > 1218$? To produce $A > 1220$? To produce $A > 1221.40$?

(d) Graph the function $A(n)$ in the window

Xmin = 0	Xmax = 52
Ymin = 1210	Ymax = 1225

Describe the graph: Is it increasing or decreasing? Concave up or down? Does it appear to have an asymptote? Give your best estimate for the asymptote.

For Problems 59-64, solve the formula for the specified variable.

59. $N = N_0 a^{kt}$, for k

Answer. $k = \dfrac{1}{t}\dfrac{\log(N/N_0)}{\log(a)}$

60. $Q = Q_0 b^{t/2}$, for t

61. $A = A_0(10^{kt} - 1)$, for t

Answer.
$t = \dfrac{1}{k}\log\left(\dfrac{A}{A_0} + 1\right)$

62. $B = B_0(1 - 10^{-kt})$, for t

63. $w = pv^q$, for q

Answer. $q = \dfrac{\log(w/p)}{\log(v)}$

64. $l = p^a q^b$, for b

In Problems 65–68 we use the laws of exponents to prove the properties of logarithms.

65. We will use the first law of exponents, $a^p \cdot a^q = a^{p+q}$, to prove the first property of logarithms.

(a) Let $m = \log_b(x)$ and $n = \log_b(y)$. Rewrite these equations in exponential form:

$x =$____________ and $y =$____________

(b) Now consider the expression $\log_b(xy)$. Replace x and y by your answers to part (a).

(c) Apply the first law of exponents to your expression in part (b).

(d) Use the definition of logarithm to simplify your answer to part (c).

(e) Refer to the definitions of m and n in part (a) to finish the proof.

Answer.

(a) $x = b^m$, $y = b^n$

(b) $\log_b(b^m \cdot b^n)$

(c) $\log_b(b^m \cdot b^n) = \log_b(b^{m+n})$

(d) $\log_b(b^{m+n}) = m + n$

(e) $\log_b(b^{m+n}) = \log_b(x) + \log_b(y)$

66. We will use the second law of exponents, $\dfrac{a^p}{a^q} = a^{p-q}$, to prove the second property of logarithms.

(a) Let $m = \log_b(x)$ and $n = \log_b(y)$. Rewrite these equations in exponential form:

$x =$____________ and $y =$____________

(b) Now consider the expression $\log_b\left(\frac{x}{y}\right)$. Replace x and y by your answers to part (a).

(c) Apply the second law of exponents to your expression in part (b).

(d) Use the definition of logarithm to simplify your answer to part (c).

(e) Refer to the definitions of m and n in part (a) to finish the proof.

67. We will use the third law of exponents, $(a^p)^q = a^{pq}$, to prove the third property of logarithms.

(a) Let $m = \log_b(x)$. Rewrite this equation in exponential form:

$$x = \underline{\hspace{3cm}}$$

(b) Now consider the expression $\log_b(x^k)$. Replace x by your answers to part (a).

(c) Apply the third law of exponents to your expression in part (b).

(d) Use the definition of logarithm to simplify your answer to part (c).

(e) Refer to the definitions of m in part (a) to finish the proof.

Answer.

(a) $x = b^m$

(b) $\log_b(b^m)^k$

(c) $\log_b(b^m)^k = \log_b(b^{mk})$

(d) $\log_b(b^{mk}) = mk$

(e) $\log_b(b^{mk}) = (\log_b(x)) \cdot k$

68.

(a) Use the laws of exponents to explain why $\log_b(1) = 0$.

(b) Use the laws of exponents to explain why $\log_b(b^x) = x$.

(c) Use the laws of exponents to explain why $b^{\log_b(x)} = x$.

4.5 Exponential Models

4.5.1 Fitting an Exponential Function through Two Points

To write a formula for an exponential function, we need to know the initial value, a, and the growth or decay factor, b. We can find these two parameters if we know any two function values.

Example 4.5.1 Find an exponential function that has the values $f(2) = 4.5$ and $f(5) = 121.5$.

Solution. We would like to find values of a and b so that the given function values satisfy $f(x) = ab^x$. By substituting the function values into the formula, we can write two equations.

$$\begin{array}{llll} f(2) = 4.5 & \text{means} & x = 2, y = 4.5, & \text{so } ab^2 = 4.5 \\ f(5) = 121.5 & \text{means} & x = 5, y = 121.5, & \text{so } ab^5 = 121.5 \end{array}$$

This is a system of equations in the two unknowns, a and b, but it is not a linear system. We can solve the system by the method of elimination, but we will divide one of the equations by the other.

$$\frac{ab^5}{ab^2} = \frac{121.5}{4.5}$$
$$b^3 = 27$$

Note that by dividing the two equations, we eliminated a, and we can now solve for b.

$$b^3 = 27$$
$$b = \sqrt[3]{27} = 3$$

Next we substitute $b = 3$ into either of the two equations and solve for a.

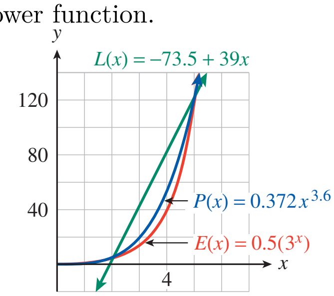
$$a(3)^2 = 4.5$$
$$a = \frac{4.5}{9}$$
$$= 0.5$$

Thus, $a = 0.5$ and $b = 3$, so the function is $f(x) = 0.5(3^x)$. □

Caution 4.5.2 Knowing only two points on the graph of f is not enough to tell us what *kind* of function f is. Through the two points in Example 4.5.1, p. 471, we can also fit a linear function or a power function.

You can check that the three functions below all satisfy $f(2) = 4.5$ and $f(5) = 121.5$. The graphs of the functions are shown at right.

$$L(x) = -73.5 + 39x$$
$$P(x) = 0.372x^{3.6}$$
$$E(x) = 0.5(3^x)$$



However, if we already know that we are looking for an exponential function, we can follow the steps below to find its formula. This method is sometimes called the **ratio method**. (Of course, if one of the known function values is the initial value, we can find b without resorting to the ratio method.)

To find an exponential function $f(x) = ab^x$ through two points:.

1 Use the coordinates of the points to write two equations in a and b.

2 Divide one equation by the other to eliminate a.

3 Solve for b.

4 Substitute b into either equation and solve for a.

Checkpoint 4.5.3 Use the ratio method to find an exponential function whose graph includes the points $(1, 20)$ and $(3, 125)$.

Answer. $f(x) = 8(2.5)^x$

We can use the ratio method to find an exponential growth or decay model if we know two function values.

Example 4.5.4 The unit of currency in Ghana is the cedi, denoted by ¢. Beginning in 1986, the cedi underwent a period of exponential inflation. In 1993, one U.S. dollar was worth ¢720, and in 1996, the dollar was worth about ¢1620. Find a formula for the number of cedi to the dollar as a function of time since 1986. What was the annual inflation rate?

Solution. We want to find a function $C(t) = ab^t$ for the number of cedi to the dollar, where $t = 0$ in 1986. We have two function values, $C(7) = 720$, and

$C(10) = 1620$, and with these values we can write two equations.

$$ab^7 = 720$$
$$ab^{10} = 1620$$

We divide the second equation by the first to find

$$\frac{ab^{10}}{ab^7} = \frac{1620}{720}$$
$$b^3 = 2.25$$

Now we can solve this last equation for b to get $b = \sqrt[3]{2.25} \approx 1.31$. Finally, we substitute $b = 1.31$ into the first equation to find a.

$$a(1.31)^7 = 720$$
$$a = \frac{720}{1.317}$$
$$= 108.75$$

Thus, $C(t) = 108.75(1.31)^t$, and the annual inflation rate was 31%. □

Checkpoint 4.5.5 The number of earthquakes that occur worldwide is a decreasing exponential function of their magnitude on the Richter scale. Between 2000 and 2005, there were 7480 earthquakes of magnitude 5 and 793 earthquakes of magnitude 6. (Source: National Earthquake Information Center, U.S. Geological Survey)

a Find a formula for the number of earthquakes, $N(m)$, in terms of their magnitude.

b It is difficult to keep an accurate count of small earthquakes. Use your formula to estimate the number of magnitude 1 earthquakes that occurred between 2000 and 2005. How many earthquakes of magnitude 8 occurred?

Answer.

a $N(m) = 558,526,329(0.106)^m$

b $59,212,751;\ \ 9$

4.5.2 Doubling Time

Instead of giving the rate of growth of a population, we can specify its rate of growth by giving the time it takes for the population to double.

Example 4.5.6 In 2005, the population of Egypt was 74 million and was growing by 2% per year.

a If it continues to grow at the same rate, how long will it take the population of Egypt to double?

b How long will it take the population to double again?

c Illustrate the results on a graph.

