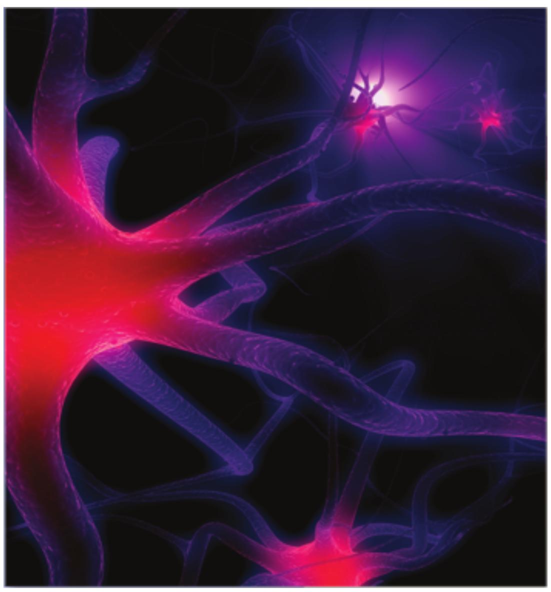
Chapter 5

Logarithmic Functions

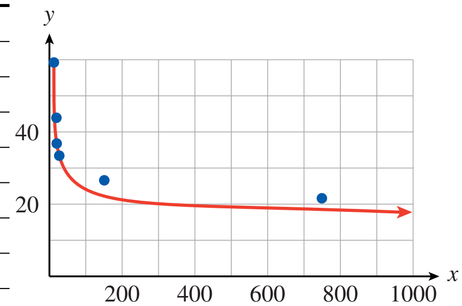


In Chapter 4, p. 391, we used logarithms to solve exponential equations. In this chapter, we consider logarithmic functions as models in their own right. We also study another base for exponential and logarithmic functions, the natural base e, which is the most useful base for many scientific applications.

In 1885, the German philosopher Hermann Ebbinghaus conducted one of the first experiments on memory, using himself as a subject. He memorized lists of nonsense syllables and then tested his memory of the syllables at intervals ranging from 20 minutes to 31 days. After one hour, he remembered less than 50% of the items, but he found that the rate of forgetting leveled off over time. He modeled his data by the function

$$y = \frac{184}{2.88\log(t) + 1.84}$$

Time elapsed	Percent remembered
20 minutes	58.2%
1 hour	44.2%
9 hours	35.8%
1 day	33.7%
2 days	27.8%
6 days	25.4%
31 days	21.1%



Ebbinghaus's model uses a logarithmic function. The graph of the data is called the "forgetting curve." Ebbinghaus's work, including his application of the scientific method to his research, provides part of the foundation of modern psychology.

5.1 Inverse Functions

5.1.1 Introduction

When you buy a house, your monthly mortgage payment is a function of the size of the loan. The table shows mortgage payments on 30-year loans of various sizes at 6% interest.

Loan amount, L	$150,000$	$175,000$	$200,000$	$225,000$	$250,000$
Mortgage payment, M	899.33	1049.21	1199.10	1348.99	1498.88

For the function $M = f(L)$, the input value is the amount of the loan, and the output is the mortgage payment.

However, when you are shopping for a house, you may think of the mortgage payment as the input variable: If you can afford a certain monthly mortgage payment, how large a loan can you finance? Now the mortgage payment is the input value, and the loan amount is the output. By interchanging the inputs and outputs, we define a new function, $L = g(M)$, shown below.

Mortgage payment, M	899.33	1049.21	1199.10	1348.99	1498.88
Loan amount, L	$150,000$	$175,000$	$200,000$	$225,000$	$250,000$

This new function gives the same information as the original function, f, but from a different point of view. We call the function g the **inverse function** for f.

The elements of the *range* of f are used as the input values for g, and the output values of g are the corresponding domain elements of f. For example, from the tables you can verify that $f(200,000) = 1199.10$, and $g(1199.10) = 200,000$. In fact, this property defines the inverse function.

Inverse Functions.

Suppose g is the **inverse function** for f. Then

$$g(b) = a \qquad \text{if and only if} \qquad f(a) = b$$

We can also use composition notation (see Subsection 1.2.8, p. 37) to define inverse functions. Suppose g is the inverse function for f. Then,

$$g(f(a)) = a \text{ and } f(g(b)) = b.$$

Example 5.1.1 Suppose g is the inverse function for f, and we know the following function values for f:

$$f(-3) = 5, \quad f(2) = 1, \quad f(5) = 0$$

Find $g(5)$ and $g(0)$.

Solution. We know that $g(5) = -3$ because $f(-3) = 5$, and $g(0) = 5$ because $f(5) = 0$. Tables may be helpful in visualizing the two functions, as shown below.

$y = f(x)$	
x	y
-3	5
2	1
5	0

$\rightarrow$ Interchange the columns $\rightarrow$

$x = g(y)$	
y	x
5	-3
1	2
0	5

For the function f, the input variable is x and the output variable is y. For the inverse function g, the roles of the variables are interchanged: y is now the input and x is the output. □

Checkpoint 5.1.2 Suppose g is the inverse function for f, and suppose we know the following function values for f :

$$f(-1) = 0, \quad f(0) = 1, \quad f(1) = 2$$

Find $g(0)$ and $g(1)$.

Answer. $g(0) = -1, \; g(1) = 0$

5.1.2 Finding a Formula for the Inverse Function

If a function is given by a table of values, we can interchange the columns (or rows) of the table to obtain the inverse function. Swapping the columns works because we are really interchanging the input and output variables. If a function is defined by an equation, we can find a formula for its inverse function in the same way: Interchange the roles of the variables in the equation so that the old output variable becomes the new input variable.

Example 5.1.3

a The function $H = f(t) = 6 + 2t$ gives the height of corn seedlings, in inches, t days after they are planted. Find a formula for the inverse function and explain its meaning in this context.

b Make a table of values for $f(t)$ and a table for its inverse function.

Solution.

a Write the equation for f in the form

$$H = 6 + 2t$$

In this equation, t is the input and H is the output. We interchange the roles of the variables by solving for t to obtain

$$t = \frac{H - 6}{2}$$

In this equation, H is the input and t is the output. The formula for the inverse function is

$$t = g(H) = \frac{H - 6}{2}$$

The function g gives the number of days it will take the corn seedlings to grow to a height of H inches.

b To make a table for f, we choose values for t and evaluate $f(t) = 6 + 2t$ at those t-values, as shown at left below.

$H = f(t)$	
t	H
0	6
1	8
2	10
3	12

$t = g(H)$	
H	t
6	0
8	1
10	2
12	3

To make a table for g, we could choose values for H and evaluate $\dfrac{H-6}{2}$, but because g is the inverse function for f, we can simply interchange the columns in our table for f, as shown at right above.

You can check that the values in the second table do satisfy the formula for the inverse function, $g(H) = \dfrac{H-6}{2}$.

□

Note 5.1.4 Note once again that the two tables show the same relationship between t and H, but the roles of input and output have been interchanged. The function f tells us the height of the seedlings after t days, and g tells us how long it will take the seedlings to grow to height H.

Checkpoint 5.1.5 Carol can burn 600 calories per hour bicycling and 400 calories per hour swimming. She would like to lose 5 pounds, which is equivalent to $16,000$ calories.

a Write an equation relating the number of hours of cycling, x, and the number of hours swimming, y, that Carol must spend to lose 5 pounds.

b Write y as a function of x, $y = f(x)$. What does $f(10)$ tell you?

c Find the inverse function, $x = g(y)$. What does $g(10)$ tell you?

Answer.

a $600x + 400y = 16,000$

b $y = f(x) = 40 - 1.5x$; $f(10) = 25$; If Carol cycles for 10 hrs, she must swim for 25 hrs.

c $x = g(y) = 26\frac{2}{3} - \frac{2}{3}y$; $g(10) = 20$; If Carol swims for 10 hrs, she must cycle for 20 hrs.

5.1.3 Inverse Function Notation

If the inverse of a function f is also a function, we denote the inverse by the symbol f^{-1}, read "f inverse." This notation makes it clear that the two functions are related in a special way. For example, the function $f(t) = 6 + 2t$ in Example 5.1.3, p. 509 has inverse function $f^{-1}(H) = \dfrac{H-6}{2}$.

Example 5.1.6 If $\quad y = f(x) = x^3 + 2, \quad$ find $\quad f^{-1}(10)$.

Solution. We first find the inverse function for $y = x^3 + 2$ by solving for x:

$$\begin{aligned} x^3 &= y - 2 && \text{Substract 2 from both sides.} \\ x &= \sqrt[3]{y-2} && \text{Take cube roots.} \end{aligned}$$

The inverse function is $x = f^{-1}(y) = \sqrt[3]{y-2}$. Now we evaluate the inverse function at $y = 10$:

$$f^{-1}(10) = \sqrt[3]{10-2} = 2$$

□

Caution 5.1.7 Although the same symbol, $^{-1}$, is used for both reciprocals and inverse functions, the two notions are *not* equivalent. That is, the inverse of a given function is usually not the same as the reciprocal of that function. In Example 5.1.3, p. 509, note that $f^{-1}(y)$ is not the same as the reciprocal of $f(y)$, because

$$\frac{1}{f(y)} = \frac{1}{y^3+2} \quad \text{but} \quad f^{-1}(y) = \sqrt[3]{y-2}$$

To avoid confusion, we use the notation $\dfrac{1}{f}$ to refer to the reciprocal of the function f.

Note 5.1.8 In Example 5.1.3, p. 509, you can check that $f(2) = 10$. In fact, the two statements

$$f^{-1}(10) = 2 \quad \text{and} \quad f(2) = 10$$

are equivalent; they convey the same information. This fact is a restatement of our earlier observation about inverse functions, this time using inverse function notation.

Inverse Functions.

Suppose the inverse of f is a function, denoted by f^{-1}. Then

$$f^{-1}(y) = x \qquad \text{if and only if} \qquad f(x) = y$$

Checkpoint 5.1.9

a If $z = f(w) = \dfrac{1}{w+3}$, find $f^{-1}(1)$.

b Write two equations about the value of $f^{-1}(1)$, one using f^{-1} and one using f.

c Show that $f^{-1}(1)$ is not equal to $\dfrac{1}{f(1)}$.

Answer.

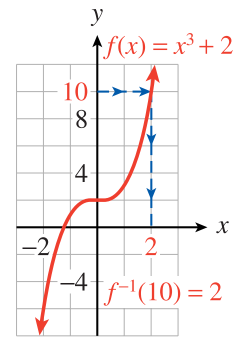
a -2

b $f^{-1}(1) = -2$, $f(-2) = 1$

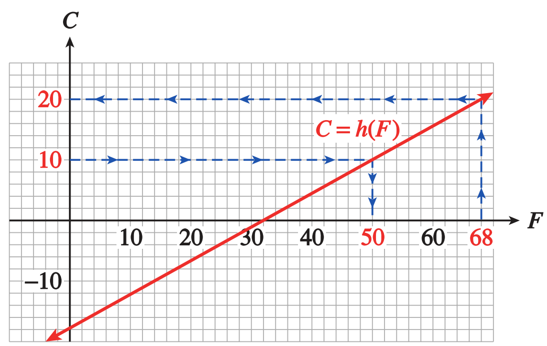
c $f^{-1}(1) = -2$, but $\dfrac{1}{f(1)} = 4$

We can use a graph of a function $y = f(x)$ to find values of the inverse function $x = f^{-1}(y)$. The figure below shows the graph of $f(x) = x^3 + 2$.

You already know how to evaluate a function from its graph: We start with the horizontal axis. For instance, to evaluate $f(-2)$, we find -2 on the x-axis, move vertically to the point on the graph with $x = -2$, in this case $(-2, -6)$, and read the y-coordinate of the point. We see that $f(-2) = -6$. To evaluate the inverse function, we start with the vertical axis. For example, to find $f^{-1}(10)$, we find 10 on the vertical axis and move horizontally to the point on the graph with $y = 10$. In this case, the point is $(2, 10)$, so $f^{-1}(10) = 2$.



Example 5.1.10 The function $C = h(F)$ gives Celsius temperature as a function of Fahrenheit temperature. The graph of the function is shown below. Use the graph to evaluate $h(68)$ and $h^{-1}(10)$, and then explain their meaning in this context.



Solution. To evaluate $h(68)$, we find the input $F = 68$ on the horizontal axis, then find the point on the graph with $F = 68$ and read its vertical coordinate. We see that the point $(68, 20)$ lies on the graph, so $h(68) = 20$. When the Fahrenheit temperature is 68°, the Celsius temperature is 20°.

The inverse function reverses the roles of input and output. Because $C = h(F)$, $F = h^{-1}(C)$, so the inverse function gives us the Fahrenheit temperature if we know the Celsius temperature. In particular, $h^{-1}(10)$ is the Fahrenheit temperature when the Celsius temperature is 10°.

To use the graph of h to find values of h^{-1}, we start with the vertical axis and find the point on the graph with $C = 10$. This point is $(50, 10)$, so $F = 50$ when $C = 10$, or $h^{-1}(10) = 50$. When the Celsius temperature is 10°, the Fahrenheit temperature is 50°. □

Checkpoint 5.1.11

a Use the graph of h in Example 5.1.10, p. 512 to find $h^{-1}(-10)$.

b Does $h^{-1}(-10) = -h^{-1}(10)$?

c Write two equations, one using h and one using h^{-1}, stating the Fahrenheit temperature when the Celsius temperature is 0°.

Answer.

a -14

b No

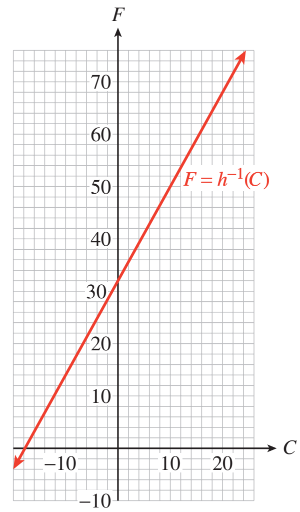
c $h(32) = 0$, $h^{-1}(0) = 32$

5.1.4 Graph of the Inverse Function

In Example 5.1.10, p. 512, we used a graph of h to read values of h^{-1}. But we can also plot the graph of h^{-1} itself. Because C is the input variable for h^{-1}, we plot C on the horizontal axis and F on the vertical axis. To find some points on the graph of h^{-1}, we interchange the coordinates of points on the graph of h. The graph of h^{-1} is shown at right.

$C = h(F)$	
F	C
14	−10
32	0
50	10
68	20

$F = h^{-1}(C)$	
C	F
−10	14
0	32
10	50
20	68



Example 5.1.12 The Park Service introduced a flock of 12 endangered pheasant into a wildlife preserve. After t years, the population of the flock was given by

$$P = f(t) = 12 + 2t^3$$

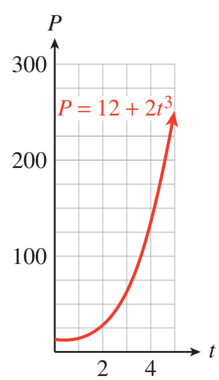
a Graph the function on the domain $[0, 5]$.

b Find a formula for the inverse function, $t = f^{-1}(P)$. What is the meaning of the inverse function in this context?

c Sketch a graph of the inverse function.

Solution.

a The graph of f is shown at right, with t on the horizontal axis and P on the vertical axis.

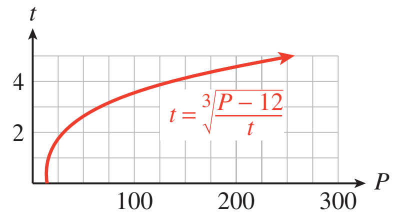


b We solve $P = 12 + 2t^3$ for t in terms of P.

$$\begin{aligned} 2t^3 &= P - 12 && \text{Substract 12 from both sides.} \\ t^3 &= \frac{P-12}{2} && \text{Divide both sides by 2.} \\ t &= \sqrt[3]{\frac{P-12}{2}} && \text{Take cube roots.} \end{aligned}$$

The inverse function is $t = f^{-1}(P) = \sqrt[3]{\frac{P-12}{2}}$. It tells us the number of years it takes for the pheasant population to grow to size P.

c The graph of f^{-1} is shown below, with P on the horizontal axis and t on the vertical axis.



□

Checkpoint 5.1.13 The formula $T = f(L) = 2\pi\sqrt{\dfrac{L}{32}}$ gives the period in seconds, T, of a pendulum as a function of its length in feet, L.

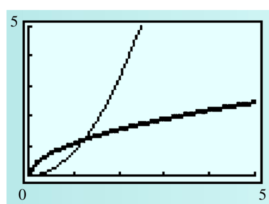
a Graph the function on the domain $[0, 5]$.

b Find a formula for the inverse function, $L = f^{-1}(T)$. What is the meaning of the inverse function in this context?

c Sketch a graph of the inverse function.

Answer.

a



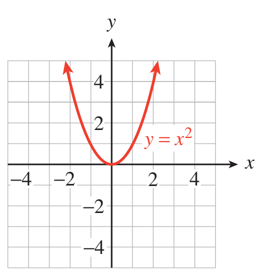
b $L = f^{-1}(T) = \dfrac{8}{\pi^2}T^2$. f^{-1} gives the length of a pendulum as a function of its period.

c See graph for part(a).

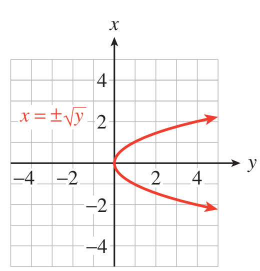
5.1.5 When Is the Inverse a Function?

We can always find the inverse of a function simply by interchanging the input and output variables. In the preceding examples, interchanging the variables created a new function. However, the inverse of a function does not always turn out to be a function itself.

For example, to find the inverse of $y = f(x) = x^2$, we solve for x to get $x = \pm\sqrt{y}$. When we regard y as the input and x as the output, the relationship does not describe a function. The graphs of f and its inverse are shown below. (Note that for the graph of the inverse, we plot y on the horizontal axis and x on the vertical axis.) Because the graph of the inverse does not pass the vertical line test, it is not a function.



Graph of f



Graph of inverse

For many applications, it is important to know whether or not the inverse of f is a function. This can be determined from the graph of f. When we interchange the roles of the input and output variables, horizontal lines of the form $y = k$ become vertical lines.

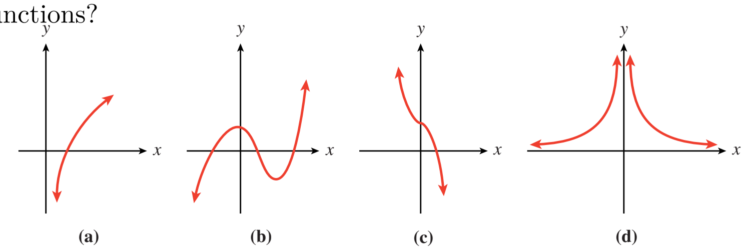
Thus, if the graph of the *inverse* is going to pass the vertical line test, the graph of the *original function* must pass the **horizontal line test**, namely, that no horizontal line should intersect the graph in more than one point.

Horizontal Line Test.

If no horizontal line intersects the graph of a function more than once, then its inverse is also a function.

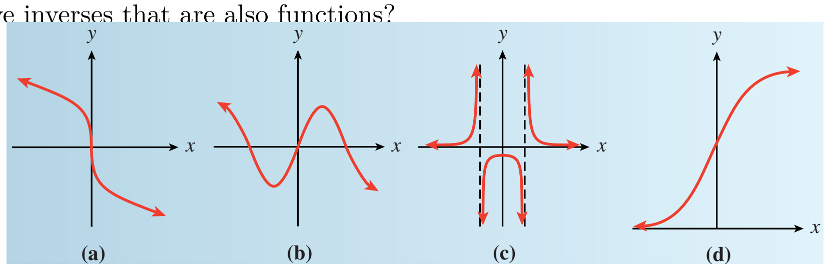
Notice that the graph of $f(x) = x^2$ does not pass the horizontal line test, so we would not expect its inverse to be a function.

Example 5.1.14 Which of the functions shown below have inverses that are also functions?



Solution. In each case, we apply the horizontal line test to determine whether the inverse is a function. Because no horizontal line intersects their graphs more than once, the functions pictured in figures (a) and (c) have inverses that are also functions. The functions in figures (b) and (d) do not have inverses that are functions. □

Checkpoint 5.1.15 Which of the functions whose graphs are shown below have inverses that are also functions?



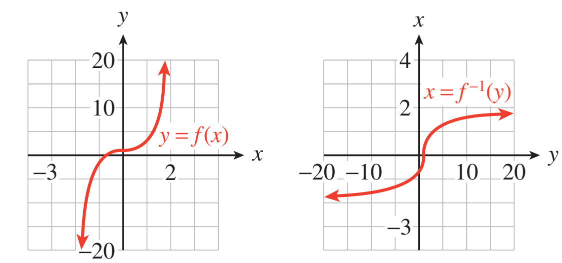
Answer. (a) and (d)

A function that passes the horizontal line test is called **one-to-one**, because each input has only one output and each output has only one input. A one-to-one function passes the horizontal line test as well as the vertical line test. With this terminology, we can state the following theorem.

One-to-one Functions.

The inverse of a function f is also a function if and only if f is one-to-one.

Caution 5.1.16 A function may have an inverse function even if we cannot find its formula. The function $f(x) = x^5 + x + 1$ shown in figure (a) is one-to-one, so it has an inverse function. We can even graph the inverse function, as shown in figure (b), by interchanging the coordinates of points on the graph of f.



(a) **(b)**

However, we cannot find a formula for the inverse function because we cannot solve the equation $y = x^5 + x + 1$ for x in terms of y.

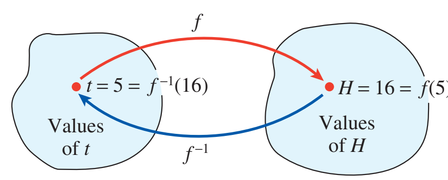
5.1.6 Mathematical Properties of the Inverse Function

The inverse function f^{-1} undoes the effect of the function f. In Example 5.1.3, p. 509, the function $f(t) = 6 + 2t$ multiplies the input by 2 and then adds 6 to the result. The inverse function $f^{-1}(H) = \dfrac{H-6}{2}$ undoes those operations in reverse order: It subtracts 6 from the input and then divides the result by 2.

If we apply the function f to a given input value and then apply the function f^{-1} to the output from f, the end result will be the original input value. For example, if we choose $t = \mathbf{5}$ as an input value, we find that

$$f(\mathbf{5}) = 6 + 2(\mathbf{5}) = \mathbf{16} \qquad \text{Multiply by 2, then add 6.}$$

$$\text{and } f^{-1}(\mathbf{16}) = \frac{\mathbf{16} - 6}{2} = \mathbf{5} \qquad \text{Subtract 6, then divide by 2.}$$



We return to the original input value, 5, as illustrated above.

Example 5.1.17, p. 516 illustrates the fact that if f^{-1} is the inverse function for f, then f is also the inverse function for f^{-1}.

Example 5.1.17 Consider the function $f(x) = x^3 + 2$ and its inverse, $f^{-1}(y) = \sqrt[3]{y-2}$.

a Show that the inverse function undoes the effect of f on $x = 2$.

b Show that f undoes the effect of the inverse function on $y = -25$.

Solution.

a We first evaluate the function f for $x = 2$:

$$f(\mathbf{2}) = \mathbf{2}^3 + 2 = \mathbf{10}$$

Then we evaluate the inverse function f^{-1} at $y = 10$:

$$f^{-1}(\mathbf{10}) = \sqrt[3]{\mathbf{10} - 2} = \sqrt[3]{8} = \mathbf{2}$$

We started and ended with 2.

b We first evaluate the function f^{-1} for $y = -25$:

$$f^{-1}(\mathbf{-25}) = \sqrt[3]{\mathbf{-25} - 2} = \mathbf{-3}$$

Then we evaluate the function f for $x = -3$:

$$f(\mathbf{-3}) = (\mathbf{-3})^3 + 2 = \mathbf{-25}$$

We started and ended with -25.

□

Checkpoint 5.1.18

a Find a formula for the inverse of the function $f(x) = \dfrac{2}{x-1}$

b Show that f^{-1} undoes the effect of f on $x = 3$.

c Show that f undoes the effect of f^{-1} on $y = -2$.

Answer.

a $f^{-1}(y) = 1 + \dfrac{2}{y}$

b $f(3) = 1$, and $f^{-1}(1) = 3$

c $f^{-1}(-2) = 0$ and $f(0) = -2$

Functions and Inverse Functions.

Suppose f^{-1} is the inverse function for f. Then

$$f^{-1}\left(f(x)\right) = x \quad \text{and} \quad f\left(f^{-1}(y)\right) = y$$

as long as x is in the domain of f, and y is in the domain of f^{-1}.

5.1.7 Symmetry

So far we have been careful to keep track of the input and output variables when we work with inverse functions. This is important when we are dealing with applications; the names of the variables are usually chosen because they have a meaning in the context of the application, and it would be confusing to change them.

However, we can also study inverse functions purely as mathematical objects. There is a relationship between the graph of a function and the graph of its inverse that is easier to see if we plot them both on the same set of axes.

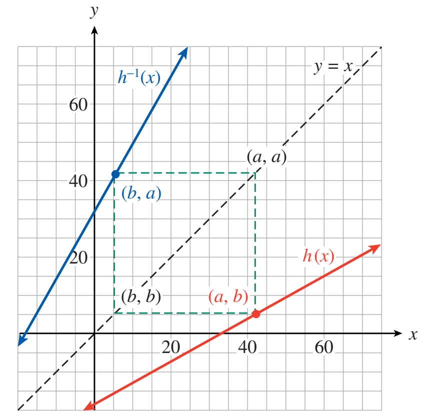
A graph does not change if we change the names of the variables, so we can let x represent the input for both functions, and let y represent the output. Consider the function $C = h(F)$ from Example 5.1.10, p. 512, and its inverse function, $F = h^{-1}(C)$. The formulas for these functions are

$$C = h(F) = \frac{5}{9}(F - 32)$$
$$F = h^{-1}(C) = 32 + \frac{9}{5}C$$

But their graphs are the same if we write them as

$$y = h(x) = \frac{5}{9}(x - 32)$$
$$y = h^{-1}(x) = 32 + \frac{9}{5}x$$

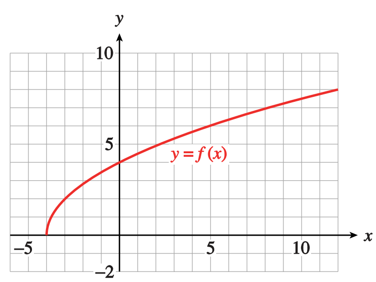
The graphs are shown below.

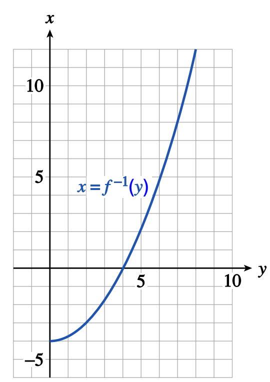


Now, for every point (a, b) on the graph of f, the point (b, a) is on the graph of the inverse function. Observe that the points (a, b) and (b, a) are always located symmetrically across the line $y = x$. The graphs are **symmetric about the line** $y = x$, which means that if we were to place a mirror along the line $y = x$, each graph would be the reflection of the other.

Example 5.1.19 Graph the function $f(x) = 2\sqrt{x+4}$ on the domain $[-4, 12]$. Graph its inverse function f^{-1} on the same grid.

Solution. The graph of f has the same shape as the graph of $y = \sqrt{x}$, shifted 4 units to the left and stretched vertically by a factor of 2. The figure at left below shows the graph of f, along with a table of values. By interchanging the rows of the table, we obtain points on the graph of the inverse function, shown at right.

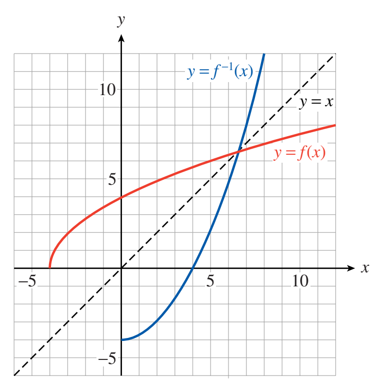




x	−4	−3	0	5	12
$y = f(x)$	0	2	4	6	8

y	0	2	4	6	8
$y = f^{-1}(y)$	−4	−3	0	5	12

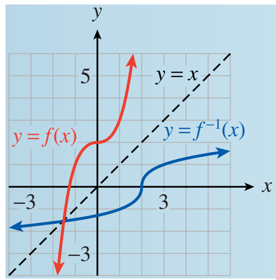
If we use x as the input variable for both functions, and y as the output, we can graph f and f^{-1} on the same grid, as shown at right. The two graphs are symmetric about the line $y = x$.



□

Checkpoint 5.1.20 Graph the function $f(x) = x^3 + 2$ and its inverse $f^{-1}(x) = \sqrt[3]{x-2}$ on the same set of axes, along with the line $y = x$.

Answer.



5.1.8 Domain and Range

When we interchange the input and output variables to obtain the inverse function, we interchange the domain and range of the function. For the functions graphed in Example 5.1.19, p. 518, you can see that

$$\begin{aligned} \text{Domain}(f) &= [-4, 12] \quad \text{and} \quad & \text{Domain}(f^{-1}) &= [0, 8] \\ \text{Range}(f) &= [0, 8] & \text{Range}(f-1) &= [-4, 12] \end{aligned}$$

This relationship between the domain and range of a function and its inverse holds in general.

Domain and Range of the Inverse Function.

If f^{-1} is the inverse function for $f$then

$$\textbf{Domain}\left(f^{-1}\right) = \textbf{Range}\,(f)$$
$$\textbf{Range}\left(f^{-1}\right) = \textbf{Domain}\,(f)$$

Example 5.1.21

a Graph the function $y = f(x) = \dfrac{1}{x+3}$ in the window

$$\begin{aligned} \text{Xmin} &= -6 & \text{Xmax} &= 3.4 \\ \text{Ymin} &= -6 & \text{Ymax} &= 3 \end{aligned}$$

b Graph the inverse function in the same window, along with the line $y = x$.

c State the domain and range of f, and of f^{-1}.

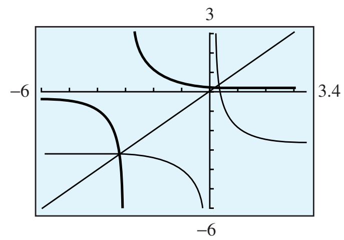
Solution.

a The graph of f is shown below. It looks like the graph of $y = \dfrac{1}{x}$, shifted 3 units to the left.

b To find the inverse function, we solve for x. Take the reciprocal of both sides of the equation.

$$\frac{1}{y} = x + 3 \qquad \text{Subtract 3 from both sides.}$$
$$x = \frac{1}{y} - 3$$

The inverse function is $x = f^{-1}(y) = \dfrac{1}{y} - 3$, or, using x for the input variable, $f^{-1}(x) = \dfrac{1}{x} - 3$. The graph of f^{-1} looks like the graph of $y = \dfrac{1}{x}$, shifted down 3 units, as shown below.



c Because f is undefined at $x = -3$, the domain of f is all real numbers except -3. The graph has a horizontal asymptote at $y = 0$, so the range is all real numbers except 0.

The inverse function $f^{-1}(x) = \dfrac{1}{x} - 3$ is undefined at $x = 0$, so its domain is all real numbers except 0. The graph of f^{-1} has a horizontal asymptote at $y = -3$, so its range is all real numbers except -3.

□

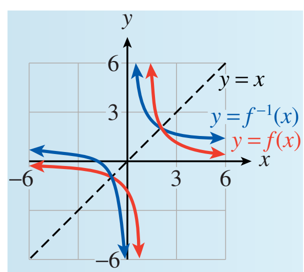
Checkpoint 5.1.22

a Graph the function $f(x) = \dfrac{2}{x-1}$ and its inverse function, f^{-1} (which you found in Checkpoint 5.1.18, p. 517), on the same set of axes, along with the line $y = x$.

b State the domain and range of f, and of f^{-1}.