Solution.

a The population of Egypt is growing according to the formula $P(t) = 74(1.02)^t$, where t is in years and $P(t)$ is in millions. We would like to know when the population will reach 148 million (twice 74 million), so we solve the equation

$$\begin{aligned} 74(1.02)^t &= 148 && \text{Divide both sides by } \mathbf{74.} \\ 1.02^t &= 2 && \text{Take the log of both sides.} \\ t\log(1.02) &= \log(2) && \text{Divide both sides by log 1.02.} \\ t &= \frac{\log(2)}{\log(1.02)} \\ &\approx 35 \text{ years} \end{aligned}$$

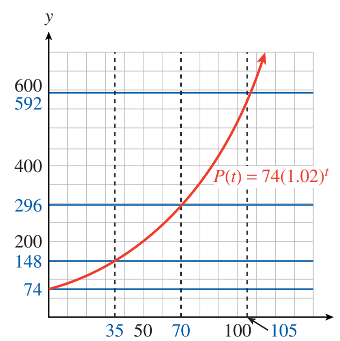
It will take the population about 35 years to double.

b Twice 148 million is 296 million, so we solve the equation

$$\begin{aligned} 148(1.02)^t &= 296 && \text{Divide both sides by } \mathbf{148.} \\ 1.02^t &= 2 && \text{Take the log of both sides.} \\ t\log(1.02) &= \log(2) && \text{Divide both sides by log 1.02.} \\ t &= \frac{\log(2)}{\log(1.02)} \\ &\approx 35 \text{ years} \end{aligned}$$

It will take the population about 35 years to double again.

c A graph of $P(t) = 74(1.02)^t$ is shown below. Note that the population doubles every 35 years.



□

In Example 4.5.6, p. 473, it took the population 35 years to double. Notice that the calculations in parts (a) and (b) are identical after the first step. In fact, we can start at any point, and it will take the population 35 years to double. We say that 35 years is the **doubling time** for this population. In the Homework problems, you will show that any increasing exponential function has a constant doubling time.

Checkpoint 4.5.7 In 2005, the population of Uganda was 26.9 million people and was growing by 3.2% per year.

a Write a formula for the population of Uganda as a function of years since 2005.

b How long will it take the population of Uganda to double?

c Use your formula from part (a) to verify the doubling time for three doubling periods.

Answer.

a $P(t) = 26.9(1.032)^t$ million

b 22 years

c $P(0) = 26.9$; $P(22) \approx 53.8$, $P(44) \approx 107.6$, $P(66) \approx 215.1$

If we know the doubling time for a population, we can immediately write down its growth law. Because the population of Egypt doubles in 35 years, we can write

$$P(t) = 74 \cdot 2^{t/35}$$

In this form, the growth factor for the population is $2^{1/35}$, and you can check that, to five decimal places, $2^{1/35} = 1.02$.

Doubling Time.

If D is the doubling time for an exponential function $P(t)$, then

$$P(t) = P_0 2^{t/D}$$

So, from knowing the doubling time, we can easily find the growth rate of a population.

Example 4.5.8 At its current rate of growth, the population of the United States will double in 115.87 years.

a Write a formula for the population of the United States as a function of time.

b What is the annual percent growth rate of the population?

Solution.

a The current population of the United States is not given, so we represent it by P_0. With t expressed in years, the formula is then

$$P(t) = P_0 2^{t/115.87}$$

b We write $2^{t/115.87}$ in the form $\left(2^{1/115.87}\right)^t$ to see that the growth factor is $b = 2^{1/115.87}$, or 1.006. For exponential growth, $b = 1 + r$, so $r = 0.006$, or 0.6%.

□

Checkpoint 4.5.9 At its current rate of growth, the population of Mexico will double in 36.8 years. What is its annual percent rate of growth?

Answer. 1.9%

4.5.3 Half-Life

The **half-life** of a decreasing exponential function is the time it takes for the output to decrease to half its original value. For example, the half-life of a radioactive isotope is the time it takes for half of the substance to decay. The half-life of a drug is the time it takes for half of the drug to be eliminated from the body. Like the doubling time, the half-life is constant for a particular function; no matter where you start, it takes the same amount of time to reach half that value.

Example 4.5.10 If you take 200 mg of ibuprofen to relieve sore muscles, the amount of the drug left in your body after t hours is $Q(t) = 200(0.73)^t$.

a What is the half-life of ibuprofen?

b When will 50 mg of ibuprofen remain in your body?

c Use the half-life to sketch a graph of $Q(t)$.

Solution.

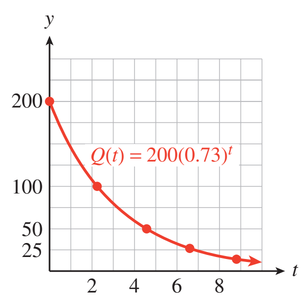
a To find the half-life, we calculate the time elapsed when only half the original amount, or 100 mg, is left.

$$\begin{aligned} 200(0.73)^t &= 100 && \text{Divide both sides by 200.}\\ 0.73^t &= 0.5 && \text{Take the log of both sides.}\\ t\log(0.73) &= \log(0.5) && \text{Divide both sides by } \log 0.73.\\ t &= \frac{\log(0.5)}{\log(0.73)}\\ &= 2.2 \end{aligned}$$

The half-life is 2.2 hours.

b After 2.2 hours, 100 mg of ibuprofen is left in the body. After another 2.2 hours, half of that amount, or 50 mg, is left. Thus, 50 mg remain after 4.4 hours.

c We locate multiples of 2.2 hours on the horizontal axis. After each interval of 2.2 hours, the amount of ibuprofen is reduced to half its previous value. The graph is shown below.



t	0	2.2	4.4	6.6	8.8
$Q(t)$	200	100	50	25	12.5

□

Checkpoint 4.5.11 Alcohol is eliminated from the body at a rate of 15% per hour.

a Write a decay formula for the amount of alcohol remaining in the body.

b What is the half-life of alcohol in the body?

Answer.

a $A(t) = A_0(0.85)^t$

b 4.3 hours

Just as we can write an exponential growth law in terms of its doubling time, we can use the half-life to write a formula for exponential decay. For example, the half-life of ibuprofen is 2.2 hours, so every 2.2 hours the amount remaining is reduced by a factor of 0.5. After t hours a 200-mg dose will be reduced to

$$Q(t) = 200(0.5)^{t/2.2}$$

Once again, you can check that this formula is equivalent to the decay function given in Example 4.5.10, p. 476.

Half-Life.

If H is the half-life for an exponential function $Q(t)$, then

$$Q(t) = Q_0(0.5)^{t/H}$$

Radioactive isotopes are molecules that decay into more stable molecules, emitting radiation in the process. Although radiation in large doses is harmful to living things, radioactive isotopes are useful as tracers in medicine and industry, and as treatment against cancer. The decay laws for radioactive isotopes are often given in terms of their half-lives.

Example 4.5.12 Cobalt-60 is used in cold pasteurization to sterilize certain types of food. Gamma rays emitted by the isotope during radioactive decay kill any bacteria present without damaging the food. The half-life of cobalt-60 is 5.27 years.

a Write a decay law for cobalt-60.

b What is the annual decay rate for cobalt-60?

Solution.

a We let $Q(t)$ denote the amount of cobalt-60 left after t years, and let Q_0 denote the initial amount. Every 5.27 years, $Q(t)$ is reduced by a factor of 0.5, so

$$Q(t) = Q_0(0.5)^{t/5.27}$$

b We rewrite the decay law in the form $Q(t) = Q_0(1 - r)^t$ as follows:

$$Q(t) = Q_0(0.5)^{t/5.27} = Q_0\left((0.5)^{1/5.27}\right)^t = Q_0(0.8768)^t$$

Thus, $1 - r = 0.8768$, so $r = 0.1232$, or 12.32%.

□

Checkpoint 4.5.13 Cesium-137, with a half-life of 30 years, is one of the most dangerous by-products of nuclear fission. What is the annual decay rate for cesium-137?

Answer. 2.28%

4.5.4 Annuities and Amortization

An **annuity** is a sequence of equal payments or deposits made at equal time intervals. A retirement fund is an example of an annuity. For ordinary annuities, payments are made at the end of each compounding period. The **future value** of an annuity is the sum of all the payments plus all the interest earned.

Future Value of an Annuity.

If you make n payments per year for t years into an annuity that pays interest rate r compounded n times per year, the **future value**, FV, of the annuity is

$$FV = \frac{P\left[\left(1+\dfrac{r}{n}\right)^{nt} - 1\right]}{\dfrac{r}{n}}$$

where each payment is P dollars.

Example 4.5.14 Greta plans to contribute \$200 a month to a retirement fund that pays 5% interest compounded monthly.

a What is the future value of Greta's retirement fund after 15 years?

b For how many years must she contribute in order to accumulate \$100,000?

Solution.

a We evaluate the formula for FV when $P = 200$, $r = 0.05$, $n = 12$, and $t = 15$. Substituting these values into the formula, we find

$$\begin{aligned} FV &= \frac{200\left[\left(1+\frac{0.05}{12}\right)^{12(15)} - 1\right]}{\frac{0.05}{12}} \\ &= \frac{200[(1.00416)^{180} - 1]}{0.0041\overline{6}} \\ &= 53,457.79 \end{aligned}$$

In 15 years, Greta's retirement fund will be worth $\$53,457.79$.

b We would like to find the value of t when $P = 200$, $r = 0.05$, $n = 12$, and $FV = 100,000$, so we must solve the equation

$$100,000 = \frac{200\left[\left(1+\frac{0.05}{12}\right)^{12t} - 1\right]}{\frac{0.05}{12}}$$

Isolate the expression in brackets.

$$\frac{1}{200}\left(\frac{0.05}{12}\right)100,000 = \left(1+\frac{0.05}{12}\right)^{12t} - 1$$

Simplify. Add 1 to both sides.

$$2.08\overline{3} + 1 = (1.0041\overline{6})^{12t}$$

Take the log of both sides.

$$\log(3.08\overline{3}) = 12t\log(1.0041\overline{6})$$ Solve for t.

$$\begin{aligned} t &= \frac{\log(3.08\overline{3})}{12\log(1.0041\overline{6})} \\ &\approx 22.6 \end{aligned}$$

Greta must contribute for over 22 years in order to accumulate $\$100,000$.

□

Checkpoint 4.5.15 Rufus is saving for a new car. He puts \$2500 a year into an account that pays 4% interest compounded annually. How many years will it take him to accumulate \$20,000? (Round up to the next whole year.)

Answer. 8 years

In Example 4.5.14, p. 478, we knew the monthly deposits into the annuity and calculated how much the sum of all the deposits (plus interest) would be in the future. Now imagine that you have just retired and you want to begin drawing monthly payments from your retirement fund. The total amount accumulated in your fund is now its **present value**, and that amount must cover your future withdrawal payments.

Present Value of an Annuity.

If you wish to receive n payments per year for t years from a fund that earns interest rate r compounded n times per year, the **present value**, PV, of the annuity must be

$$PV = \frac{P\left[1-\left(1+\dfrac{r}{n}\right)^{-nt}\right]}{\dfrac{r}{n}}$$

where each payment is P dollars.

Example 4.5.16 Candace Welthy is setting up a college fund for her nephew Delbert that will provide \$400 a month for the next 5 years. If the interest rate is 4% compounded monthly, how much money should she deposit now to cover the fund?

Solution. We would like to find the present value of an annuity in which $P = 400$, $r = 0.04$, $n = 12$, and $t = 5$. Substituting these values into the formula gives

$$\begin{aligned} PV &= \frac{400\left[1-\left(1+\frac{0.04}{12}\right)^{-(12)(5)}\right]}{\frac{0.04}{12}} \\ &= \frac{400[1-(1.00\overline{3})]^{-60}}{0.00\overline{3}} \\ &= 21,719.63 \end{aligned}$$

Delbert's Aunt Welthy should deposit \$21,719.63. ☐

Payments on a loan, such as a home mortgage, are also an annuity, but in this case the monthly payments do not collect interest; instead, we must pay interest on the present value of the loan. Repaying a loan (plus interest) by making a sequence of equal payments is called **amortizing** the loan.

Checkpoint 4.5.17 Use the formula for the present value of an annuity to calculate your monthly mortgage payment on a home loan of \$250,000 amortized over 30 years at 6% interest compounded monthly.

Answer. \$1498.88

4.5.5 Section Summary

4.5.5.1 Vocabulary

Look up the definitions of new terms in the Glossary.

- Doubling time
- Half-life
- Amortization
- Annuity

4.5.5.2 CONCEPTS

1 We can use the ratio method to fit an exponential function through two points.

To find an exponential function $f(x) = ab^x$ through two points:.

1 Use the coordinates of the points to write two equations in a and b.

2 Divide one equation by the other to eliminate a.

3 Solve for b.

4 Substitute b into either equation and solve for a.

2 Every increasing exponential has a fixed **doubling time**. Every decreasing exponential function has a fixed **half-life**.

3 If D is the doubling time for a population, its growth law can be written as $P(t) = P_0 2^{t/D}$.

4 If H is the half-life for a quantity, its decay law can be written as $Q(t) = Q_0(0.5)^{t/H}$.

5 **Future Value of an Annuity.**

If you make n payments per year for t years into an annuity that pays interest rate r compounded n times per year, the **future value**, FV, of the annuity is

$$FV = \frac{P\left[\left(1+\dfrac{r}{n}\right)^{nt} - 1\right]}{\dfrac{r}{n}}$$

where each payment is P dollars.

6 **Present Value of an Annuity.**

If you wish to receive n payments per year for t years from a fund that earns interest rate r compounded n times per year, the **present value**, PV, of the annuity must be

$$PV = \frac{P\left[1 - \left(1+\dfrac{r}{n}\right)^{-nt}\right]}{\dfrac{r}{n}}$$

where each payment is P dollars.

4.5.5.3 STUDY QUESTIONS

1 Compare the methods for fitting a line through two points and fitting an exponential function through two points.

2 A population of 3 million people has a doubling time of 15 years. What is the population 15 years from now? 30 years from now? 60 years from now?

3 Francine says that because the half-life of radium-223 is 11.7 days, after 23.4 days it will have all decayed. Is she correct? Why or why not?

4 Which is larger: the sum of all the deposits you make into your retirement fund, or the future value of the fund? Why?

5 Which is larger: the sum of all the payments you make towards your mortgage, or the amount of the loan? Why?

4.5.5.4 SKILLS

Practice each skill in the Homework 4.5.6, p. 481 problems listed.

1 Fit an exponential function through two points: #1–18

2 Find the doubling time or half-life: #19–26

3 Write an exponential function, given the doubling time or half-life: #27–34, #39–42

4 Use the formula for future value of an annuity: #43 and 44

5 Use the formula for present value of an annuity: #45 and 46

4.5.6 Exponential Models (Homework 4.5)

For Problems 1-8, find an exponential function that has the given values.

1. $A(0) = 0.14,\ A(3) = 7$

Answer.
$A(x) = 0.14(50)^{x/3}$

2. $B(0) = 28,\ B(5) = 0.25$

3. $f(7) = 12,\ f(8) = 9$

Answer.
$f(x) = \dfrac{65,536}{729}\left(\dfrac{3}{4}\right)^x$

4. $g(2) = 2.6,\ g(3) = 3.9$

5. $M(4) = 100,\ M(7) = 0.8$

Answer.
$M(x) = 62,500(0.2)^x$

6. $N(12) = 512,000,$
$N(14) = 1,024,000$

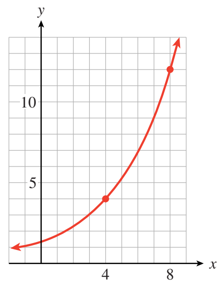
7. $s(3.5) = 16.2,\ s(6) = 3936.6$

Answer. $s(x) = \dfrac{1}{135}(9)^x$

8. $T(1.2) = 15,\ T(1.8) = 1.875$

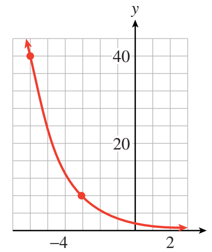
For Problems 9-12, find a formula for the exponential function shown.

9.

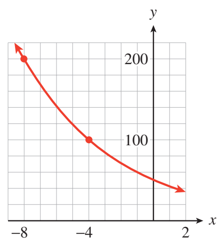


Answer. $y = \dfrac{4}{3}(3)^{x/4}$

10.

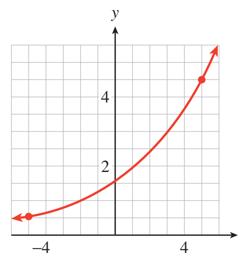


11.



Answer. $y = 50(2)^{-x/4}$

12.



For Problems 13–18,

a Fit a linear function to the points.

b Fit an exponential function to the points.

c Graph both functions on the same axes.

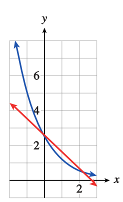
13. $(0, 2.6),\ (1, 1.3)$

Answer.

(a) $y = 2.6 - 1.3x$

(b) $y = 2.6(0.5)^x$

(c)



14. $(0, 0.48),\ (1, 0.16)$

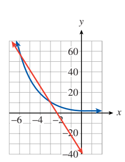
15. $(-6, 60),\ (-3, 12)$

Answer.

(a) $y = -36 - 16x$

(b) $y = \dfrac{12}{5}(5)^{-x/3}$

(c)



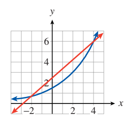
16. $(2, 1.5),\ (4, 4.5)$

17. $(-2, 0.75),\ (4, 6)$

Answer.

(a) $y = 2.5 + 0.875x$

(b) $y = 1.5(2)^{x/2}$

(c) 

18. $(-1, 0.5),\ (1, 1)$

19. Nevada was the fastest growing state in the nation between 1990 and 2000, with an annual growth rate of 5.2%.

a Write a function for the population of Nevada as a function of time. Let the initial population be P_0.

b How long will it take for the population to double?

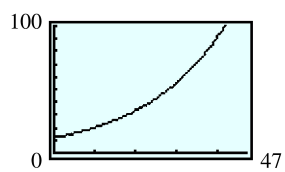
c In 1990, the population of Nevada was 12 hundred thousand. Write this in function notation.

d $P(5)$ represents the population of what year?

Answer.

(a) $P = P_0(1.052)^t$; t is the number of years since 1990.