Answer.

a



b Domain of f: all real numbers except 1, Range of f: all real numbers except 0, Domain of f^{-1}: all real numbers except 0, Range of f^{-1}: all real numbers except 1

5.1.9 Section Summary

5.1.9.1 Vocabulary

Look up the definitions of new terms in the Glossary.

- Inverse function
- Horizontal line test
- One-to-one

5.1.9.2 CONCEPTS

1 The **inverse** of a function describes the same relationship between two variables but interchanges the roles of the input and output.

2 Inverse Functions.

If the inverse of a function f is also a function, then the inverse is denoted by the symbol f^{-1}, and

$$f^{-1}(b) = a \quad \text{if and only if} \quad f(a) = b$$

3 We can make a table of values for the inverse function, f^{-1}, by interchanging the columns of a table for f.

4 If a function is defined by a formula in the form $y = f(x)$, we can find a formula for its inverse function by solving the equation for x to get $x = f^{-1}(y)$.

5 The inverse function f^{-1} undoes the effect of the function f, that is, if we apply the inverse function to the output of f, we return to the original input value.

6 If f^{-1} is the inverse function for f, then f is also the inverse function for f^{-1}.

7 The graphs of f and its inverse function are **symmetric about the line** $y = x$.

8 **Horizontal line test**: If no horizontal line intersects the graph of a function more than once, then the inverse is also a function.

9 A function that passes the horizontal line test is called **one-to-one**.

10 The inverse of a function f is also a function if and only if f is one-to-one.

11 Functions and Inverse Functions.

Suppose f^{-1} is the inverse function for f. Then

$$f^{-1}\left(f(x)\right) = x \quad \text{and} \quad f\left(f^{-1}(y)\right) = y$$

as long as x is in the domain of f, and y is in the domain of f^{-1}.

12 Domain and Range of the Inverse Function.

If f^{-1} is the inverse function for $f$then

$$\textbf{Domain}\left(f^{-1}\right) = \textbf{Range}\,(f)$$
$$\textbf{Range}\left(f^{-1}\right) = \textbf{Domain}\,(f)$$

5.1.9.3 STUDY QUESTIONS

1 Explain how the terms **inverse function**, **one-to-one**, and **horizontal line test** are related.

2 If you know that $f^{-1}(3) = -7$, what can you say about the values of f ?

3 Explain how to use a graph of the function g to evaluate $g^{-1}(2)$.

4 Evaluate $f(f^{-1}(5))$.

5 Delbert says that if $f(x) = x^{3/5}$, then $f^{-1}(x) = x^{-3/5}$. Is he correct? Why or why not?

5.1.9.4 SKILLS

Practice each skill in the Homework 5.1.10, p. 522 problems listed.

1 Given certain function values, find values of the inverse function: #1–4

2 Interpret values of the inverse function: #5–12

3 Find a formula for the inverse function: #9–22, 27–34

4 Graph the inverse function: #15 and 16, 23–34

5 Find the domain and range of the inverse function: #33 and 34

6 Use the horizontal line test to identify one-to-one functions: #35–42

5.1.10 Inverse Functions (Homework 5.1)

1. Let $f(-1) = 0$, $f(0) = 1$, $f(1) = -2$, and $f(2) = -1$.

(a) Make a table of values for $f(x)$ and another table for its inverse function.

(b) Find $f^{-1}(1)$

(c) Find $f^{-1}(-1)$

Answer.

(a)

x	-1	0	1	2
$f(x)$	0	1	-2	-1

y	0	1	-2	-1
$f^{-1}(y)$	-1	0	1	2

(b) $f^{-1}(1) = 0$

(c) $f^{-1}(-1) = 2$

2. Let $f(-1) = 1$, $f(-1) = -2$, $f(0) = 0$, and $f(1) = -1$.

(a) Make a table of values for $f(x)$ and another table for its inverse function.

(b) Find $f^{-1}(-1)$

(c) Find $f^{-1}(1)$

3. $f(x) = x^3 + x + 1$

(a) Make a table of values for $f(x)$ and another table for its inverse function.

(b) Find $f^{-1}(1)$

(c) Find $f^{-1}(3)$

Answer.

(a)

x	-1	0	1	2
$f(x)$	-1	1	3	11

y	-1	1	3	11
$f^{-1}(y)$	-1	0	1	2

(b) $f^1(1) = 0$

(c) $f^{-1}(3) = 1$

4. $f(x) = x^5 + x^3 + 7$

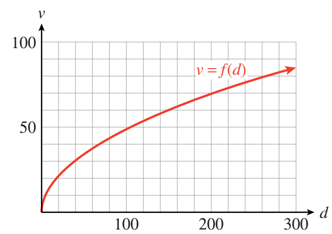
(a) Make a table of values for $f(x)$ and another table for its inverse function.

(b) Find $f^{-1}(7)$

(c) Find $f^{-1}(5)$

For Problems 5-8, use the graph to evaluate each expression.

5. An insurance investigator measures the length, d, of the skid marks at an accident scene, in feet. The graph shows the function $v = f(d)$, which gives the velocity, v (mph), at which a car was traveling when it hit the brakes.



(a) Use the graph to estimate $f(60)$ and explain its meaning in this context.

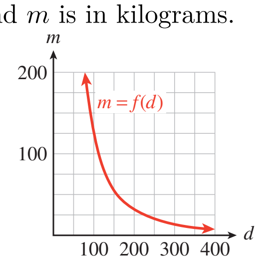
(b) Use the graph to estimate $f^{-1}(60)$ and explain its meaning in this context.

Answer.

(a) $f(60) \approx 38$. The car that left the 60-foot skid marks was traveling at 38 mph.

(b) $f^{-1}(60) \approx 150$. The car traveling at 60 mph left 150-foot skid marks

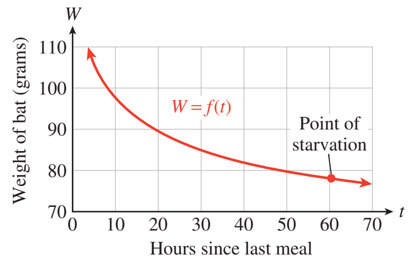
6. The weight, m, of a missile launched from a catapult is a function of the distance, d, to the target. The graph shows the function $m = f(d)$, where d is in meters and m is in kilograms.



(a) Use the graph to estimate $f(100)$ and explain its meaning in this context.

(b) Use the graph to estimate $f^{-1}(100)$ and explain its meaning in this context.

7. After eating, the weight of a vampire bat drops steadily until its next meal. The graph shows the function $W = f(t)$, which gives the weight, W, of the bat in grams t hours since its last meal.



(a) What are the coordinates of the point of starvation? Include units in your answer.

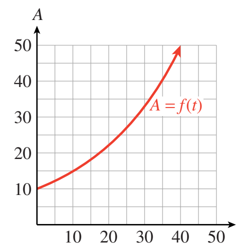
(b) Use the graph to estimate $f^{-1}(90)$ and explain its meaning in this context.

Answer.

(a) (60 hours, 78 grams)

(b) $f^{-1}(90) \approx 19$, so that the vampire bat's weight has dropped to 90 grams about 19 hours after its last meal.

8. The amount of money, A, in an interest-bearing savings account is a function of the number of years, t, it remains in the account. The graph shows $A = f(t)$, where A is in thousands of dollars.



(a) Use the graph to estimate $f(30)$ and explain its meaning in this context.

(b) Use the graph to estimate $f^{-1}(30)$ and explain its meaning in this context.

9. The function $I = g(r) = (1 + r)^5 - 1$ gives the interest, I, that a dollar earns in 5 years in terms of the interest rate, r.

(a) Evaluate $g(0.05)$ and explain its meaning in this context.

(b) Find the interest rate needed to earn \$0.50 by substituting $I = 0.50$ in the formula and solving for r.

(c) Find a formula for the inverse function.

(d) Write your answer to part (b) with inverse function notation.

Answer.

(a) $g(0.05) = 0.28$. At 5% interest, \$1 earns \$0.28 interest in 5 years.

(b) 8.45%

(c) $g^{-1}(I) = (I + 1)^{1/5} - 1$

(d) $g^{-1}(0.50) \approx 0.0845$

10. The function $C = h(F) = \dfrac{5}{9}(F - 32)$ gives the Celsius temperature C in terms of the Fahrenheit temperature F.

(a) Evaluate $h(104)$ and explain its meaning in this context.

(b) Find the Fahrenheit temperature of 37° Celsius by substituting $C = 37$ in the formula and solving for F.

(c) Find a formula for the inverse function.

(d) Write your answer to part (b) with inverse function notation.

11. If you are flying in an airplane at an altitude of h miles, on a clear day you can see a distance of d miles to the horizon, where $d = f(h) = \sqrt{7920h}$.

(a) Evaluate $f(0.5)$ and explain its meaning in this context.

(b) Find the altitude needed in order to see a distance of 10 mile by substituting $d = 10$ in the formula and solving for h.

(c) Find a formula for the inverse function.

(d) Write your answer to part (b) with inverse function notation.

Answer.

(a) $f(0.5) \approx 62.9$. At an altitude of 0.5 miles, you can see 62.9 miles to the horizon.

(b) 0.0126 mile, or 66.7 feet

(c) $h = f^{-1}(d) = \dfrac{d^2}{7920}$

(d) $f^{-1}(10) \approx 0.0126$

12. A moving ship creates waves that impede its own speed. The function $v = f(L) = 1.3\sqrt{L}$ gives the ship's maximum speed in knots in terms of its length, L, in feet.

(a) Evaluate $f(400)$ and explain its meaning in this context.

(b) Find the length needed for a maximum speed of 35 knots by substituting $v = 35$ in the formula and solving for L.

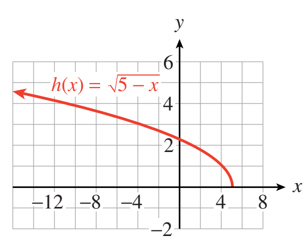
(c) Find a formula for the inverse function.

(d) Write your answer to part (b) with inverse function notation.

13.

(a) Use the graph of $h(x) = \sqrt{5 - x}$ to find $h^{-1}(3)$.

(b) Find a formula for $h^{-1}(x)$ and evaluate $h^{-1}(3)$.



Answer.

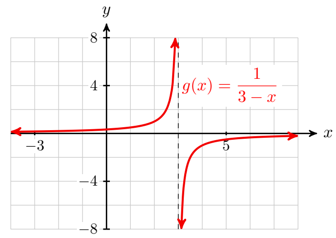
(a) $h^{-1}(3) \approx -4$

(b) $h^{-1}(x) = 5 - x^2$; $h^{-1}(3) = -4$

14.

(a) Use the graph of $g(x) = \dfrac{1}{3-x}$ to find $g^{-1}(-2)$.

(b) Find a formula for $g^{-1}(x)$ and evaluate $g^{-1}(-2)$.



15.

(a) Find f^{-1} for the function $f(x) = (x-2)^3$.

(b) Show that f^{-1} undoes the effect of f on $x = 4$.

(c) Show that f undoes the effect of f^{-1} on $x = -8$.

(d) Graph the function and its inverse on the same grid, along with the graph of $y = x$.

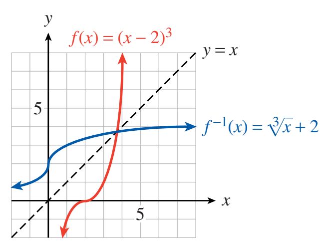
Answer.

(a) $f^{-1}(y) = 3\sqrt[3]{y} + 2$

(b) $f^{-1}(f(4)) = f^{-1}(8) = 4$

(c) $f(f^{-1}(-8)) = f(0) = -8$

(d)



16.

(a) Find f^{-1} for the function $f(x) = \dfrac{2}{x+1}$.

(b) Show that f^{-1} undoes the effect of f on $x = 3$.

(c) Show that f undoes the effect of f^{-1} on $x = -1$.

(d) Graph the function and its inverse on the same grid, along with the graph of $y = x$.

17. If $F(t) = \dfrac{2}{3}t + 1$, find $F^{-1}(5)$.

Answer. 6

18. If $G(s) = \dfrac{s-3}{4}$, find $G^{-1}(-2)$.

19. If $m(v) = 6 - \dfrac{2}{v}$, find $m^{-1}(-3)$.

Answer. $\dfrac{2}{9}$

20. If $p(z) = 1 - 2z^3$, find $p^{-1}(7)$.

21. If $f(x) = \dfrac{x+2}{x-1}$, find $f^{-1}(2)$.

Answer. 4

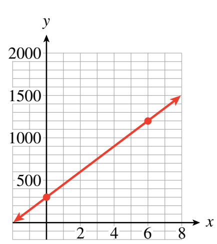
22. If $g(n) = \dfrac{3n+1}{n-3}$, find $g^{-1}(-2)$.

For Problems 23–26,

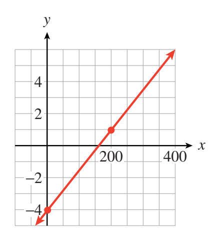
a Use the graph to make a table of values for the function $y = f(x)$.

b Make a table of values and a graph of the inverse function.

23.



24.



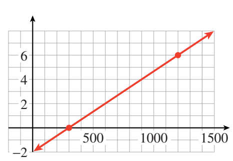
Answer.

(a)

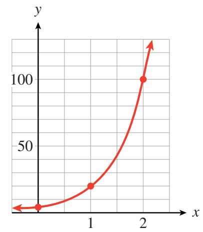
x	0	6
y	300	1200

(b)

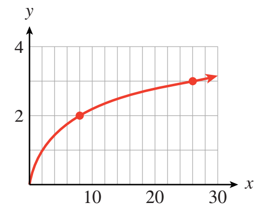
x	300	1200
y	0	6



25.



26.



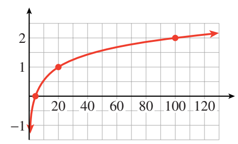
Answer.

(a)

x	0	1	2
y	5	20	100

(b)

x	5	20	100
y	0	1	2



For Problems 27–32,

a Find a formula for the inverse of the function.

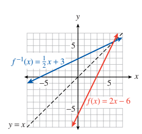
b Graph the function and its inverse on the same set of axes, along with the graph of $y = x$.

27. $f(x) = 2x - 6$

Answer.

(a) $f^{-1}(x) = \dfrac{x+6}{2}$

(b)



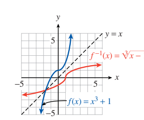
28. $f(x) = 3x - 1$

29. $f(x) = x^3 + 1$

Answer.

(a) $f^{-1}(x) = \sqrt[3]{x-1}$

(b)



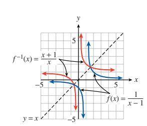
30. $f(x) = \sqrt[3]{x+1}$

31. $f(x) = \dfrac{1}{x-1}$

Answer.

(a) $f^{-1}(x) = \dfrac{1}{x} + 1$

(b)



32. $f(x) = \dfrac{1}{x} - 3$

33.

(a) Find the domain and range of the function $g(x) = \sqrt{4-x}$.

(b) Find a formula for $g^{-1}(x)$.

(c) State the domain and range of $g^{-1}(x)$.

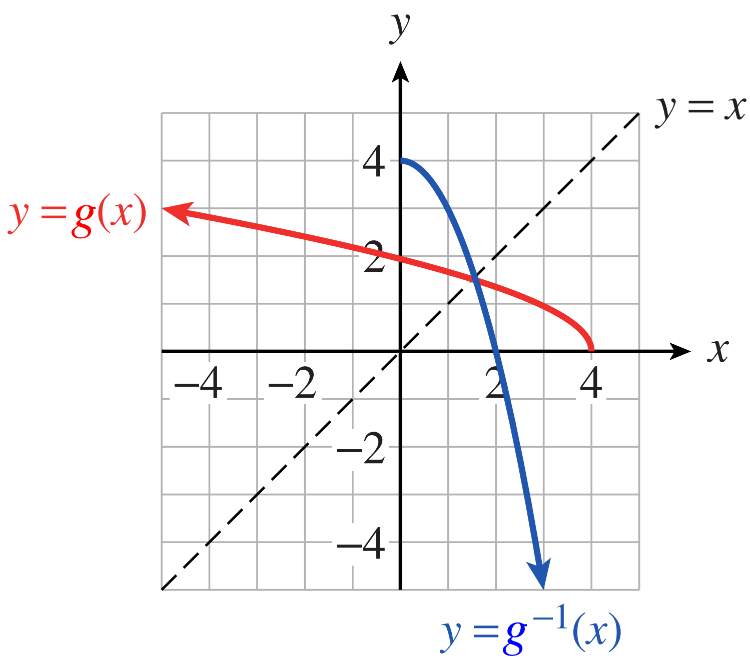
(d) Graph g and g^{-1} on the same grid.