(b) $\dfrac{\log(2)}{\log(1.052)} \approx 13.7$ years

(c) 

20. In 1986, the inflation rate in Bolivia was 8000% annually. The unit of currency in Bolivia is the boliviano.

a Write a formula for the price of an item as a function of time. Let P_0 be its initial price.

b How long did it take for prices to double? Give both an exact value and a decimal approximation rounded to two decimal places.

c Suppose $P_0 = 5$ bolivianos. Graph your function in the window Xmin $= 0$, Xmax $= 0.94$, Ymin $= 0$, Ymax $= 100$.

d Use **intersect** to verify that the price of the item doubles from 5 to 10 bolivianos, from 10 to 20, and from 20 to 40 in equal periods of time.

21. The gross domestic product (GDP) of the United Kingdom was 1 million pounds in the year 2000 and is growing at a rate of 2.8% per year. (The unit of currency in the U.K. is the pound, denoted by £.)

a Write a formula for the GDP as a function of years since 2000.

b How long will it take for the GDP to grow to 2 million pounds? Give both an exact value and a decimal approximation rounded to two decimal places.

c How long should it take for the GDP to 4 million pounds?

d Using your answers to (b) and (c), make a rough sketch of the function.

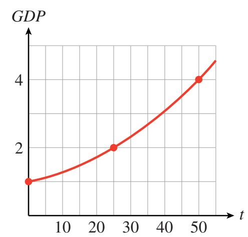
Answer.

(a) $GDP = 1.028^t$ million pounds

(b) $\dfrac{\log(2)}{\log(1.028)} \approx 25.1$ years

(c) 50.2 years

(d)



22. The number of phishing Web sites (fraudulent Web sites designed to trick victims into revealing personal financial information) is growing by 15% each month. In June 2005, there were 4000 phishing Web sites. (Source: www.itnews.com.au/newsstory)

a Write a formula for the number of phishing Web sites as a function of months since June 2005.

b How long will it take for the number of sites to reach 8000? Give both an exact value and a decimal approximation rounded to two decimal places.

c How long should it take for the number of sites to reach $16,000$?

d Using your answers to (b) and (c), make a rough sketch of the function.

23. Radioactive potassium-42, which is used by cardiologists as a tracer, decays at a rate of 5.4% per hour.

a Find the half-life of potassium-42.

b How long will it take for three-fourths of the sample to decay? For seven-eighths of the sample?

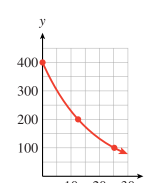
c Suppose you start with 400 milligrams of potassium-42. Using your answers to (a) and (b), make a rough sketch of the decay function.

Answer.

a $\dfrac{\log(0.5)}{\log(0.946)} \approx 12.5$ hours

b 25 hours

c



24. In October 2005, the *Los Angeles Times* published an article about efforts to save the endangered Channel Island foxes. "Their population declined by

95% to about 120 between 1994 and 2000, according to the park service."

a What was the fox population in 1994?

b Write a formula for the fox population as a function of time since 1994, assuming that their numbers declined exponentially.

c How long did it take for the fox population to be reduced to half its 1994 level? To one-quarter of the 1994 level?

d Using your answers to part (c), make a rough sketch of the decay function.

25. Caffeine leaves the body at a rate of 15.6% each hour. Your first cup of coffee in the morning has 100 mg of caffeine.

a How long will it take before you have 50 mg of that caffeine in your body?

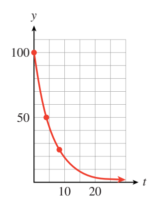
b How long will it take before you have 25 mg of that caffeine in your body?

c Using your answers to (a) and (b), make a rough sketch of the decay function.

Answer.

a $\dfrac{\log(0.5)}{\log(0.844)} \approx 4.1$ hours

b 8.2 hours

c 

26. Pregnant women should monitor their intake of caffeine, because it leaves the body more slowly during pregnancy and can be absorbed by the unborn child through the bloodstream. Caffeine leaves a pregnant woman's body at a rate of 6.7% each hour.

a How long will it take before the 100 mg of caffeine in a cup of coffee is reduced to 50 mg?

b How long will it take before the 100 mg of caffeine in a cup of coffee is reduced to 25 mg?

c Make a rough sketch of the decay function, and compare with the graph in Problem 25.

For Problems 27–30,

a Write a growth or decay formula for the exponential function.

b Find the percent growth or decay rate.

27. A population starts with 2000 and has a doubling time of 5 years.

Answer.

a $P = 2000(2)^{t/5}$

b 14.87%

28. You have 10 grams of a radioactive isotope whose half-life is 42 years.

29. A certain medication has a half-life of 18 hours in the body. You are given an initial dose of D_0 mg.

Answer.

a $D = D_0\left(\dfrac{1}{2}\right)^{t/18}$

b 3.78%

30. The doubling time of a certain financial investment is 8 years. You invest an amount M_0.

31. The half-life of radium-226 is 1620 years.

a Write a decay law for radium-226.

b What is the annual decay rate for radium-226?

Answer.

(a) $A = A_0 \left(\frac{1}{2}\right)^{t/1620}$ (b) 0.043%

32. Dichloro-diphenyl-trichloroethane (DDT) is a pesticide that was used in the middle decades of the twentieth century to control malaria. After 1945, it was also widely used on crops in the United States, and as much as one ton might be sprayed on a single cotton field. However, after the toxic effects of DDT on the environment began to appear, the chemical was banned in 1972.

a A common estimate for the half-life of DDT in the soil is 15 years. Write a decay law for DDT in the soil.

b In 1970, many soil samples in the United States contained about 0.5 mg of DDT per kg of soil. The NOAA (National Oceanic and Atmospheric Administration) safe level for DDT in the soil is 0.008 mg/kg. When will DDT content in the soil be reduced to a safe level?

33. In 1798, the English political economist Thomas R. Malthus claimed that human populations, unchecked by environmental or social constraints, double every 25 years, regardless of the initial population size.

a Write a growth law for human populations under these conditions.

b What is the growth rate in unconstrained conditions?

Answer.

(a) $P = P_0(2)^{t/25}$ (b) 2.81%

34. David Sifry observed in 2005 that over the previous two years, the number of Weblogs, or blogs, was doubling every 5 months. (Source: www.sifry.com/alerts/archives)

a Write a formula for the number of blogs t years after January 2005, assuming it continues to grow at the same rate.

b What is the growth rate for the number of blogs?

35. Let $y = f(t) = ab^t$ be an exponential growth function, with $a > 0$ and $b > 1$.

a Suppose that the value of y doubles from $t = 0$ to $t = D$, so that $f(D) = 2 \cdot f(0)$. Rewrite this fact as an equation in terms of a, b, and D.

b What does your answer to (a) tell you about the value of b^D?

c Use the first law of exponents and your result from (b) to rewrite $f(t + D)$ in terms of $f(t)$.

d Explain why your result from (c) shows that the doubling time is constant.

Answer.

(a) $ab^D = 2 \cdot ab^0 = 2a$

(b) $b^D = 2$

(c) $f(t+D) = ab^{t+D} = a \cdot b^t \cdot b^D = ab^t \cdot 2 = 2f(t)$

(d) For any value of t, after D units of time, the new value of f is 2 times the old value.

36. Let $y = g(t) = ab^t$ be an exponential decay function, with $a > 0$ and $0 < b < 1$.

a Suppose that the value of y is halved from $t = 0$ to $t = H$, so that $g(H) = \dfrac{1}{2} \cdot g(0)$. Rewrite this fact as an equation in terms of a, b, and H.

b What does your answer to (a) tell you about the value of b^H?

c Use the first law of exponents and your result from (b) to rewrite $g(t+H)$ in terms of $g(t)$.

d Explain why your result from (c) shows that the half-life is constant.

37. Let $y = g(t) = ab^t$ be an exponential decay function, with $a > 0$ and $0 < b < 1$. In this problem, we will show that there is a fixed value R such that y is decreased by a factor of $\dfrac{1}{3}$ every R units.

a Suppose that $g(R) = \dfrac{1}{3} \cdot g(0)$. Rewrite this fact as an equation in terms of a, b, and R.

b What does your answer to (a) tell you about the value of b^R?

c Use the first law of exponents and your result from (b) to rewrite $g(t+R)$ in terms of $g(t)$.

d Explain why your result from (c) shows that an exponential decay function has a constant "one-third-life."

Answer.

(a) $ab^R = \dfrac{1}{3} \cdot ab^0 = \dfrac{1}{3}a$

(b) $b^R = \dfrac{1}{3}$

(c) $g(t+R) = ab^{t+R} = a \cdot b^t \cdot b^R = ab^t \cdot \dfrac{1}{3} = \dfrac{1}{3}g(t)$

(d) For any value of t, after R units of time, the new value of g is $\frac{1}{3}$ times the old value.

38. Let $y = f(t) = ab^t$ be an exponential decay function, with $a > 0$ and $b > 1$. In this problem, we will show that there is a fixed value T such that y triples every T units.

a Suppose that $f(T) = 3 \cdot f(0)$. Rewrite this fact as an equation in terms of a, b, and T.

b What does your answer to (a) tell you about the value of b^T?

c Use the first law of exponents and your result from (b) to rewrite $f(t+T)$ in terms of $f(t)$.

d Explain why your result from (c) shows that an exponential decay function has a constant tripling time.

In Problems 39–42,

a Write a decay law for the isotope.

b Use the decay law to answer the question. (Round to the nearest ten years.)

39. Carbon-14 occurs in living organisms with a fixed ratio to nonradioactive carbon-12. After a plant or animal dies, the carbon-14 decays into stable carbon with a halflife of 5730 years. When samples from the Shroud of Turin were analyzed in 1988, they were found to have 91.2% of their original carbon-14. How old were those samples in 1988?

Answer.

(a) $A = A_0 \left(\dfrac{1}{2}\right)^{t/5730}$ (b) About 760 years old

40. Rubidium-strontium radioactive dating is used in geologic studies to measure the age of minerals. Rubidium-87 decays into strontium-87 with a half-life of 48.8 billion years. Several meteors were found to have 93.7% of their original rubidium. How old are the meteors?

41. Americium-241 (Am-241) is used in residential smoke detectors. Particles emitted as Am-241 decays cause the air in a smoke alarm to ionize, allowing current to flow between two electrodes. If smoke absorbs the particles, the current changes and sets off the alarm. The half-life of Am-241 is 432 years. How long will it take for 30% of the Am-241 to decay?

Answer.

(a) $A = A_0 \left(\dfrac{1}{2}\right)^{t/432}$ (b) About 220 years

42. Doctors can measure the amount of blood in a patient by injecting a known volume of red blood cells tagged with chromium-51. After allowing the blood to mix, they measure the percentage of tagged cells in a sample of the patient's blood and use a proportion to compute the original blood volume. Chromium-51 has a half-life of 27.7 days. How much of the original chromium-51 will still be present after 2 days?

For Problems 43 and 44, use the formula for future value of an annuity.

43. You want to retire with a nest egg of one million dollars. You plan to make fixed monthly payments of \$1000 into a savings account until then. How long will you need to make payments if the account earns 6% interest compounded monthly? What if the annual interest rate is 5%?

Answer. ≈ 30 years; ≈ 33 years

44. Francine plans to make monthly payments into an account to save up for a cruise vacation. She wants to save \$25,000 for the trip. How many \$200 payments will she need if the account pays 3% interest compounded monthly? What if the rate is 4%?

For Problems 45 and 46, use the formula for present value of an annuity.

45. You want to finance $25,000 to purchase a new car, and your financing institution charges an annual interest rate of 2.7%, compounded monthly. How large will your monthly payment be to pay off the loan in 5 years? In 6 years?

Answer. $445.89; $376.50

46. Delbert has accumulated $5000 in credit card debt. The account charges an annual interest rate of 17%, compounded monthly. Delbert decides not to make any further charges to his account and to pay it off in equal monthly payments. What will the payment be if Delbert decides to pay off the entire amount in 5 years? In 10 years?

47. Moore's law predicts that the number of transistors per computer chip will continue to grow exponentially, with a doubling time of 18 months.

(a) Write a formula for Moore's law, with t in years and $M_0 = 2200$ in 1970.

(b) From 1970 to 1999, the number of transistors per chip was actually modeled approximately by $N(t) = 2200(1.356)^t$. How does this function compare with your answer to part (a)?

(c) Complete the table showing the number of transistors per chip in recent years, the number predicted by Moore's law, and the number predicted by $N(t)$.

Name of chip	Year	Moore's law	$N(t)$	Actual number
Pentium IV	2000			42,000,000
Pentium M (Banias)	2003			77,000,000
Pentium M (Dothan)	2004			140,000,000

(d) What is the doubling time for $N(t)$?

Answer.

(a) $N(t) = 2200(2)^{t/1.5}$

(b) The given model has a smaller growth factor, 1.356, than $2^{1/1.5} \approx 1.59$.

(c)

Name of chip	Year	Moore's law	$N(t)$	Actual number
Pentium IV	2000	2,306,867,200	20,427,413	42,000,000
Pentium M (Banias)	2003	9,227,468,800	50,932,200	77,000,000
Pentium M (Dothan)	2004	14,647,693,680	69,064,063	140,000,000

(d) About 2.3 years

48. If the population of a particular animal is very small, inbreeding will cause a loss of genetic diversity. In a population of N individuals, the percent of the species' original genetic variation that remains after t generations is given by

$$V = V_0 \left(1 - \frac{1}{2N}\right)^t$$

(Source: Chapman and Reiss, 1992)

(a) Assuming $V_0 = 100$, graph V as a function of t for three different values of N: $N = 1000$, 100, and 10.

(b) Fill in the table to compare the values of V after 5, 50, and 100 generations.

Population size	Number of generations		
	5	50	100
1000			
100			
10			

(c) Studies of the cheetah have revealed variation at only 3.2% of its genes. (Other species show variation at 10% to 43% of their genes.) The population of cheetah may be less than 5000. Assuming the population can be maintained at its current level, how many generations will it take before the cheetah's genetic variation is reduced to 1%?

4.6 Chapter Summary and Review

4.6.1 Key Concepts

1 If a quantity is multiplied by a constant factor, b, in each time period, we say that it undergoes **exponential growth** or **decay** . The constant b is called the **growth factor** if $b > 1$ and the **decay factor** if $0 < b < 1$.

2 Quantities that increase or decrease by a constant percent in each time period grow or decay exponentially.

3 **Exponential Growth and Decay.**

The function
$$P(t) = P_0 b^t$$
models exponential growth and decay.

$P_0 = P(0)$ is the initial value of P;

b is the growth or decay factor.

1 If $b > 1$, then $P(t)$ is increasing, and $b = 1 + r$, where r represents percent increase.

2 If $0 < b < 1$, then $P(t)$ is decreasing, and $b = 1 - r$, where r represents percent decrease.

4 **Interest Compounded Annually.**

The amount $A(t)$ accumulated (principal plus interest) in an account bearing interest compounded annually is

$$A(t) = (1 + r)^t$$

where

P	is the principal invested,
r	is the interest rate,
t	is the time period, in years.

5 In linear growth, a constant amount is *added* to the output for each unit

increase in the input. In exponential growth, the output is *multiplied* by a constant factor for each unit increase in the input.

6 An **exponential function** has the form

$$f(x) = ab^x, \quad \text{where} \quad b > 0 \quad \text{and } b \neq 1, \quad a \neq 0$$

7 Properties of Exponential Functions, $f(x) = ab^x, \quad a > 0$**.**

1 Domain: all real numbers

2 Range: all positive numbers

3 If $b > 1$, the function is increasing and concave up; if $0 < b < 1$, the function is decreasing and concave up.

4 The y-intercept is $(a, 0)$. There is no x-intercept

8 The graphs of exponential functions can be transformed by shifts, stretches, and reflections.

9 Reflections of Graphs.

1 The graph of $y = -f(x)$ is the reflection of the graph of $y = f(x)$ about the x-axis.

2 The graph of $y = f(-x)$ is the reflection of the graph of $y = f(x)$ about the y-axis.

10 Exponential functions $f(x) = ab^x$ have different properties than power functions $f(x) = kx^p$.

11 We can solve **exponential equations** by writing both sides with the same base and equating the exponents.

12 We can use graphs to find approximate solutions to exponential equations

13 We use logrithms to help us solve exponential equations.

14 The **base** b **logarithm of** x, written $\log_b(x)$, is the exponent to which b must be raised in order to yield x.

15 If $b > 0$, $b \neq 1$, and $x > 0$,

$$y = \log_b(x) \quad \text{if and only if} \quad x = b^y$$

16 The operation of taking a base b logarithm is the inverse operation for raising the base b to a power.

17 Base 10 logarithms are called **common logarithms**, and $\log(x)$ means $\log_{10}(x)$.

18 Steps for Solving Base 10 Exponential Equations.

1 Isolate the power on one side of the equation.

2 Rewrite the equation in logarithmic form.

3 Use a calculator, if necessary, to evaluate the logarithm.

4 Solve for the variable.

19 Properties of Logarithms.

If x, y, $b > 0$, and $b \neq 1$, then

1 $\log_b(xy) = \log_b(x) + \log_b(y)$

2 $\log_b\left(\dfrac{x}{y}\right) = \log_b(x) - \log_b(y)$

3 $\log_b(x^k) = k\log_b(x)$

20 We can use the properties of logarithms to solve exponential equations with any base.

21 Compounded Interest.

The amount $A(t)$ accumulated (principal plus interest) in an account bearing interest compounded n times annually is

$$A(t) = \left(1 + \frac{r}{n}\right)^{nt}$$

where

P	is the principal invested,
r	is the interest rate,
t	is the time period, in years.

22 We can use the ratio method to fit an exponential function through two points.

To find an exponential function $f(x) = ab^x$ through two points:.

1 Use the coordinates of the points to write two equations in a and b.

2 Divide one equation by the other to eliminate a.

3 Solve for b.

4 Substitute b into either equation and solve for a.

23 Every increasing exponential has a fixed **doubling time**. Every decreasing exponential function has a fixed **half-life**.

24 If D is the doubling time for a population, its growth law can be written as $P(t) = P_0 2^{t/D}$.

25 If H is the half-life for a quantity, its decay law can be written as $Q(t) = Q_0(0.5)^{t/H}$.

26 **Future Value of an Annuity.**

If you make n payments per year for t years into an annuity that pays interest rate r compounded n times per year, the **future value**, FV, of the annuity is

$$FV = \frac{P\left[\left(1+\frac{r}{n}\right)^{nt} - 1\right]}{\frac{r}{n}}$$

where each payment is P dollars.