Answer.

(a) Domain: $(-\infty, 4]$; Range: $[0, \infty)$

(b) $g^{-1}(x) = 4 - x^2$

(c) Domain: $[0, \infty)$; Range: $(-\infty, 4]$

(d) 

34.

(a) Find the domain and range of the function $g(x) = 8 - \sqrt{x}$.

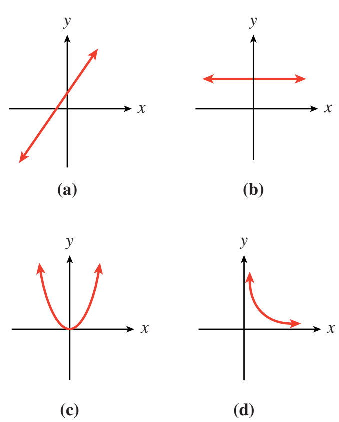
(b) Find a formula for $g^{-1}(x)$.

(c) State the domain and range of $g^{-1}(x)$.

(d) Graph g and g^{-1} on the same grid.

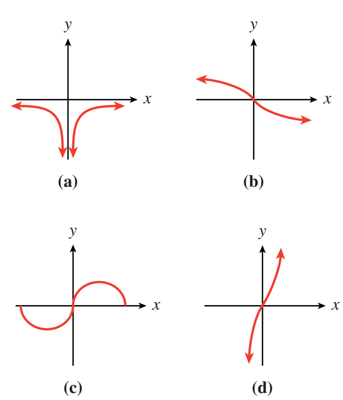
Which of the functions in Problems 35–42 have inverses that are also functions?

35.



Answer. (a) and (d)

36.



37.

(a) $f(x) = x$

(b) $f(x) = x^2$

Answer. (a)

38.

(a) $f(x) = x^3$

(b) $f(x) = |x|$

39.

(a) $f(x) = \dfrac{1}{x}$

(b) $f(x) = \dfrac{1}{x^2}$

Answer. (a)

40.

(a) $f(x) = \sqrt{x}$

(b) $f(x) = \sqrt[3]{x}$

41.

(a) $f(x) = 2^x$

(b) $f(x) = \left(\dfrac{1}{2}\right)^x$

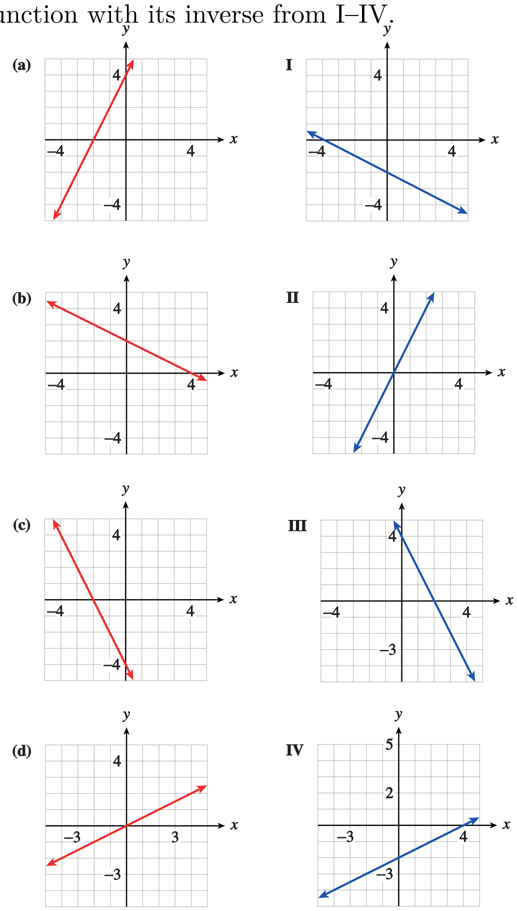
Answer. (a) and (b)

42.

(a) $f(x) = x^3 + x^2$

(b) $f(x) = x^3 + x$

43. Match each function with its inverse from I–IV.



Answer.

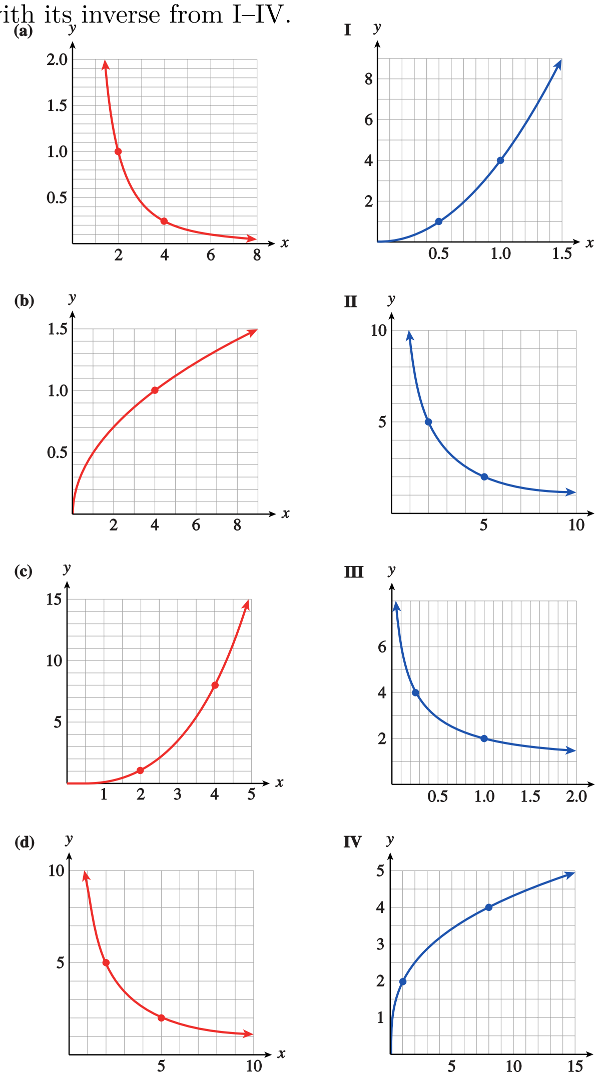
(a) $f(x) = 4 + 2x$; IV

(b) $f(x) = 2 - \dfrac{x}{2}$; III

(c) $f(x) = -4 - 2x$; I

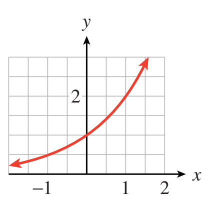
(d) $f(x) = \dfrac{x}{2}$; II

44. Find a formula for each function shown in (a)–(d). Then match each function with its inverse from I–IV.



For Problems 45 and 46, use the graph of f to match the other graphs with the appropriate function. (*Hint*: Look at the coordinates of some specific points.)

45.

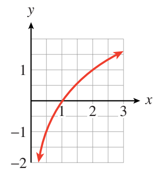


a $-f$

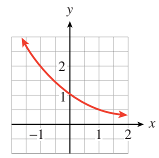
b $\dfrac{1}{f}$

c f^{-1}

I



II



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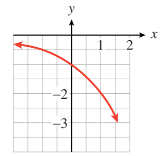
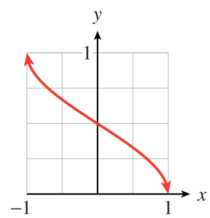


Answer.

(a) III (b) II (c) I

46.

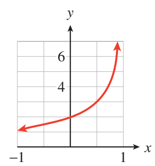


a $-f$

b $\dfrac{1}{f}$

c f^{-1}

I



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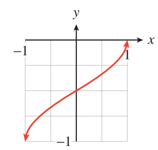

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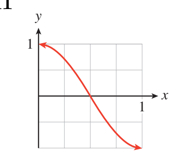


5.2 Logarithmic Functions

5.2.1 Inverse of the Exponential Function

Inverse functions are really a generalization of inverse operations. For example, raising to the nth power and taking nth roots are inverse operations. In fact, we use the following rule to define cube roots:

$$b = \sqrt[3]{a} \qquad \text{if and only if} \qquad a = b^3$$

Compare this rule to the definition of inverse functions from Section 5.1, p. 508. In this case, if $f(x) = x^3$ and $g(x) = \sqrt[3]{x}$, we see that

$$b = g(a) \qquad \text{if and only if} \qquad a = f(b)$$

We have shown that the two functions $f(x) = x^3$ and $g(x) = \sqrt[3]{x}$ are inverse functions.

In Chapter 4, p. 391, we saw that a similar rule relates the operations of raising a base b to a power and taking a base b logarithm, because they are inverse operations.

Conversion Formulas for Logarithms.

For any base $b > 0, b \neq 1$,

$$\boldsymbol{y = \log_b(x)} \qquad \text{if and only if} \qquad \boldsymbol{x = b^y}$$

We can now define the **logarithmic function**, $g(x) = \log_b(x)$, that takes the log base b of its input values. The conversion formulas tell us that the log function, $g(x) = \log_b(x)$, is the inverse of the exponential function, $f(x) = b^x$.

Logarithmic Function.

The **logarithmic function** base b, $g(x) = \log_b(x)$, is the inverse of the exponential function of the same base, $f(x) = b^x$.

For example, the function $g(x) = \log_2(x)$ is the inverse of $f(x) = 2^x$. Each function undoes the effect of the other. So, if we start with $x = 3$, apply f, and then apply g to the result, we return to the original number, 3.

$$x = \mathbf{3} \xrightarrow[\text{exponential function}]{\text{Apply the}} f(\mathbf{3}) = 2^{\mathbf{3}} = 8 \xrightarrow[\text{log function}]{\text{Apply the}} g(8) = \log_2(8) = \underset{\text{number}}{\overset{\text{Original}}{\mathbf{3}}}$$

We can write both calculations together as

$$\log_2\left(2^{\mathbf{3}}\right) = \mathbf{3}$$

A similar equation holds for any value of x and for any base $b > 0$. In other words, applying first the exponential function and then the log function returns the original input value, so that

$$\boldsymbol{\log_b(b^x) = x}$$

Example 5.2.1 Simplify each expression.

a $\log_4(4^6)$

b $\log_8(8^{2a+3})$

Solution.

a In this expression, we start with 6, apply the exponential function with base 4, and then take a logarithm base 4. Because the logarithm is the inverse of the exponential function, we return to the original number, 6.

$$x = \mathbf{6} \xrightarrow[\text{exponential function}]{\text{Apply the}} 4^{\mathbf{6}} \xrightarrow[\text{log function}]{\text{Apply the}} \log_4\left(4^{\mathbf{6}}\right) = \underset{\text{number}}{\overset{\text{Original}}{\mathbf{6}}}$$

b The input of the exponential function is the expression $2a+3$. Because the bases of the log and the exponential function are both 8, they are inverse functions, and applying them in succession returns us to the original input. Thus, $\log_8(8^{2a+3}) = 2a + 3$.

□

Checkpoint 5.2.2 Simplify each expression.

a $\log(10^6)$

b $\log_w(w^{x+1})$, for $w > 0$, $w \neq 1$

Answer.

a 6

b $x + 1$

We can also apply the two functions in the opposite order. For example,

$$2^{\log_2(\mathbf{8})} = \mathbf{8}$$

To see that this equation is true, we simplify the exponent first. We start with 8, and apply the log base 2 function. Because $\log_2(8) = 3$, we have

$$\mathbf{8} \xrightarrow[\text{log function}]{\text{Apply the}} \log_2(\mathbf{8}) = 3 \xrightarrow[\text{exponential function}]{\text{Apply the}} 2^{\log_2(\mathbf{8})} = \underset{\text{number}}{\overset{\text{Original}}{\mathbf{8}}}$$

Of course, a similar equation holds for any positive value of x and any base $b > 0, b \neq 1$:

$$b^{\log_b(x)} = x$$

Example 5.2.3 Simplify each expression.

a $10^{\log(1000)}$

b $Q^{\log_Q(25)}$, for $Q > 0, Q \neq 1$

Solution.

a In this expression, we start with 1000, take the logarithm base 10, and then apply the exponential function base 10 to the result. We return to the original input, so

$$10^{\log(1000)} = 1000$$

b The log function, $\log_Q(x)$, and the exponential function, Q^x, are inverse functions, so

$$Q^{\log_Q(25)} = 25$$

□

Checkpoint 5.2.4 Simplify each expression.

a $4^{\log_4(64)}$

b $2^{\log_2(x^2+1)}$

Answer.

a 64

b $x^2 + 1$

We summarize these relationships as follows.

Exponential and Logarithmic Functions.

Because $f(x) = b^x$ and $g(x) = \log_b(x)$ are inverse functions for $b > 0, b \neq 1$,

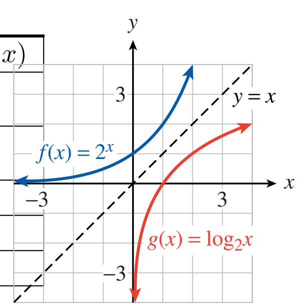
$$\log_b(b^x) = x, \text{ for all } x \qquad \text{and} \qquad b^{\log_b(x)} = x, \text{ for } x > 0$$

5.2.2 Graphs of Logarithmic Functions

We can obtain a table of values for $g(x) = \log_2(x)$ by making a table for $f(x) = 2^x$ and then interchanging the columns, as shown in the tables below. You can see that the graphs of $f(x) = 2^x$ and $g(x) = \log_2(x)$, shown in the figre, are symmetric about the line $y = x$.

x	$f(x) = 2^x$
-2	$\frac{1}{4}$
-1	$\frac{1}{2}$
0	1
1	2
2	4

x	$g(x) = \log_2(x)$
$\frac{1}{4}$	-2
$\frac{1}{2}$	-1
1	0
2	1
4	2



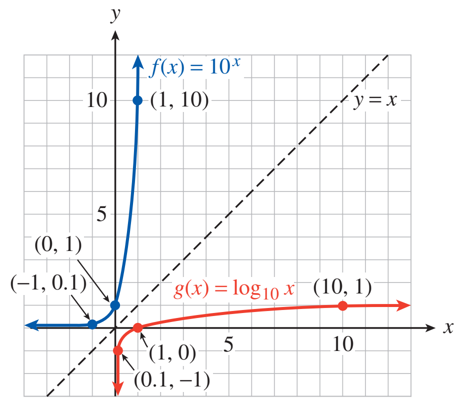
Example 5.2.5 Graph the function $f(x) = 10^x$ and its inverse $g(x) = \log(x)$ on the same axes.

Solution. We start by making a table of values for the function $f(x) = 10^x$.

We can make a table of values for the inverse function, $g(x) = \log(x)$, by interchanging the components of each ordered pair in the table for f.

x	$f(x)$
-2	0.01
-1	0.1
0	1
1	10
2	100

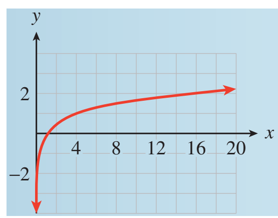
x	$g(x)$
0.01	-2
0.1	-1
1	0
10	1
100	2



We plot each set of points and connect them with smooth curves to obtain the graphs shown above. □

Checkpoint 5.2.6 Make a table of values and graph the function $h(x) = \log_4(x)$.

Answer.



While an exponential growth function increases very rapidly for positive values, its inverse, the logarithmic function, grows extremely slowly, as you can see in Example 5.2.5, p. 534. In addition, the logarithmic function $y = \log_b(x)$ for any base $b > 0, b \neq 1$, has the following properties.

Logarithmic Functions $y = \log_b(x)$**.**

1 Domain: all positive real numbers

2 Range: all real numbers

3 x-intercept: $(1, 0)$

4 y-intercept: none

5 Vertical asymptote at $x = 0$

6 The graphs of $y = \log_b(x)$ and $y = b^x$ are symmetric about the line $y = x$.

Example 5.2.7

a Find the inverse of the function $f(x) = 2^{x-3} - 4$.

b Graph f and f^{-1} on the same grid.

c State the domain and range of f and of f^{-1}.