27 **Present Value of an Annuity.**

If you wish to receive n payments per year for t years from a fund that earns interest rate r compounded n times per year, the **present value**, PV, of the annuity must be

$$PV = \frac{P\left[1-\left(1+\frac{r}{n}\right)^{-nt}\right]}{\frac{r}{n}}$$

where each payment is P dollars.

.

4.6.2 Chapter 4 Review Problems

For Problems 1–4,

a Write a function that describes exponential growth or decay.

b Evaluate the function at the given values.

1. The number of computer science degrees awarded by Monroe College has increased by a factor of 1.5 every 5 years since 1984. If the college granted 8 degrees in 1984, how many did it award in 1994? In 2005?

Answer.
a $D = 8(1.5)^{t/5}$ b 18; 44

2. The price of public transportation has been rising by 10% per year since 1975. If it cost \$0.25 to ride the bus in 1975, how much did it cost in 1985? How much will it cost in the year 2010 if the current trend continues?

3. A certain medication is eliminated from the body at a rate of 15% per hour. If an initial dose of 100 milligrams is taken at 8 a.m., how much is left at 12 noon? At 6 p.m.?

Answer.
a $M = 100(0.85)^t$ b 52.2 mg; 19.7 mg

4. After the World Series, sales of T-shirts and other baseball memorabilia decline 30% per week. If \$200,000 worth of souvenirs were sold during the Series, how much will be sold 4 weeks later? After 6 weeks?

For Problems 5-8, use the laws of exponents to simplify.

5. $(4n^{x+5})^2$

Answer. $16n^{2x+10}$

6. $9^x \cdot 3^{x-3}$

7. $\dfrac{m^{x+2}}{m^{2x+4}}$

Answer. $\dfrac{1}{m^{x+2}}$

8. $\sqrt[3]{8^{2x+1} \cdot 8^{x-2}}$

For Problems 5-8, find a growth or decay law for the function.

9.

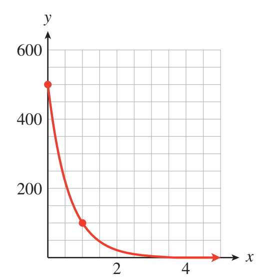
t	0	1	2	3
$g(t)$	16	13.6	11.56	9.83

Answer. $g(t) = 16(0.85)^t$

10.

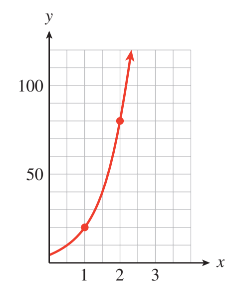
t	0	1	2	3
$f(t)$	12	19.2	30.72	49.15

11.



Answer. $f(x) = 500\left(\dfrac{1}{5}\right)^x$

12.



13. The president's approval rating increased by 12% and then decreased by 15%. What was the net change in his approval rating?

Answer. 4.8% loss

14. The number of students at Salt Creek Elementary School fell by 18% last year but increased by 26% this year. What was the net change in the number of students?

15. Enviroco's stock is growing exponentially in value and increased by 33.8% over the past 5 years. What was its annual rate of increase?

Answer. 6% loss

16. Sales of the software package Home Accountant 3.0 fell exponentially when the new version came out, decreasing by 60% over the past 3 months. What was the monthly rate of decrease?

For Problems 17–20,

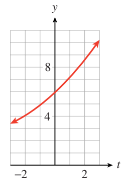
(a) Graph the function.

(b) List all intercepts and asymptotes.

(c) Give the range of the function on the domain $[-3, 3]$.

17. $f(t) = 6(1.2)^t$

Answer.

(a) 

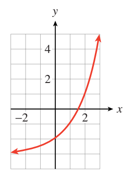
(b) y-intercept $(0, 6)$;
asymptote: $y = 0$

(c) $[3.472, 10.368]$

18. $g(t) = 35(0.6)^{-t}$

19. $P(x) = 2^x - 3$

Answer.

(a) 

(b) x-intercept $\left(\frac{\log(3)}{\log(2)}, 0\right)$;
y-intercept $(0, -2)$;
asymptote: $y = -3$

(c) $[-2.875, 5]$

20. $R(x) = 2^{x+3}$

For Problems 21-24, solve the equation.

21. $3^{x+2} = 9^{1/3}$

Answer. $\dfrac{-4}{3}$

22. $2^{x-1} = 8^{-2x}$

23. $4^{2x+1} = 8^{x-3}$

Answer. -11

24. $3^{x^2-4} = 27$

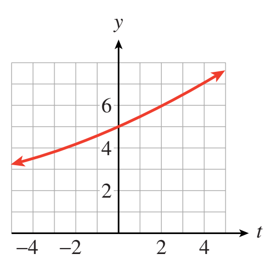
For Problems 25–28,

a Graph both functions in the same window. Are they equivalent?

b Justify your answer to part (a) algebraically.

25. $P(t) = 5(2^{t/8})$, $\quad Q(t) = 5(1.0905)^t$

Answer.

(a) 

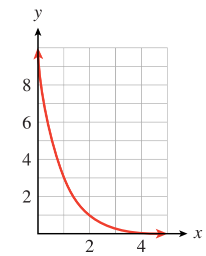
Not (quite) equivalent

(b) $2^{1/8} \approx 1.090507733 > 1.0905$

26. $M(x) = 4(3^{x/5})$, $\quad N(x) = 4(1.2457)^x$

27. $H(x) = \left(\dfrac{1}{3}\right)^{x-2}$, $\quad G(x) = 9\left(\dfrac{1}{3}\right)^{x}$

Answer.

(a) 

Equivalent

(b) $\left(\dfrac{1}{3}\right)^{x-2} = \left(\dfrac{1}{3}\right)^{x} \cdot \left(\dfrac{1}{3}\right)^{-2} = \left(\dfrac{1}{3}\right)^{x} \cdot 9$

28. $F(x) = \left(\dfrac{1}{2}\right)^{2x-3}$, $\quad L(x) = 8\left(\dfrac{1}{4}\right)^{x}$

For Problems 29–32, $f(x) = 2^x$.

(a) Write a formula for the function.

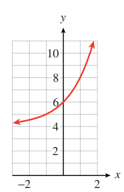
(b) Use transformations to sketch the graph, indicating any intercepts and asymptotes.

29. $y = 4 + f(x+1)$

Answer.

(a) $y = 4 + 2^{x+1}$

(b) Shift the graph of f 1 unit left, 4 units up.



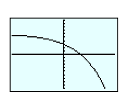
30. $y = -3 + f(x-2)$

31. $y = 6 - 3f(x)$

Answer.

(a) $y = 6 - 3 \cdot 2^x$

(b) Scale vertically by 3, reflect about x-axis, shift 6 units up.



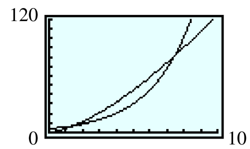
32. $y = 10 - 4f(x)$

In Problems 33–36, we compare power and exponential functions. Let

$$f(x) = 4x^{1.5}, \quad g(x) = 4(1.5)^x$$

33. Graph both functions in the window Xmin = 0, Xmax = 10, Ymin = 0, Ymax = 120. Which function grows more rapidly for large values of x?

Answer.



g eventually grows faster.

34. Estimate the solutions of $f(x) = g(x)$. For what values of x is $f(x) > g(x)$?

35. When x doubles from 2 to 4, $f(x)$ grows by a factor of ____, and $g(x)$ grows by a factor of ____.

Answer. $2^{1.5} \approx 2.83$; 2.25

36. What is the range of $f(x)$ on the domain $[0, 100]$? What is the range of $g(x)$ on the same domain?

37. "Within belts of uniform moisture conditions and comparable vegetation, the organic matter content of soil decreases exponentially with increasing temperature." Data indicate that the organic content doubles with each 10°C decrease in temperature. Write a formula for this function, stating clearly what each variable represents. (Source: Leopold, Wolman, Gordon, and Miller, 1992)

Answer. $M = M_0(2)^{t/10}$, where M is the organic content, M_0 is the organic content at 0°C, and t is the temperature in ° Celsius.

38. In 1951, a study of barley yields under diverse soil conditions led to the formula

$$Y = cV^aG^b$$

where V is a soil texture rating, G is a drainage rating, and a, b, and c are constants. In fields with similar drainage systems, the formula gives barley yields, Y, as a function of V, the soil texture. What type of function is it? If it is an increasing function, what can you say about a? (Source: Briggs and Courtney, 1985)

For Problems 39-44, find the logarithm.

39. $\log_2(16)$

Answer. 4

40. $\log_4(2)$

41. $\log_3\left(\dfrac{1}{3}\right)$

Answer. -1

42. $\log_7(7)$

43. $\log(10^{-3})$

Answer. -3

44. $\log(0.0001)$

For Problems 45-46, write the equation in logarithmic form.

45. $0.3^{-2} = x + 1$

Answer. $\log_{0.3}(x + 1) = -2$

46. $4^{0.3t} = 3N_0$

For Problems 47-50, solve.

47. $4 \cdot 10^{1.3x} = 20.4$

Answer. $\dfrac{\log(5.1)}{1.3} \approx 0.5433$

48. $127 = 2(10^{0.5x}) - 17.3$

49. $3(10^{-0.7x}) + 6.1 = 9$

Answer. $\dfrac{\log(2.9/3)}{-0.7} \approx 0.21$

50. $40(1 - 10^{-1.2x}) = 30$

For Problems 51-54, write the expression in terms of simpler logarithms. (Assume that all variables and variable expressions denote positive real numbers.)

51. $\log_b\left(\dfrac{xy^{1/3}}{z^2}\right)$

Answer. $\log_b(x) + \dfrac{1}{3}\log_b(y) - 2\log_b(z)$

52. $\log_b\left(\sqrt{\dfrac{L^2}{2R}}\right)$

53. $\log\left(x\sqrt[3]{\dfrac{x}{y}}\right)$

Answer. $\dfrac{4}{3}\log(x) - \dfrac{1}{3}\log(y)$

54. $\log\left(\sqrt{(s-a)(s-g)^2}\right)$

For Problems 55-58, write the expression as a single logarithm with coefficient 1.

55. $\dfrac{1}{3}\big(\log(x) - 2\log(y)\big)$

Answer. $\log\left(\sqrt[3]{\dfrac{x}{y^2}}\right)$

56. $\dfrac{1}{2}\log(3x) - \dfrac{2}{3}\log(y)$

57. $\dfrac{1}{3}\log(8) - 2\big(\log(8) - \log(2)\big)$

Answer. $\log\left(\dfrac{1}{8}\right)$

58. $\dfrac{1}{2}(\log(9) + 2\log(4)) + 2\log(5)$

For Problems 59–62, solve the equation by using base 10 logarithms.

59. $3^{x-2} = 7$

Answer. $\dfrac{\log(63)}{\log(3)} \approx 3.77$

60. $4 \cdot 2^{1.2x} = 64$

61. $1200 = 24 \cdot 6^{-0.3x}$

Answer. $\dfrac{\log(50)}{-0.3\log(6)} \approx -7.278$

62. $0.08 = 12 \cdot 3^{-1.5x}$

63. Solve $N = N_0(10^{kt})$ for t.

Answer. $\dfrac{\log\left(N/N_0\right)}{k}$

64. Solve $Q = R_0 + R\log(kt)$ for t.

65. The population of Dry Gulch has been declining according to the function

$$P(t) = 3800 \cdot 2^{-t/20}$$

where t is the number of years since the town's heyday in 1910.

(a) What was the population of Dry Gulch in 1990?

(b) In what year did the population dip below 120 people?

Answer.

(a) 238

(b) 2010

66. The number of compact discs produced each year by Delta Discs is given by the function

$$N(t) = 8000 \cdot 3^{t/4}$$

where t is the number of years since discs were introduced in 1980.

(a) How many discs did Delta produce in 1989?

(b) In what year did Delta first produce over 2 million discs?

67.

(a) Write a formula for the cost of a camera t years from now if it costs \$90 now and the inflation rate is 6% annually.

(b) How much will the camera cost 10 months from now?

(c) How long will it be before the camera costs \$120?

Answer.

(a) $C = 90(1.06)^t$ (b) \$94.48 (c) 5 years

68.

(a) Write a formula for the cost of a sofa t years from now if it costs \$1200 now and the inflation rate is 8% annually.

(b) How much will the sofa cost 20 months from now?

(c) How long will it be before the sofa costs \$1500?

69. Francine inherited \$5000 and plans to deposit the money in an account that compounds interest monthly.

(a) If she can get 5.5% interest, how long will it take for the money to grow to \$7500?

(b) What interest rate will she need if she would like the money to grow to \$6000 in 3 years?

Answer.

(a) 7.4 years (b) 6.1

70. Delbert received a signing bonus of \$2500 and wants to invest the money in a certificate of deposit (CD) that compounds interest quarterly.

(a) If the CD pays 4.8% interest, how long will it take his money to grow to \$3000?

(b) What interest rate will he need if he would like the money to grow to \$3000 in 1 year?

For Problems 71-74, find an exponential growth or decay function that fits the data.

71. $f(2) = 1714, \quad f(4) = 1836$

Answer.
$f(x) \approx 1600(1.035)^x$

72. $g(1) = 10,665, \quad g(6) = 24,920$

73. $g(1) = 45, \quad g(5) = 0.00142$

Answer.
$g(x) \approx 600(0.075)^x$

74. $f(2) = 17,464, \quad f(5) = 16.690$

75. The population of Sweden is growing at 0.1% annually.

(a) What is the doubling time for Sweden's population?

(b) In 2005, the population of Sweden was 9 million. At the current rate of growth, how long will it take the population to reach 10 million?

Answer.

(a) $\dfrac{\log(2)}{\log(1.001)} \approx 693$ years (b) 105 years

76. The bacteria *E. sakazakii* is found in powdered infant formula and can has a doubling time of 4.98 hours even if kept chilled to 50°F.

(a) What is the hourly growth rate for *E. sakazakii?*

(b) How long would it take a colony of *E. sakazakii* to increase by 50%?

77. Manganese-53 decays to chromium-53 with a half-life of 3.7 million years and is used to estimate the age of meteorites. What is the decay rate of manganese-53, with time expressed in millions of years?

Answer. 17%

78. The cold medication pseudoephedrine decays at a rate of 5.95% per hour in the body. What is the half-life of pseudoephedrine?

79. You would like to buy a house with a 20-year mortgage for $300,000, at an interest rate of 6.25%, compounded monthly. Use the formula for the present value of an annuity to calculate your monthly payment.

Answer. $2192.78

80. Rosalie's retirement fund pays 7% interest compounded monthly. Use the formula for the future value of an annuity to calculate how much should she contribute monthly in order to have $500,000 in 25 years.

81. An eccentric millionaire offers you a summer job for the month of June. She will pay you 2 cents for your first day of work and will double your wages every day thereafter. (Assume that you work every day, including weekends.)

(a) Make a table showing your wages on each day. Do you see a pattern?

(b) Write a function that gives your wages in terms of the number of days you have worked.

(c) How much will you make on June 15? On June 30?

Answer.

(a)

Day	1	2	3	$\cdots$	t	$\cdots$	30
Wage (cent)	2	4	8	$\cdots$	2^t	$\cdots$	2^{30}

(b) $W(t) = 2^t$ cents

(c) $327.68; $10,737,418.24

82. The king of Persia offered one of his subjects anything he desired in return for services rendered. The subject requested that the king give him an amount of grain calculated as follows: Place one grain of wheat on the first square of a chessboard, two grains on the second square, four grains on the third square, and so on, until the entire chessboard is covered.

(a) Make a table showing the number of grains of wheat on each square of the chessboard

(b) Write a function for the amount of wheat on each square.

(c) How many grains of wheat should be placed on the last (64th) square?

4.7 Projects for Chapter 4

Project 25 Bode's Law. In 1772, the astronomer Johann Bode promoted a formula for the orbital radii of the six planets known at the time. This formula calculated the orbital radius, r, as a function of the planet's position, n, in line from the Sun. (Source: Bolton, 1974)

a Evaluate Bode's law, $r(n) = 0.4 + 0.3(2^{n-1})$, for the values in the table. (Use a large negative number, such as $n = -100$, to approximate $r(-\infty)$.)

n	$-\infty$	1	2	3	4	5	6
$r(n)$							

b How do the values of $r(n)$ compare with the actual orbital radii of the planets shown in the table? (The radii are given in astronomical units (AU). One AU is the distance from the Earth to the Sun, about 149.6×10^6 kilometers.) Assign values of n to each of the planets so that they conform to Bode's law.

Planet	Mercury	Venus	Earth	Mars	Jupiter	Saturn
Orbitan radius (AU)	0.39	0.72	1.00	1.52	5.20	9.54
n						

c In 1781, William Herschel discovered the planet Uranus at a distance of 19.18 AU from the Sun. If $n = 7$ for Uranus, what does Bode's law predict for the orbital radius of Uranus?

d None of the planets' orbital radii corresponds to $n = 2$ in Bode's law. However, in 1801 the first of a group of asteroids between the orbits of Mars and Jupiter was discovered. The asteroids have orbital radii between 2.5 and 3.0 AU. If we consider the asteroids as one planet, what orbital radius does Bode's law predict?

e In 1846, Neptune was discovered 30.6 AU from the Sun, and Pluto was discovered in 1930 39.4 AU from the Sun. What orbital radii does Bode's law predict for these planets?

Project 26 Plague. In 1665, there was an outbreak of the plague in London. The table shows the number of people who died of plague during each week of the summer that year. (Source: Bolton, 1974)

Week	Deaths
0, May 9	9
1, May 16	3
2, May 23	14
3, May 30	17
4, June 6	43
5, June 13	112
6, June 20	168
7, June 27	267
8, July 4	470
9, July 11	725
10, July 18	1089
11, July 25	1843

Week	Deaths
12, August 1	2010
13, August 8	2817
14, August 15	3880
15, August 22	4237
16, August 29	6102
17, September 5	6988
18, September 12	6544
19, September 19	7165
20, September 26	5533
21, October 3	4929
22, October 10	4327

1. Scale horizontal and vertical axes for the entire data set, but plot only the data for the first 8 weeks of the epidemic, from May 9 through July 4. On the same axes, graph the function $f(x) = 2.18(1.83)^x$.