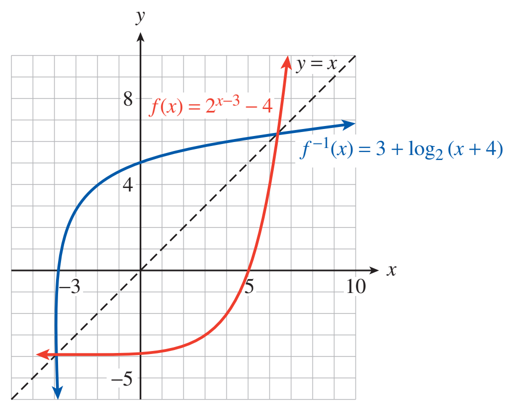
Solution.

a We write the function as $y = 2^{x-3} - 4$, and solve for x in terms of y. First, we isolate the power:

$$\begin{aligned} y + 4 &= 2^{x-3} && \text{Take logs base 2.} \\ \log_2(y+4) &= \log_2(2^{x-3}) = x - 3 \\ x &= 3 + \log_2(y+4) \end{aligned}$$

The inverse function is $f^{-1}(y) = 3 + \log_2(y+4)$. However, to graph both f and f^{-1} on the same grid, we write the inverse function as $f^{-1}(x) = 3 + \log_2(x+4)$.

b To graph f, we translate the graph of $y = 2^x$ by 3 units to the right and 4 units down. The graph of f^{-1} looks like the graph of $y = \log_2(x)$, but shifted 4 units to the left and 3 units up. The graphs are shown below, along with the line $y = x$.



c The function f is a translation of an exponential function, and its domain consists of all real numbers. Because the graph is shifted 4 units down, the range of f is $(-4, \infty)$. Because the log of a negative number or zero is undefined, for $f^{-1}(x) = 3 + \log_2(x+4)$, we must have $x + 4 > 0$, or $x > -4$. We can verify on the graph that the range of f^{-1} includes all real numbers. Thus,

$$\begin{aligned} &\text{Domain}(f) = \text{all real numbers} = \text{Range}(f^{-1}) \\ &\text{Range}(f) = (-4, \infty) = \text{Domain}(f^{-1}) \end{aligned}$$

□

Checkpoint 5.2.8

a Find the inverse function for $f(x) = 2\log(x+1)$.

b Graph f and f^{-1} in the window

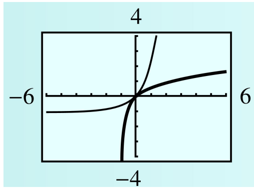
$$\begin{array}{ll} \text{Xmin} = -6 & \text{Xmax} = 6 \\ \text{Ymin} = -4 & \text{Ymax} = 4 \end{array}$$

c State the domain and range of f and f^{-1}.

Answer.

a $f^{-1}(x) = 10^{x/2} - 1$

b



c Domain of f: $(-1,\infty)$, Range of f: all real numbers, Domain of f^{-1}: all real numbers, Range of f^{-1}: $(-1,\infty)$

5.2.3 Evaluating Logarithmic Functions

We can use the `LOG` key on a calculator to evaluate the function $f(x) = \log(x)$.

Example 5.2.9 Let $f(x) = \log(x)$. Evaluate the following expressions.
a $f(35)$ b $f(-8)$ c $2f(16) + 1$

Solution.

a $f(35) = \log(35) \approx 1.544$

b Because -8 is not in the domain of f, $f(-8)$, or $\log(-8)$, is undefined.

c $2f(16) + 1 = 2(\log(16)) + 1 \approx 2(1.204) + 1 = 3.408$

□

Checkpoint 5.2.10 The formula $T = \dfrac{\log(2 \cdot t_i)}{3\log\left(D_f/D_0\right)}$ is used by X-ray technicians to calculate the doubling time of a malignant tumor. D_0 is the diameter of the tumor when first detected, D_f is its diameter at the next reading, and t_i is the time interval between readings, in days. Calculate the doubling time of the following tumor: its diameter when first detected was 1 cm, and 7 days later its diameter was 1.05 cm.

Answer. 33 days

5.2.4 Logarithmic Equations

A **logarithmic equation** is one in which the variable appears inside of a logarithm. For example,

$$\log_4(x) = 3$$

is a log equation. To solve a log equation, remember that logarithms and exponentials with the same base are inverse functions. Therefore,

$$\boldsymbol{y = \log_b(x)} \qquad \text{if and only if} \qquad \boldsymbol{x = b^y}$$

Thus, we can rewrite a logarithmic equation in exponential form.

Example 5.2.11 Solve for x.
a $2(\log_3(x)) - 1 = 4$ b $\log(2x + 100) = 3$

Solution.

a We isolate the logarithm, then rewrite the equation in exponential form:

$$\begin{aligned} 2(\log_3(x)) &= 5 && \text{Divide both sides by 5.} \\ \log_3(x) &= \frac{5}{2} && \text{Convert to exponential form.} \\ x &= 3^{5/2} \end{aligned}$$

b First, we convert the equation to exponential form.

$$2x + 100 = 10^3 = 1000$$

Now we can solve for x to find $2x = 900$, or $x = 450$.

□

Checkpoint 5.2.12 Solve for the unknown value in each equation.

a $\log_b(2) = \dfrac{1}{2}$

b $\log_3(2x - 1) = 4$

Answer.

a $b = 4$

b $x = 41$

Example 5.2.13 If $f(x) = \log(x)$, find x so that $f(x) = -3.2$.

Solution. We must solve the equation $\log(x) = -3.2$. Rewriting the equation in exponential form yields

$$x = 10^{-3.2} \approx 0.00063$$

□

Note 5.2.14 Evaluating 10^x. In Example 5.2.13, p. 538, the expression $10^{-3.2}$ can be evaluated in two different ways with a calculator. We can use the `^` key and press

10 `^` `(-)` 3.2 `ENTER`

which gives `6.30957344 E -4`, or approximately 0.00063. Alternatively, because 10^x is the inverse function for $\log(x)$, we can press

`2nd` `LOG` `(-)` 3.2 `ENTER`

which gives the same answer as before.

Checkpoint 5.2.15 Imagine the graph of $f(x) = \log(x)$. How far must you travel along the x-axis until the y-coordinate reaches a height of 5.25?

Answer. $x = 177,827.941$

5.2.5 Using the Properties of Logarithms

The properties of logarithms are useful in solving both exponential and logarithmic equations. To solve logarithmic equations, we first combine any expressions involving logs into a single logarithm.

Example 5.2.16 Solve $\log(x + 1) + \log(x - 2) = 1$.

Solution. We use Property (1) of logarithms to rewrite the left-hand side as a single logarithm:

$$\log\big((x+1)(x-2)\big) = 1$$

Once the left-hand side is expressed as a single logarithm, we can rewrite the equation in exponential form as

$$(x + 1)(x - 2) = 10^1$$

from which

$$
\begin{aligned}
x^2 - x - 2 &= 10 && \text{Subtract 10 from both sides.} \\
x^2 - x - 12 &= 0 && \text{Factor the left side.} \\
(x-4)(x+3) &= 0 && \text{Apply the zero-factor principle.}
\end{aligned}
$$

Thus, $x = 4$ or $x = -3$. The number -3 is not a solution of the original equation, because neither $\log(x+1)$ nor $\log(x-2)$ is defined for $x = -3$. The solution of the original equation is 4. □

In Example 5.2.16, p. 538, the apparent solution $x = -3$ is called **extraneous** because it does not solve the original equation. We should always check for extraneous solutions when solving log equations. The following steps give a rough outline for solving log equations.

Steps for Solving Logarithmic Equations.

1 Use the properties of logarithms to combine all logs into one log.

2 Isolate the log on one side of the equation.

3 Convert the equation to exponential form.

4 Solve for the variable.

5 Check for extraneous solutions.

Checkpoint 5.2.17 Solve $\log_2(x) + \log_2(x-2) = 3$.

Hint. Rewrite the left side as a single logarithm.
Rewrite the equation in exponential form.
Solve for x.
Check for extraneous solutions.

Answer. $x = 4$

5.2.6 Section Summary

5.2.6.1 Vocabulary

Look up the definitions of new terms in the Glossary.

- Logarithmic function
- Logarithmic equation
- Extraneous solution

5.2.6.2 CONCEPTS

1 We define the **logarithmic function**, $g(x) = \log_b(x)$, which takes the log base b of its input values. The log function $g(x) = \log_b(x)$ is the inverse of the exponential function $f(x) = b^x$.

2 Exponential and Logarithmic Functions.

Because $f(x) = b^x$ and $g(x) = \log_b(x)$ are inverse functions for $b > 0, b \neq 1$,

$$\log_b(b^x) = x, \text{ for all } x \qquad \text{and} \qquad b^{\log_b(x)} = x, \text{ for } x > 0$$

3 **Logarithmic Functions** $y = \log_b(x)$**.**

1 Domain: all positive real numbers

2 Range: all real numbers

3 x-intercept: $(1, 0)$

4 y-intercept: none

5 Vertical asymptote at $x = 0$

6 The graphs of $y = \log_b(x)$ and $y = b^x$ are symmetric about the line $y = x$.

4 A **logarithmic equation** is one in which the variable appears inside of a logarithm. We can solve logarithmic equations by converting to exponential form.

5 **Steps for Solving Logarithmic Equations.**

1 Use the properties of logarithms to combine all logs into one log.

2 Isolate the log on one side of the equation.

3 Convert the equation to exponential form.

4 Solve for the variable.

5 Check for extraneous solutions.

5.2.6.3 STUDY QUESTIONS

1 Can the output of the function $y = \log_b(x)$ be negative?

2 Francine says that $\log_2\left(\dfrac{1}{x}\right) = -\log_2(x)$. Is she correct? Why or why not?

3 Sketch a typical logarithmic function.

4 Simplify:

a $10^{\log(13)}$

b $7^{\log_7(13)}$

5 Why is the following attempt to solve the equation incorrect?

Solve:
$$\begin{aligned} \log(x) + \log(x+1) &= 2 \\ x + x + 1 &= 10^2 \end{aligned}$$

5.2.6.4 SKILLS

Practice each skill in the Homework 5.2.7, p. 541 problems listed.

1 Evaluate log functions: #1–16, 27 and 28

2 Simplify expressions involving logs: #15 and 16, 19, and 20

3 Graph logarithmic functions and transformations of log functions: #1–4, 25–28

4 Find formulas for inverse functions: #17–24

5 Solve logarithmic equations: #29–54

6 Solve formulas involving logs: #55–60

5.2.7 Logarithmic Functions (Homework 5.2)

In Problems 1–4,

a Make tables of values for each exponential function and its inverse logarithmic function.

b Graph both functions on the same set of axes.

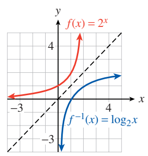
1. $f(x) = 2^x$

Answer.

(a)

x	-1	0	1	2
2^x	$\frac{1}{2}$	1	2	4

x	$\frac{1}{2}$	1	2	4
$\log_2(x)$	-1	0	1	2

(b) 

2. $f(x) = 3^x$

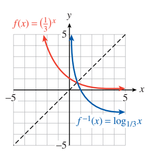
3. $f(x) = \left(\dfrac{1}{3}\right)^x$

Answer.

(a)

x	-2	-1	0	1
$\left(\frac{1}{3}\right)^x$	9	3	1	$\frac{1}{3}$

x	9	3	1	$\frac{1}{3}$
$\log_{1/3}(x)$	-2	-1	0	1

(b) 

4. $f(x) = \left(\dfrac{1}{2}\right)^x$

5.

(a) How large must x be before the graph of $y = \log(x)$ reaches a height of 4?

(b) How large must x be before the graph of $y = \log(x)$ reaches a height of 8?

Answer.

(a) $x = 10,000$

(b) $x = 10^8$

6.

(a) How large must x be before the graph of $y = \log_2(x)$ reaches a height of 5?

(b) How large must x be before the graph of $y = \log(x)$ reaches a height of 10?

7. For what values of x is $y = \log(x) < -2$?

Answer. $0 < x < 0.01$

8. For what values of x is $y = \log_2(x) < -3$?

In Problems 9–14, $f(x) = \log(x)$. Evaluate.

9.

(a) $f(487) + f(206)$

(b) $f(487 + 206)$

10.

(a) $f(93) + f(1500)$

(b) $f(93 + 1500)$

Answer.

(a) $\log(100,322) \approx 5.001$

(b) $\log(693) \approx 2.841$

11.

(a) $f(-7)$

(b) $6f(28)$

12.

(a) $f(0)$

(b) $f(-7)$

Answer.

(a) $\log(-7)$ is undefined.

(b) $6\log(28) \approx 8.683$

13.

(a) $18 - 5f(3)$

(b) $\dfrac{2}{5 + f(0.6)}$

14.

(a) $15 - 4f(7)$

(b) $\dfrac{3}{2 + f(0.2)}$

Answer.

(a) 15.614

(b) 0.419

15. Let $f(x) = 3^x$ and $g(x) = \log_3(x)$.

(a) Compute $f(4)$.

(b) Compute $g[f(4)]$.

(c) Explain why $\log_3(3^x) = x$ for any x.

(d) Compute $\log_3(3^{1.8})$.

(e) Simplify $\log_3(3^a)$.

Answer.

(a) 81

(b) 4

(c) Definition of logarithm base 3

(d) 1.8

(e) a

16. Let $f(x) = 2^x$ and $g(x) = \log_2(x)$.

(a) Compute $f(32)$.

(b) Compute $g[f(32)]$.

(c) Explain why $2^{\log_2(x)} = x$ for any $x > 0$.

(d) Compute $2^{\log_2(6)}$.

(e) Simplify $2^{\log_2(Q)}$.

17.

(a) If $h(r) = \log_2(r)$, find $h^{-1}(8)$.

(b) If $H(w) = 3^w$, find $H^{-1}\left(\dfrac{1}{9}\right)$.

Answer.

(a) 2^8

(b) -2

18.

(a) If $g(z) = \log_3(z)$, find $g^{-1}(-3)$.

(b) If $G(q) = 2^q$, find $G^{-1}(1)$.

For Problems 19–20, simplify.

19.

(a) $10^{\log(2k)}$

(b) $10^{3\log(x)}$

(c) $(\sqrt{10})^{\log(x)}$

(d) $\log(100^m)$

Answer.

(a) $2k$

(b) x^3

(c) $\sqrt{x}$

(d) $2m$

20.

(a) $\log(10^{(1-x)})$

(b) $100^{\log(2x)}$

(c) $(0.1)^{\log(x-1)}$

(d) $\log\left(10^{\log(10)}\right)$

21.

(a) What is the domain of the function $f(x) = 4 + \log_3(x - 9)$?

(b) Find a formula for $f^{-1}(x)$.

Answer.

(a) $(9, \infty)$

(b) $f^{-1}(x) = 3^{x-4} + 9$

22.

(a) What is the domain of the function $f(x) = 1 - \log_2(16 - 4x)$?

(b) Find a formula for $f^{-1}(x)$.

23.

(a) Find the inverse of the function $f(x) = 100 - 4^{x+2}$.

(b) Show that f^{-1} undoes the effect of f on $x = 1$.

(c) Show that f undoes the effect of f^{-1} on $x = 84$.

Answer.

(a) $f^{-1}(x) = \log_4(100 - x) - 2$

(b) $f^{-1}(f(1)) = f^{-1}(36) = \log_4(64) - 2 = 1$

(c) $f\left(f^{-1}(84)\right) = f(0) = 100 - 4^2 = 84$

24.

(a) Find the inverse of the function $f(x) = 5 + 2^{-x}$.

(b) Show that f^{-1} undoes the effect of f on $x = -2$.

(c) Show that f undoes the effect of f^{-1} on $x = 6$.

For Problems 25–26, match each graph to its equation.

25.

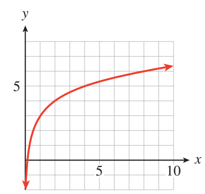
(a) $y = \log_2(x - 3)$

(b) $y = 3 + \log_2(x)$

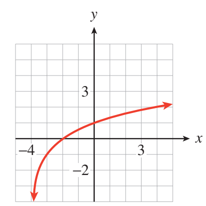
(c) $y = 2 - \log_2(x)$

(d) $y = \log_2(x + 4) - 1$

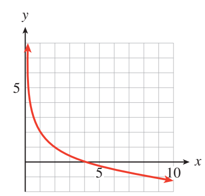
I



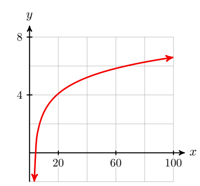
III



II



IV



Answer.

(a) IV (b) I (c) II (d) III

26.

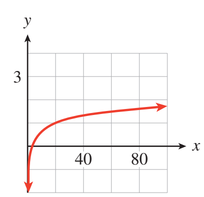
(a) $y = 5\log(x)$

(b) $y = \log\left(\dfrac{x}{2}\right)$

(c) $y = \log\left(\dfrac{1}{x}\right)$

(d) $y = \log(-x)$

I



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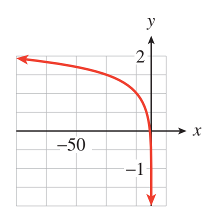

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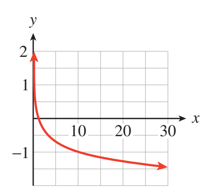
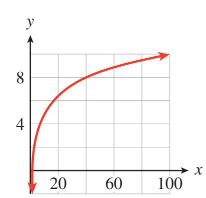


IV



27. In a psychology experiment, volunteers were asked to memorize a list of nonsense words, then 24 hours later were tested to see how many of the words they recalled. On average, the subjects had forgotten 20% of the words. The researchers found that the more lists their volunteers memorized, the larger the fraction of words they were unable to recall. (Source: Underwood, *Scientific American*, vol. 210, no. 3)

Number of lists, n	1	4	8	12	16	20
Percent forgotten, F	20	40	55	66	74	80

(a) Plot the data. What sort of function seems to fit the data points?

(b) Psychologists often describe rates of forgetting by logarithmic functions. Graph the function

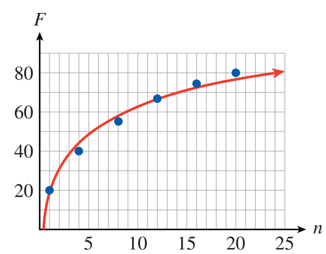
$$f(n) = 16.6 + 46.3\log(n)$$

on the same graph with your data. Comment on the fit.

(c) What happens to the function $f(n)$ as n grows increasingly large? Does this behavior accurately reflect the situation being modeled?

Answer.

(a)



(b) The graph resembles a logarithmic function. The (translated) log function is close to the points but appears too steep at first and not steep enough after $n = 15$. Overall, it is a good fit.

(c) f grows (more and more slowly) without bound. f will eventually exceed 100 per cent, but no one can forget more than 100% of what is learned.

28. The water velocity at any point in a stream or river is related to the logarithm of the depth at that point. For the Hoback River near Bondurant, Wyoming,

$$v = 2.63 + 1.03\log(d)$$

where v is the velocity of the water, in feet per second, and d is the vertical distance from the stream bed, in feet, at that point. For Pole Creek near Pinedale, Wyoming,

$$v = 1.96 + 0.65\log(d)$$

Both streams are 1.2 feet deep at the locations mentioned. (Source: Leopold, Luna, Wolman, and Gordon, 1992)

(a) Complete the table of values for each stream.

Distance from bed (feet)	0.2	0.4	0.6	0.8	1.0	1.2
Velocity, Hoback River, (ft/sec)						
Velocity, Pole Creek (ft/sec)						

(b) If you double the distance from the bed, by how much does the velocity increase in each stream?

(c) Plot both functions on the same graph.

(d) The average velocity of the entire stream can be closely approximated as follows: Measure the velocity at 20% of the total depth of the stream from the surface and at 80% of the total depth, then average these two values. Find the average velocity for the Hoback River and for Pole Creek.

In Problems 29–30, $f(x) = \log(x)$. Solve for x.

29.

(a) $f(x) = 1.41$ (b) $f(x) = -1.69$ (c) $f(x) = 0.52$

Answer.

(a) $10^{1.41} \approx 25.704$
(c) $10^{0.52} \approx 3.3113$
(b) $10^{-1.69} \approx 0.020417$

30.

(a) $f(x) = 2.3$ (b) $f(x) = -1.3$ (c) $f(x) = 0.8$

For Problems 31-38, convert the logarithmic equation to exponential form.

31. $\log_{16}(256) = w$
Answer.
$16^w = 256$

32. $\log_9(729) = y$

33. $\log_b(9) = -2$
Answer.
$b^{-2} = 9$

34. $\log_b(8) = -3$

35. $\log(A) = -2.3$
Answer.
$10^{-2.3} = A$

36. $\log(C) = -4.5$

37. $\log_u(v) = w$
Answer.
$u^w = v$

38. $\log_m(n) = p$

For Problems 39-46, solve for the unknown value.

39. $\log_b(8) = 3$
Answer. $b = 2$

40. $\log_b(625) = 4$

41. $\log_b(10) = \dfrac{1}{2}$
Answer. $b = 100$

42. $\log_b(0.1) = -1$

43. $\log_2(3x - 1) = 5$
Answer. $x = 11$

44. $\log_5(9 - 4x) = 3$

45. $3(\log_7(x)) + 5 = 7$
Answer. $x = 7^{2/3}$

46. $5(\log_2(x)) + 6 = -14$

For Problems 47-54, solve the logarithmic equation.

47. $\log(x) + \log(x + 21) = 2$
Answer. $x = 4$

48. $\log(x + 3) + \log(x) = 1$

49. $\log_8(x + 5) - \log_8(2) = 1$
Answer. $x = 11$

50. $\log(x - 1) - \log(4) = 2$

51. $\log(x + 2) + \log(x - 1) = 1$
Answer. $x = 3$

52. $\log_4(x + 8) - \log_4(x + 2) = 2$

53. $\log_3(x-2) - \log_3(x+1) = 3$

Answer. No solution

54. $\log(x+3) - \log(x-1) = 1$

For Problems 55-60, solve for the indicated variable.

55. $t = T\log\left(1+\dfrac{A}{k}\right)$, for A

Answer. $A = k(10^{t/T} - 1)$

56. $\log(R) = \log(R_0) + kt$, for R

57. $N = N_0 \log_b(ks)$, for s

Answer. $s = \dfrac{b^{N/N_0}}{k}$

58. $T = \dfrac{H\log\left(\dfrac{N}{N_0}\right)}{\log\left(\dfrac{1}{2}\right)}$, for N

59. $M = \sqrt{\dfrac{\log(H)}{k\log(H_0)}}$, for H

Answer. $H = (H_0)^{kM^2}$

60. $h = a - \sqrt{\dfrac{\log(B)}{t}}$, for B

61. Choose the graph for each function described below.

(a) The area, A, of a pentagon is a quadratic function of the length, l, of its side.

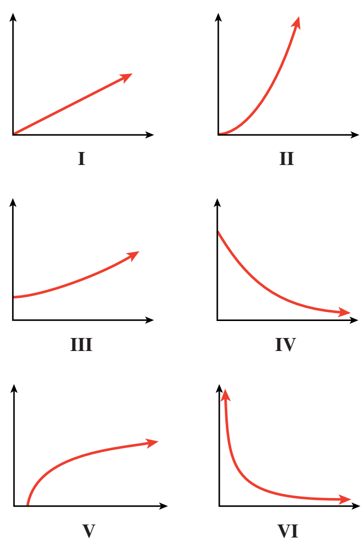
(b) The strength, F, of a hurricane varies inversely with its speed, s.

(c) The price of food has increased by 3% every year for a decade.

(d) The magnitude, M, of a star is a logarithmic function of its brightness, I.

(e) The speed of the train increased at a constant rate.

(f) If you do not practice a foreign language, you lose $\frac{1}{8}$ of the words in your working vocabulary, V, each year.



Answer.

(a) II

(b) VI

(c) III

(d) V

(e) I

(f) IV

62. For each of the functions $g(x)$ listed below, select the graph of its inverse function, if possible, from the figures labeled I–VI. (The inverse of one of the functions is not shown.)

(a) $g(x) = 2^x$

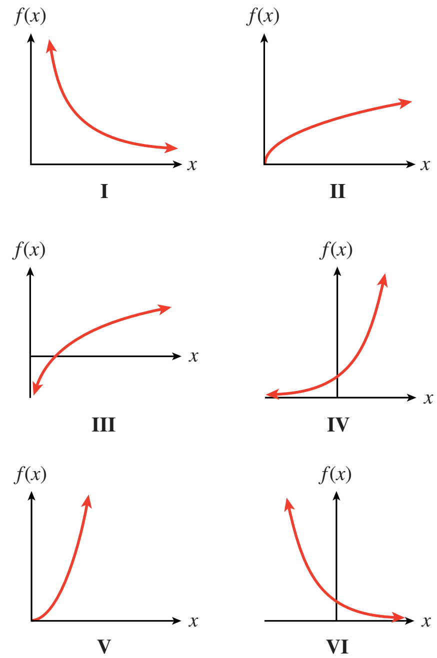
(b) $g(x) = x^2, \ \ x \geq 0$

(c) $g(x) = \dfrac{2}{x}$

(d) $g(x) = \sqrt{x}$

(e) $g(x) = \log_2(x)$

(f) $g(x) = \left(\dfrac{1}{2}\right)^x$



For Problems 63-64, graph the function on the domain [-4, 4] and a suitable range. Which have inverses that are also functions?

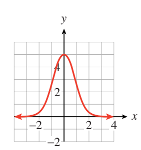
63.

(a) $f(x) = 5(2^{-x^2})$

(b) $f(x) = 2^x + 2^{-x}$

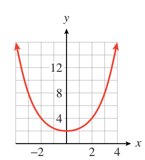
Answer.

(a)



No inverse function

(b)



No inverse function

64.

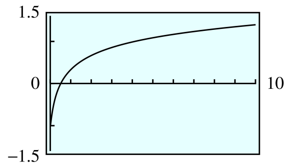
(a) $f(x) = 5(\log(x))^2 + 1$

(b) $f(x) = 5\log(x^2 + 1)$

For Problems 65-68, graph the pair of functions on your calculator. Explain the result.

65. $f(x) = \log(2x), \quad g(x) = \log(2) + \log(x)$

Answer.

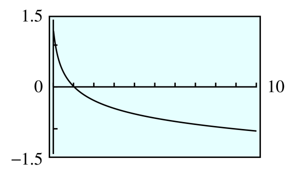


The functions are equal.

66. $f(x) = \log\left(\frac{x}{3}\right), \quad g(x) = \log(x) - \log(3)$

67. $f(x) = \log\left(\frac{1}{x}\right), \quad g(x) = -\log(x)$

Answer.



The functions are equal.

68. $f(x) = \log(x^3), \quad g(x) = 3\log(x)$

69.

(a) Complete the following table.

x	x^2	$\log(x)$	$\log(x^2)$
1			
2			
3			
4			
5			
6			

(b) Do you notice a relationship between $\log(x)$ and $\log(x^2)$? State the relationship as an equation.

Answer.

(a)

x	x^2	$\log(x)$	$\log(x^2)$
1	1	0	0
2	4	0.301	0.602
3	9	0.477	0.954
4	16	0.602	1.204
5	25	0.699	1.398
6	36	0.778	1.556

(b) $\log(x^2) = 2\log(x)$

70.

(a) Complete the following table.

x	$\dfrac{1}{x}$	$\log(x)$	$\log\left(\dfrac{1}{x}\right)$
1			
2			
3			
4			
5			
6			

(b) Do you notice a relationship between $\log(x)$ and $\log\left(\dfrac{1}{x}\right)$? State the relationship as an equation.

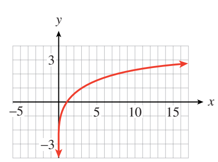
In Problems 69 and 70, you found relationships between $\log(x)$ and $\log(x^2)$, and between $\log(x)$ and $\log\left(\dfrac{1}{x}\right)$. Assumingthat those relationships hold for any base, complete the following tables and use them to graph the given functions.

71.

x	$y = \log_e(x)$
1	0
2	0.693
4	
16	
$\frac{1}{2}$	
$\frac{1}{4}$	
$\frac{1}{16}$	

Answer.

x	$y = \log_e(x)$
1	0
2	0.693
4	1.386
16	2.772
$\frac{1}{2}$	-0.693
$\frac{1}{4}$	-1.386
$\frac{1}{16}$	-2.772



72.

x	$y = \log_f(x)$
1	0
2	0.431
4	
16	
$\frac{1}{2}$	
$\frac{1}{4}$	
$\frac{1}{16}$	

5.2.8 Investigation

Investigation 33 Interest Compounded Continuously. We learned in Section 4.4, p. 455 that the amount, A (principal plus interest), accumulated in an account with interest compounded n times annually is

$$A = P\left(1 + \frac{r}{n}\right)^{nt}$$

where P is the principal invested, r is the interest rate, and t is the time period, in years.

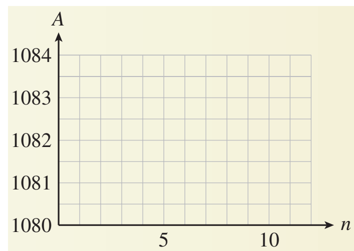
1 Suppose you keep \$1000 in an account that pays 8% interest. How much is the amount A after 1 year if the interest is compounded twice a year? Four times a year?

$$n = \mathbf{2} : A = 1000\left(1 + \frac{0.08}{\mathbf{2}}\right)^{\mathbf{2}(1)} =$$

$$n = \mathbf{4} : A = 1000\left(1 + \frac{0.08}{\mathbf{4}}\right)^{\mathbf{4}(1)} =$$

2 What happens to A as we increase n, the number of compounding periods per year? Fill in the table showing the amount in the account for different values of n.

n	A
1 (annually)	1080
2 (semiannually)	
4 (quarterly)	
6 (bimonthly)	
12 (monthly)	
365 (daily)	
1000	
$10,000$	



3 Plot the values in the table from $n = 1$ to $n = 12$, and connect them with a smooth curve. Describe the curve: What is happening to the value of A?

4 In part (2), as you increased the value of n, the other parameters in the formula stayed the same. In other words, A is a function of n, given by $A = 1000(1 + \dfrac{0.08}{n})^n$. Use your calculator to graph A on successively larger domains:

a $\text{Xmin} = 0, \text{Xmax} = 12; \text{Ymin} = 1080, \text{Ymax} = 1084$

b $\text{Xmin} = 0, \text{Xmax} = 50; \text{Ymin} = 1080, \text{Ymax} = 1084$

c $\text{Xmin} = 0, \text{Xmax} = 365; \text{Ymin} = 1080, \text{Ymax} = 1084$

5 Use the **Trace** feature or the **Table** feature to evaluate A for very large values of n. Rounded to the nearest penny, what is the largest value of A that you can find?

6 As n increases, the values of A approach a limiting value. Although A continues to increase, it does so by smaller and smaller increments and will never exceed \$1083.29. When the number of compounding periods increases without bound, we call the limiting result **continuous compounding**.

7 Is there an easier way to compute A under continues compounding? Yes! Compute $1000e^{0.08}$ on your calculator. (Press `2nd LN` to enter e^x.) Compare the value to your answer in part (5) for the limiting value. The number e is called the **natural base**. We'll compute its value shortly.

8 Repeat your calculations for two other interest rates, 15% and (an extremely unrealistic) 100%, again for an investment of \$1000 for 1 year. In each case, compare the limiting value of A, and compare to the value of $1000e^r$.

a

$r = 0.15$	
n	A
1	115
2	
4	
6	
12	
3652	
1000	
10,000	

$1000e^{0.15} =$

b

$r = 1$	
n	A
1	200
2	
4	
6	
12	
3652	
1000	
10,000	

$1000e^{1} =$

9 In part (8b), you have computed an approximation for $1000e$. What is the value of e, rounded to 5 decimal places?

10 Complete the table of values. What does $\left(1+\dfrac{1}{n}\right)^n$ appear to approach as n increases?

n	100	1000	10,000	100,000
$\left(1+\frac{1}{n}\right)^n$				

5.3 The Natural Base

There is another base for logarithms and exponential functions that is often used in applications. This base is an irrational number called e, where

$$e \approx 2.71828182845$$

The number e is essential for many advanced topics, and it is often called the **natural base**.