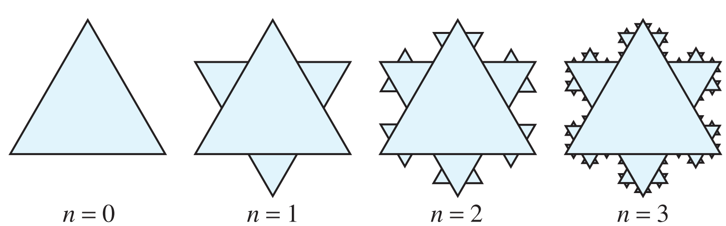
2. By what weekly percent rate did the number of victims increase during the first eight weeks?

3. Add data points for July 11 through October 10 to your graph. Describe the progress of the epidemic relative to the function f and offer an explanation.

4. Make a table showing the total number of plague victims at the end of each week and plot the data. Describe the graph.

Project 27 Koch snowflake. The Koch snowflake is an example of a fractal. It is named in honor of the Swiss mathematician Niels Fabian Helge von Koch (1870–1924). Here is how to construct a Koch snowflake:

- Draw an equilateral triangle with sides of length 1 unit. This is stage $n = 0$.
- Divide each side into 3 equal segments and draw a smaller equilateral triangle on each middle segment, as shown in the figure. The new figure (stage $n = 1$) is a 6-pointed star with 12 sides.
- Repeat the process to obtain stage $n = 2$: Trisect each of the 12 sides and draw an equilateral triangle on each middle third.
- If you continue this process forever, the resulting figure is the **Koch snowflake**.



a We will consider several functions related to the Koch snowflake:

$S(n)$ is the length of each side in stage n
$N(n)$ is the number of sides in stage n
$P(n)$ is the perimeter of the snowflake at stage n

Fill in the table describing the snowflake at each stage.

Stage n	$S(n)$	$N(n)$	$P(n)$
0			
1			
2			
3			

b Write an expression for $S(n)$.

c Write an expression for $N(n)$.

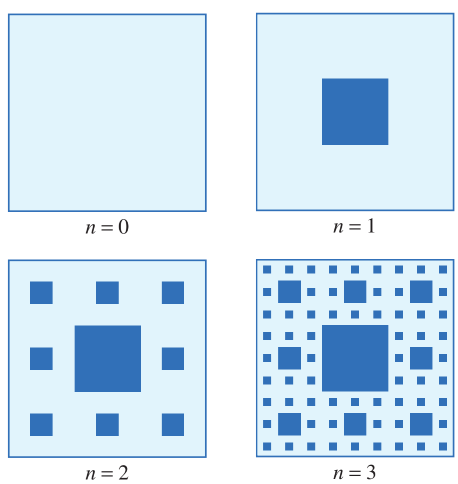
d Write an expression for $P(n)$.

e What happens to the perimeter as n gets larger and larger?

f As n increases, the area of the snowflake increases also. Is the area of the completed Koch snowflake finite or infinite?

Project 28 Sierpinski carpet. The Sierpinski carpet is another fractal. It is named for the Polish mathematician Waclaw Sierpinski (1882–1969). Here is how to build a Sierpinski carpet:

- Start with a unit square (sides of length 1 unit.)
- For stage $n = 1$, trisect each side and partition the square into 9 smaller squares. Remove the center square, leaving a hole surrounded by 8 squares, as shown in the figure.
- For stage $n = 2$, repeat the process on each of the remaining 8 squares.
- If you continue this process forever, the resulting is the **Sierpinski carpet**.



a We will consider several functions related to the Sierpinski carpet:

$S(n)$ is the side of a new square at stage n
$A(n)$ is the area of a new square at stage n
$N(n)$ is the number of new squares removed at stage n
$R(n)$ is the total area removed at stage n
$T(n)$ is the total area remaining at stage n

Fill in the table describing the carpet at each stage.

Stage n	$S(n)$	$A(n)$	$N(n)$	$R(n)$	$T(n)$
0					
1					
2					
3					

b Write an expression for $S(n)$.

c Write an expression for $A(n)$.

d Write an expression for $N(n)$.

e Write an expression for $R(n)$.

f Write an expression for $T(n)$.

g What happens to the area remaining as n approaches infinity?

Project 29 Stream order. The **order** of a stream or river is a measure of its relative size. A first-order stream is the smallest, one that has no tributaries. Second-order streams have only first-order streams as tributaries. Third-order streams may have first- and second-order streams as tributaries, and so on. The Mississippi River is an example of a tenth-order stream, and the Columbia River is ninth order.

Both the number of streams of a given order and their average length are exponential functions of their order. In this problem, we consider all streams in the United States. (Source: Leopold, Luna, Gordon, and Miller, 1992)

a Using the given values, find a function $N(x) = ab^{x-1}$ for the number of streams of a given order.

b Complete the column for number of streams of each order. (Round to the nearest whole number of streams for each order.)

c Find a function $L(x) = ab^{x-1}$ for the average lengthof streams of a given order, then complete that column.

d Find the total length of all streams of each order, hence estimating the total length of all stream channels in the United States.

Order	Number	Average Length	Total Length
1	$1,600,000$	1	
2	$339,200$	2.3	
3			
4			
5			
6			
7			
8			
9			
10			

Project 30 Species rank. Related species living in the same area often evolve in different sizes to minimize competition for food and habitat. Here are the masses of eight species of fruit pigeon found in New Guinea, ranked from smallest to largest. (Source: Burton, 1998)

Size rank	1	2	3	4
Mass (grams)	49	76	123	163

Size rank	5	6	7	8
Mass (grams)	245	414	592	802

a Plot the masses of the pigeons against their order of increasing size. What kind of function might fit the data?

b Compute the ratios of the masses of successive sizes of fruit pigeons. Are the ratios approximately constant? What does this information tell you about your answer to part (a)?

c Compute the average ratio to two decimal places. Using this ratio, estimate the mass of a hypothetical fruit pigeon of size rank 0.

d Using your answers to part (c), write an exponential function that approximates the data. Graph this function on top of the data and evaluate the fit.

In Projects 7 and 8, we will prove the formulas in Section 4.5, p. 471 for the present and future values of an annuity.

The **future value** of $\$M$ of money is its value in the future: its current value plus the interest it will accrue in the interval.

The **present value** of $\$M$ of money is the amount you would need to deposit now so that it will grow to $\$M$ in the future.

Project 31 Future value. Suppose you deposit \$100 at the end of every 6 months into an account that pays 4% compounded annually. How much money will be in the account at the end of 3 years?

a During the 3 years, you will make 6 deposits. Use the formula $F = P(1 + \frac{r}{n})^{nt}$ to write an expression for the future value (principal plus interest) of each deposit. (Do not evaluate the expression!)

Deposit number	Amount deposited	Time in account	Future value
1	100	2.5	$100(1.02)^5$
2	100	2	
3	100	1.5	
4	100	1	
5	100	0.5	
6	100	0	

b Let S stand for the sum of the future values of all the deposits. Write out the sum, without evaluating the terms you found in part (a).

$$S =$$

c You could find S by working out all the terms and adding them up, but what if there were 100 terms, or more? We will use a trick to find the sum in an easier way. Multiply both sides of the equation in part (b) by 1.02. (Use the distributive law on the right side!)

$$1.02S =$$

d Now subtract the equation in part (b) from the equation in part (c). Be sure to line up like terms on the right side.

$$1.02S =$$
$$-S =$$
$$0.02S =$$

e Finally, solve for S. If you factor 100 from the numerator on the right side, your expression should look a lot like the formula for the future value of an annuity. (To help you see this, note that, for this example, $\frac{r}{n} = ?$ and $nt = ?$)

f Try to repeat the argument above, using letters for the parameters instead of numerical values.

Project 32 Present value. You would like to set up an account that pays 4% interest compounded semiannually so that you can withdraw \$100 at the end of every 6 months for the next 3 years. How much should you deposit now?

a During the 3 years, you will make 6 withdrawals. Use the formula $P = A(1 + \frac{r}{n})^{-nt}$ to write an expression for the present value of those withdrawals. (Do not evaluate the expression!)

Withdrawal number	Amount withdrawn	Time in account	Present value
1	100	0.5	$100(1.02)^{-1}$
2	100	1	
3	100	1.5	
4	100	2	
5	100	2.5	
6	100	3	

b Let S stand for the sum of the present values of all the withdrawals. Write out the sum, without evaluating the terms you found in part (a).

$$S =$$

c We will use a trick to evaluate the sum. Multiply both sides of the equation in part (b) by 1.02. (Use the distributive law on the right side!)

$$1.02S =$$

d Now subtract the equation in part (b) from the equation in part (c). Be sure to line up like terms on the right side.

$$1.02S =$$
$$-S =$$
$$0.02S =$$

e Finally, solve for S. If you factor 100 from the numerator on the right side, your expression should look a lot like the formula for the present value of an annuity. (To help you see this, note that, for this example, $\frac{r}{n} = ?$ and $nt = ?$)

f Try to repeat the argument above, using letters for the parameters instead of numerical values.