5.3.1 The Natural Exponential Function

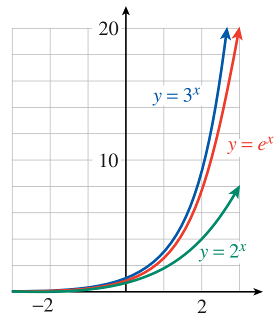
The **natural exponential function** is the function $f(x) = e^x$. Values for e^x can be obtained with a calculator using the $\boxed{e^x}$ key (`2nd` `LN` on most calculators). For example, you can evaluate e^1 by pressing

`2nd` `LN` `1`

to confirm the value of e given above.

Because e is a number between 2 and 3, the graph of $f(x) = e^x$ lies between the graphs of $y = 2^x$ and $y = 3^x$. Compare the tables of values and the graphs of the three functions below. As with other exponential functions, the domain of the natural exponential function includes all real numbers, and its range is the set of positive numbers.

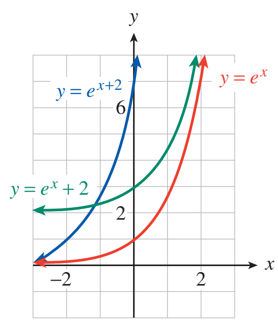
x	$y=2^x$	$y=e^x$	$y=3^x$
-3	0.125	0.050	0.037
-2	0.250	0.135	0.111
-1	0.500	0.368	0.333
0	1	1	1
1	2	2.718	3
2	4	7.389	9
3	8	20.086	27



Example 5.3.1 Graph each function. How does each graph differ from the graph of $y=e^x$?

a $g(x)=e^{x+2}$ b $h(x)=e^x+2$

Solution.



If $f(x)=e^x$, then $g(x)=f(x+2)$, so the graph of g is shifted 2 units to the left of $y=e^x$. Also, $h(x)=f(x)+2$, so the graph of h is shifted 2 units up from $y=e^x$. The graphs are shown above. □

Checkpoint 5.3.2 Use your calculator to evaluate the following powers.

a e^2 b $e^{3.5}$ c $e^{-0.5}$

Answer.

a $e^2\approx 7.389$ b $e^{3.5}\approx 33.115$ c $e^{-0.5}\approx 0.6065$

5.3.2 The Natural Logarithmic Function

The base e logarithm of a number x, or $\log_e(x)$, is called the **natural logarithm** of x and is denoted by $\ln(x)$.

The Natural Logarithm.

The natural logarithm is the logarithm base e.

$$\ln(x)=\log_e(x), \qquad x>0$$

The natural logarithm of x is the exponent to which e must be raised to produce x. For example, the natural logarithm of 10, or $\ln(10)$, is the solution of the equation

$$e^y=10$$

You can verify on your calculator that

$$e^{\mathbf{2.3}}\approx 10 \text{ or } \ln(10)\approx \mathbf{2.3}$$

In general, natural logs obey the same conversion formulas that work for logs to other bases.

Conversion Formulas for Natural Logs.

$$\boldsymbol{y = \ln(x)} \quad \text{if and only if} \quad \boldsymbol{e^y = x}$$

In particular,

$$\ln(e) = 1 \text{ because } e^1 = e$$
$$\ln(1) = 0 \text{ because } e^0 = 1$$

The conversion formulas tell us that the **natural log function**, $g(x) = \ln(x)$, is the inverse function for the natural exponential function, $f(x) = e^x$.

Example 5.3.3

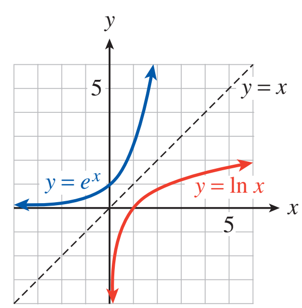
a Graph $f(x) = e^x$ and $f^{-1}(x) = \ln(x)$ on the same grid.

b Give the domain and range of the natural log function.

Solution.

a We can make a table of values for $f^{-1}(x) = \ln(x)$ by interchanging the columns in the table for $f(x) = e^x$. Plotting the points gives us the graph below.

x	$y = \ln(x)$
0.050	-3
0.135	-2
0.368	-1
1	0
2.718	1
7.389	2
20.086	3



b The domain of the natural log function is the same as the range of $y = e^x$, or all positive numbers. The range of $y = \ln(x)$ is the same as the domain of $y = e^x$, or all real numbers. These results are confirmed by the graph of $y = \ln(x)$.

□

Caution 5.3.4 Observe that the natural log of a number greater than 1 is positive, while the logs of fractions between 0 and 1 are negative. In addition, *the natural logs of negative numbers and zero are undefined.*

Checkpoint 5.3.5 Use your calculator to evaluate each logarithm. Round your answers to four decimal places.

a $\ln(100)$ b $\ln(0.01)$ c $\ln(e^3)$

Answer.

a $\ln(100) \approx 4.6052$

b $\ln(0.01) \approx -4.6052$

c $\ln(e^3) = 3$

5.3.3 Properties of the Natural Logarithm

We use natural logarithms in the same way that we use logs to other bases. The properties of logarithms that we studied in Section 4.4, p. 455 also apply to logarithms base e.

Properties of Natural Logarithms.

If $x, y > 0$, then

1 $\ln(xy) = \ln(x) + \ln(y)$

2 $\ln\left(\dfrac{x}{y}\right) = \ln(x) - \ln(y)$

3 $\ln(x^k) = k\ln(x)$

Because the functions $y = e^x$ and $y = \ln(x)$ are inverse functions, the following properties are also true.

The Natural log and e^x.

$$\ln(e^x) = x, \quad \text{for all } x, \qquad \text{and} \qquad e^{\ln(x)} = x, \quad \text{for } x > 0$$

Example 5.3.6 Simplify each expression.

a $\ln(e^{0.3x})$

b $e^{2\ln(x+3)}$

Solution.

a The natural log is the log base e, and hence the inverse of e^x. Therefore,

$$\ln(e^{0.3x}) = 0.3x$$

b First, we simplify the exponent using the third property of logs to get

$$2\ln(x+3) = \ln(x+3)^2$$

Then $e^{2\ln(x+3)} = e^{\ln\left((x+3)^2\right)} = (x+3)^2$.

□

Checkpoint 5.3.7 Simplify each expression.

a $e^{(\ln(x))/2}$

b $\ln\left(\dfrac{1}{e^{4x}}\right)$

Answer.

a $\sqrt{x}$

b $-4x$

5.3.4 Solving Equations

We use the natural logarithm to solve exponential equations with base e. The techniques we've learned for solving other exponential equations also apply to equations with base e.

Example 5.3.8 Solve each equation for x.

a $e^x = 0.24$

b $\ln(x) = 3.5$

Solution.

a We convert the equation to logarithmic form and evaluate using a calculator.

$$x = \ln(0.24) \approx -1.427$$

b We convert the equation to exponential form and evaluate.

$$x = e^{3.5} \approx 33.1155$$

□

Checkpoint 5.3.9 Solve each equation. Round your answers to four decimal places.

a $\ln(x) = -0.2$ b $e^x = 8$

Answer.

a 0.8187 b 2.0794

To solve more complicated exponential equations, we isolate the power on one side of the equation before converting to logarithmic form.

Example 5.3.10 Solve $140 = 20e^{0.4x}$.

Solution. First, we divide each side by 20 to obtain

$$7 = e^{0.4x}$$

Then we convert the equation to logarithmic form.

$$\begin{aligned} 0.4x &= \ln(7) && \text{Divide both sides by 0.4.} \\ x &= \frac{\ln(7)}{0.4} \end{aligned}$$

Rounded to four decimal places, $x \approx 4.8648$. □

Note 5.3.11 We can also solve the equation in Example 5.3.10, p. 556,

$$7 = e^{0.4x}$$

by taking the natural logarithm of both sides. This gives us

$$\begin{aligned} \ln(7) &= \ln\left(e^{0.4x}\right) && \text{Simplify the right side.} \\ \ln(7) &= 0.4x \end{aligned}$$

because $\ln(e^a) = a$ for any number a. We then proceed with the solution as before.

Checkpoint 5.3.12 Solve

$$80 - 16e^{-0.2x} = 70.3$$

Hint. Subtract from both sides and divide by **− 16.**
Take the natural log of both sides.
Divide by **− 0.2.**

Answer. $x = -5\ln\left(\dfrac{9.6}{16}\right) \approx 2.5023$

Example 5.3.13 Solve $P = \dfrac{a}{1+be^{-kt}}$ for t.

Solution. We multiply both sides of the equation by the denominator, $1 + be^{-kt}$, to get

$$P(1+be^{-kt}) = a$$

Then we isolate the power, e^{-kt}, as follows:

$$\begin{aligned}
1+be^{-kt} &= \frac{a}{P} && \text{Subtract 1 from both sides.}\\
be^{-kt} &= \frac{a}{P} - 1 && \text{Rewrite the right side.}\\
be^{-kt} &= \frac{a-P}{P} && \text{Divide both sides by } \boldsymbol{b}\textbf{.}\\
e^{-kt} &= \frac{a-P}{bP}
\end{aligned}$$

Next, we take the natural logarithm of both sides to get

$$\ln(e^{-kt}) = \ln\left(\frac{a-P}{bP}\right)$$

and recall that $\ln(e^x) = x$ to simplify the left side.

$$-kt = \ln\left(\frac{a-P}{bP}\right)$$

Finally, we divide both sides by $-k$ to solve for t.

$$t = \frac{-1}{k}\ln\left(\frac{a-P}{bP}\right)$$

□

Checkpoint 5.3.14 Solve $N = Ae^{-kt}$ for k.

Hint. Divide both sides by $\boldsymbol{A}$.
Take the natural log of both sides.
Divide both sides by $\boldsymbol{-t}$.

Answer. $k = \dfrac{-\ln(N/A)}{t}$

5.3.5 Exponential Growth and Decay

In Section 4.1, p. 394, we considered functions of the form

$$P(t) = P_0 \cdot b^t$$

which describe exponential growth when $b > 1$ and exponential decay when $0 < b < 1$. Exponential growth and decay can also be modeled by functions of the form

$$\boldsymbol{P(t) = P_0 \cdot e^{kt}}$$

where we have substituted e^k for the growth factor b, so that

$$\begin{aligned}
P(t) &= P_0 \cdot b^t\\
&= P_0 \cdot \left(e^k\right)^t = P_0 \cdot e^{kt}
\end{aligned}$$

We can find the value of k by solving the equation $b = e^k$ for k, to get $k = \ln(b)$.

For instance, in Example 4.1.1, p. 395 in Section 4.1, p. 394 we found that a colony of bacteria grew according to the formula

$$P(t) = 100 \cdot \mathbf{3^t}$$

We can express this function in the form $P(t) = 100 \cdot \boldsymbol{e^{kt}}$ if we set

$$3 = e^k \quad \text{or} \quad k = \ln(3) \approx 1.0986$$

Thus, the growth law for the colony of bacteria can be written

$$P(t) \approx 100 \cdot e^{1.0986t}$$

By graphing both functions on your calculator, you can verify that

$$P(t) = 100 \cdot 3t \quad \text{and} \quad P(t) = 100 \cdot e^{1.0986t}$$

are just two ways of writing the same function.

Example 5.3.15 From 1990 to 2000, the population of Clark County, Nevada, grew by 6.4% per year.

a What was the growth factor for the population of Clark County from 1990 to 2000? If the population of Clark County was 768, 000 in 1990, write a formula for the population t years later.

b Write a growth formula for Clark County using base e.

Solution.

a The growth factor was $b = 1 + r = 1.064$. The population t years later was

$$P(t) = 768,000(1.064)^t$$

b We use the formula $P(t) = P_0 \cdot e^{kt}$, where $e^k = 1.064$. Solving for k, we find

$$k = \ln(1.064) = 0.062$$

so $P(t) = 768,000e^{0.062t}$.

□

Checkpoint 5.3.16 From 1994 to 1998, the number of personal computers connected to the Internet grew according to the formula $N(t) = 2.8e^{0.85t}$, where $t = 0$ in 1994 and N is in millions. (Source: Los Angeles Times, September 6, 1999)

a Evaluate $N(1)$. By what percent did the number of Internet users grow in one year?

b Express the growth law in the form $N(t) = N_0(1 + r)^t$.

Hint. $e^k = 1 + r$

Answer.

a $N(1) \approx 6.55,\ \ 134\%$ b $N(t) \approx 2.8(1.3396)^t$

If k is negative, then e^k is a fraction less than 1. For example, if $k = -2$,

$$e^{-2} = \frac{1}{e^2} \approx \frac{1}{7.3891} \approx 0.1353$$

Thus, for negative values of k, the function $P(t) = P_0e^{kt}$ describes exponential decay.

Exponential Growth and Decay.

The function

$$P(t) = P_0 e^{kt}$$

describes exponential growth if $k > 0$, and exponential decay if $k < 0$.

Example 5.3.17 Express the decay law $N(t) = 60(0.8)^t$ in the form $N(t) = N_0 e^{kt}$.

Solution. For this decay law, $N_0 = 60$ and $b = 0.8$. We would like to find a value for k so that $e^k = b = 0.8$, that is, we must solve the equation

$$\begin{aligned} e^k &= 0.8 && \text{Take natural log of both sides.} \\ \ln(e^k) &= \ln(0.8) && \text{Simplify.} \\ k &= \ln(0.8) \approx -0.2231 \end{aligned}$$

Replacing b with e^k, we find that the decay law is

$$N(t) \approx 60e^{-0.2231t}$$

□

Checkpoint 5.3.18 A scientist isolates 25 grams of krypton-91, which decays according to the formula

$$N(t) = 25e^{-0.07t},$$

where t is in seconds.

a Complete the table of values showing the amount of krypton-91 left at 10-second intervals over the first minute.

t	0	10	20	30	40	50	60
$N(t)$							

b Use the table to choose a suitable window and graph the function $N(t)$.

c Write and solve an equation to answer the question: How long does it take for 60% of the krypton-91 to decay?

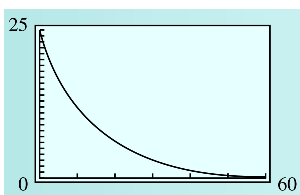
Hint. If 60% of the krypton-91 has decayed, 40% of the original 25 grams remains.

Answer.

a

t	0	10	20	30	40	50	60
$N(t)$	25	12.41	6.16	3.06	1.52	0.75	0.37

b



c $25e^{-0.07t} = 0.40(25);\quad t = \dfrac{\ln(0.4)}{-0.07} \approx 13.09$ seconds

5.3.6 Continuous Compounding

Some savings institutions offer accounts on which the interest is **compounded continuously**. The amount accumulated in such an account after t years at interest rate r is given by the function

$$A(t) = Pe^{rt}$$

where P is the principal invested.

Example 5.3.19 Suppose you invest \$500 in an account that pays 8% interest compounded continuously. You leave the money in the account without making any additional deposits or withdrawals.

a Write a formula that gives the value of your account $A(t)$ after t years.

b Make a table of values showing $A(t)$ for the first 5 years.

c Graph the function $A(t)$.

d How much will the account be worth after 10 years?

e How long will it be before the account is worth \$1000?

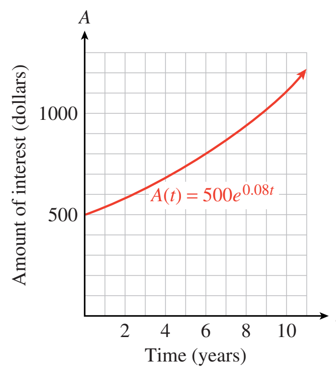
Solution.

a We substitute 500 for P, and 0.08 for r to find

$$A(t) = 500e^{0.08t}$$

b We evaluate the formula for $A(t)$ to obtain a table.

t	$A(t)$
0	500
1	541.64
2	586.76
3	635.62
4	688.56
5	745.91



c The graph of $A(t)$ is shown above.

d We evaluate $A(t)$ for $t = 10$.

$$\begin{aligned} A(10) &= 500e^{0.08(10)} \\ &= 500e^{0.8} \\ &\approx 500(2.2255) = 1112.77 \end{aligned}$$

The account will be worth \$1112.77 after 10 years.

e We substitute 1000 for $A(t)$ and solve the equation.

$$\begin{aligned} 1000 &= 500e^{0.08t} && \text{Divide both sides by 500.} \\ 2 &= e^{0.08t} && \text{Take natural log of both sides.} \\ \ln(2) &= \ln(e^{0.08t}) = 0.08t && \text{Divide both sides by 0.08.} \\ t &= \frac{\ln(2)}{0.08} \approx 8.6643 \end{aligned}$$

The account will be worth \$1000 after approximately 8.7 years.

□

Checkpoint 5.3.20 Zelda invested \$1000 in an account that pays 4.5% interest compounded continuously. How long will it be before the account is worth \$2000?

Answer. About 15.4 years

5.3.7 Section Summary

5.3.7.1 Vocabulary

Look up the definitions of new terms in the Glossary.

- Natural exponential function
- Natural logarithm
- Continuous compounding

5.3.7.2 CONCEPTS

1 The **natural base** is an irrational number called e, where

$$e \approx 2.71828182845$$

2 The **natural exponential function** is the function $f(x) = e^x$. The **natural log function** is the function $g(x) = \ln(x) = \log_e(x)$.

3 Conversion Formulas for Natural Logs.

$$\boldsymbol{y = \ln(x)} \quad \text{if and only if} \quad \boldsymbol{e^y = x}$$

4 Properties of Natural Logarithms.

If $x, y > 0$, then

1 $\ln(xy) = \ln(x) + \ln(y)$

2 $\ln\left(\dfrac{x}{y}\right) = \ln(x) - \ln(y)$

3 $\ln(x^k) = k\ln(x)$

5 We use the natural logarithm to solve exponential equations with base e.

6 Exponential Growth and Decay.

The function

$$P(t) = P_0e^{kt}$$

describes exponential growth if $k > 0$, and exponential decay if $k < 0$.

7 **Continuous compounding**: The amount accumulated in an account after t years at interest rate r compounded continuously is given by

$$A(t) = Pe^{rt}$$

where P is the principal invested.

5.3.7.3 STUDY QUESTIONS

1 State the value of e to 3 decimal places. Memorize this value.

2 Explain why $\ln(e^x) = x$.

3 State the formula for exponential growth using base e.

4 How is the formula for exponential decay in base e different from the formula for exponential growth?

5.3.7.4 SKILLS

Practice each skill in the Homework 5.3.8, p. 562 problems listed.

1 Graph exponential functions base e: #1–4

2 Simplify expressions: #5 and 6

3 Solve exponential and log equations base e: #7–10, 23–30

4 Use the properties of logs and exponents with the natural base: #19–22, 37–40

5 Use the natural exponential function in applications: #11–14, 47–58

6 Convert between $P(t) = P_0(1+r)^t$ and $P(t) = P_0e^{kt}$: #15–18, 41–46

5.3.8 The Natural Logarithm (Homework 5.3)

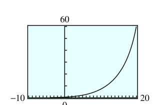
For Problems 1-4, use your calculator to complete the table for each function. Then choose a suitable window and graph the function.

x	-10	-5	0	5	10	15	20
$f(x)$							

1. $f(x) = e^{0.2x}$

Answer.

x	-10	-5	0	5	10	15	20
$f(x)$	0.135	0.368	1	2.718	7.389	20.086	54.598

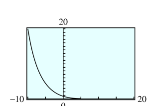


2. $f(x) = e^{0.6x}$

3. $f(x) = e^{-0.3x}$

Answer.

x	-10	-5	0	5	10	15	20
$f(x)$	20.086	4.482	1	0.223	0.05	0.011	0.00248



4. $f(x) = e^{-0.1x}$

For Problems 5-6, simplify.

5.

(a) $\ln(e^2)$ (b) $e^{\ln(5t)}$ (c) $e^{-\ln(x)}$ (d) $\ln(\sqrt{e})$

Answer.

(a) 2 (b) $5t$ (c) $\dfrac{1}{x}$ (d) $\dfrac{1}{2}$

6.

(a) $\ln(e^{x^4})$ (b) $e^{3\ln(x)}$ (c) $e^{\ln(x)-\ln(y)}$ (d) $\ln\left(\dfrac{1}{e^{2t}}\right)$

For Problems 7-10, solve for x. Give the exact solution and the solution rounded to the nearest 2 decimal places.

7.

(a) $e^x = 1.9$ (b) $e^x = 45$ (c) $e^x = 0.3$

Answer.

(a) 0.64 (b) 3.81 (c) -1.20

8.

(a) $e^x = 2.1$ (b) $e^x = -60$ (c) $e^x = 0.9$

9.

(a) $\ln(x) = 1.42$ (b) $\ln(x) = 0.63$ (c) $\ln(x) = -2.6$

Answer.

(a) 4.14 (b) 1.88 (c) 0.07

10.

(a) $\ln(x) = 2.03$ (b) $\ln(x) = 0.59$ (c) $\ln(x) = -3.4$

11. The number of bacteria in a culture grows according to the function

$$N(t) = N_0 e^{0.04t}$$

where N_0 is the number of bacteria present at time $t = 0$ and t is the time in hours.

(a) Write a growth law for a sample in which 6000 bacteria were present initially.

(b) Make a table of values for $N(t)$ in 5-hour intervals over the first 30 hours. Round to one decimal place.

(c) Graph $N(t)$.

(d) How many bacteria were present at $t = 24$ hours?

(e) How much time must elapse (to the nearest tenth of an hour) for the original 6000 bacteria to increase to $100,000$?

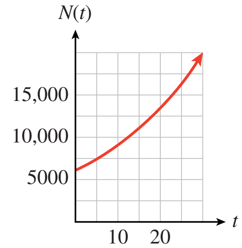
Answer.

(a) $N(t) = 6000e^{0.04t}$

(b)

t	0	5	10	15	20	25	30
$N(t)$	6000	7328	8951	$10,933$	$13,353$	$16,310$	$19,921$

(c)



(d) $15,670$

(e) 70.3 hrs

12. Hope invests \$2000 in a savings account that pays $5\frac{1}{2}\%$ annual interest compounded continuously.

(a) Write a formula that gives the amount of money $A(t)$ in Hope's account after t years.

(b) Make a table of values for $A(t)$ in 2-year intervals over the first 10 years.

(c) Graph $A(t)$.

(d) How much will Hope's account be worth after 7 years?

(e) How long will it take for the account to grow to \$5000?

13. The intensity, I (in lumens), of a light beam after passing through t centimeters of a filter having an absorption coefficient of 0.1 is given by the function

$$I(t) = 1000e^{-0.1t}$$

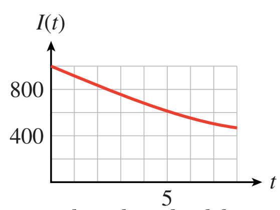
(a) Graph $I(t)$.

(b) What is the intensity (to the nearest tenth of a lumen) of a light beam that has passed through 0.6 centimeter of the filter?

(c) How many centimeters (to the nearest tenth) of the filter will reduce the illumination to 800 lumens?

Answer.

(a)



(b) 941.8 lumens

(c) 2.2 cm

14. X-rays can be absorbed by a lead plate so that

$$I(t) = I_0e^{-1.88t}$$

where I_0 is the X-ray count at the source and $I(t)$ is the X-ray count behind a lead plate of thickness t inches.

(a) Graph $I(t)$.

(b) What percent of an X-ray beam will penetrate a lead plate $\frac{1}{2}$ inch thick?

(c) How thick should the lead plate be in order to screen out 70% of the X-rays?

For problems 15-18, express each exponential function in the form $P(t) = P_0 b^t$. Is the function increasing or decreasing? What is itsinitial value?

15. $P(t) = 20e^{0.4t}$

Answer.
$P(t) = 20\left(e^{0.4}\right)^t \approx 20 \cdot 1.492^t$;
increasing; initial value 20

16. $P(t) = 0.8e^{1.3t}$

17. $P(t) = 6500e^{-2.5t}$

Answer. $P(t) =$
$6500\left(e^{-2.5}\right)^t \approx 6500 \cdot 0.082^t$;
decreasing; initial value 6500

18. $P(t) = 1.7e^{-0.02t}$

19.

(a) Fill in the table, rounding your answers to four decimal places.

x	0	0.5	1	1.5	2	2.5
e^x						

(b) Compute the ratio of each function value to the previous one. Explain the result.

Answer.

(a)

x	0	0.5	1	1.5	2	2.5
e^x	1	1.6487	2.7183	4.4817	7.3891	12.1825

(b) Each ratio is $e^{0.5} \approx 1.6487$: Increasing x-values by a constant $\Delta x = 0.5$ corresponds to multiplying the y-values of the exponential function by a constant factor of $e^{\Delta x}$.

20.

(a) Fill in the table, rounding your answers to four decimal places.

x	0	2	4	6	8	10
e^x						

(b) Compute the ratio of each function value to the previous one. What do you notice about the ratios?

21.

(a) Fill in the table, rounding your answers to the nearest integer.

x	0	0.6931	1.3863	2.0794	2.7726	3.4657	4.1589
e^x							

(b) Subtract each x-value from the next one. Explain the result.

Answer.

(a)

x	0	0.6931	1.3863	2.0794	2.7726	3.4657	4.1589
e^x	1	2	4	8	16	32	64

(b) Each difference in x-values is approximately $\ln(2) \approx 0.6931$: Increasing x-values by a constant $\Delta x = \ln(2)$ corresponds to multiplying the y-values of the exponential function by a constant factor of $e^{\Delta x} = e^{\ln(2)} = 2$. That is, each function value is approximately equal to double the previous one.

22.

(a) Fill in the table, rounding your answers to the nearest integer.

x	0	1.0986	2.1972	3.2958	4.3944	5.4931	6.5917
e^x							

(b) Subtract each x-value from the next one. Explain the result.

For Problems 23–30, solve. Give the exact solution and the solution rounded to the nearest 2 decimal places.

23. $6.21 = 2.3e^{1.2x}$

Answer. 0.8277

24. $22.26 = 5.3e^{0.4x}$

25. $6.4 = 20e^{0.3x} - 1.8$

Answer. -2.9720

26. $4.5 = 4e^{2.1x} + 3.3$

27. $46.52 = 3.1e^{1.2x} + 24.2$

Answer. 1.6451

28. $1.23 = 1.3e^{2.1x} - 17.1$

29. $16.24 = 0.7e^{-1.3x} - 21.7$

Answer. -3.0713

30. $55.68 = 0.6e^{-0.7x} + 23.1$

For Problems 31-36, solve the equation for the specified variable.

31. $y = e^{kt}$, for t

Answer. $t = \dfrac{1}{k}\ln(y)$

32. $\dfrac{T}{R} = e^{t/2}$, for t

33. $y = k(1 - e^{-t})$, for t

Answer. $t = \ln\left(\dfrac{k}{k-y}\right)$

34. $B - 2 = (A + 3)e^{-t/3}$, for t

35. $T = T_0 \ln(k + 10)$, for k

Answer. $k = e^{T/T_0} - 10$

36. $P = P_0 + \ln(10k)$, for k

37.

(a) Fill in the table, rounding your answers to three decimal places.

n	0.39	3.9	39	390
$\ln(n)$				

(b) Subtract each natural logarithm in your table from the next one. (For example, compute $\ln(3.9) - \ln(0.39)$.) Explain the result.

Answer.

(a)

n	0.39	3.9	39	390
$\ln(n)$	-0.942	1.361	3.664	5.966

(b) Each difference in function values is approximately $\ln(10) \approx 2.303$: Multiplying x-values by a constant factor of 10 corresponds to adding a constant value of ln 10 to the y-values of the natural log function.

38.

(a) Fill in the table, rounding your answers to three decimal places.

n	0.64	6.4	64	640
$\ln(n)$				

(b) Subtract each natural logarithm in your table from the next one. (For example, compute $\ln(6.4) - \ln(0.64)$.) Explain the result.

39.

(a) Fill in the table, rounding your answers to three decimal places.

n	2	4	8	16
$\ln(n)$				

(b) Divide each natural logarithm in your table by $\ln(2)$. Explain the result.

Answer.

(a)

n	2	4	8	16
$\ln(n)$	0.693	1.386	2.079	2.773

(b) Each quotient equals k, where $n = 2^k$. Because $\ln(n) = \ln(2^k) = k \cdot \ln(2)$, $k = \dfrac{\ln(n)}{\ln(2)}$.

40.

(a) Fill in the table, rounding your answers to three decimal places.

n	5	25	125	625
$\ln(n)$				

(b) Divide each natural logarithm in your table by $\ln(5)$. Explain the result.

For Problems 41–46,

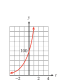
a Express each growth or decay law in the form $N(t) = N_0e^{kt}$.

b Check your answer by graphing both forms of the function on the same axes. Do they have the same graph?

41. $N(t) = 100 \cdot 2^t$

Answer.

(a) $N(t) = 100e^{(\ln(2))t} \approx 100e^{0.6931t}$

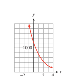
(b) 

42. $N(t) = 50 \cdot 3^t$

43. $N(t) = 1200(0.6)^t$

Answer.

(a) $N(t) = 1200e^{(\ln(0.6))t} \approx 1200e^{-0.5108t}$

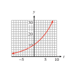
(b) 

44. $N(t) = 300(0.8)^t$

45. $N(t) = 10(1.15)^t$

Answer.

(a) $N(t) = 10e^{(\ln(1.15))t} \approx 10e^{0.1398t}$

(b) 

46. $N(t) = 1000(1.04)^t$

47. The population of Citrus Valley was 20,000 in 2000. In 2010, it was 35,000.

(a) What is P_0 if $t = 0$ in 2000?

(b) Use the population in 2010 to find the growth factor e^k.

(c) Write a growth law of the form $P(t) = P_0e^{kt}$ for the population of Citrus Valley.

(d) If it continues at the same rate of growth, what will the population be in 2030?

Answer.

(a) $20,000$

(b) $\left(\dfrac{35,000}{20,000}\right)^{1/10} \approx e^{0.056}$

(c) $P(t) = 20,000e^{0.056t}$

(d) $107,188$

48. A copy of *Time* magazine cost \$1.50 in 1981. In 1988, the cover price had increased to \$2.00.

(a) What is P_0 if $t = 0$ in 1981?

(b) Use the price in 1988 to find the growth factor e^k.

(c) Find a growth law of the form $P(t) = P_0e^{kt}$ for the price of *Time*.

(d) In 1999, a copy of *Time* cost \$3.50. Did the price of the magazine continue to grow at the same rate from 1981 to 1999?

49. Cobalt-60 is a radioactive isotope used in the treatment of cancer. A 500-milligram sample of cobalt-60 decays to 385 milligrams after 2 years.

(a) Using $P_0 = 500$, find the decay factor e^k for cobalt-60.

(b) Write a decay law $N(t) = N_0e^{kt}$ for cobalt-60.

(c) How much of the original sample will be left after 10 years?

Answer.

(a) $\left(\dfrac{385}{500}\right)^{1/2} \approx e^{-0.1307}$

(b) $N(t) = 500e^{-0.1307t}$

(c) 135.3 mg

50. Weed seeds can survive for a number of years in the soil. An experiment on cultivated land found 155 million weed seeds per acre, and in the following years the experimenters prevented the seeds from coming to maturity and producing new weeds. Four years later, there were 13.6 million seeds per acre. (Source: Burton, 1998)

(a) Find the annual decay factor e^k for the number of weed seeds in the soil.

(b) Write an exponential formula with base e for the number of weed seeds that survived after t years.

Problems 51–58 are about doubling time and half-life.

51. Delbert invests \$500 in an account that pays 9.5% interest compounded continuously.

(a) Write a formula for $A(t)$ that gives the amount of money in Delbert's account after t years.

(b) How long will it take Delbert's investment to double to \$1000?

(c) How long will it take Delbert's money to double again, to \$2000?

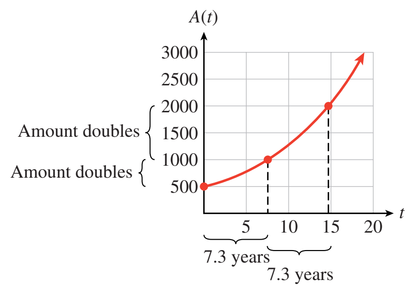
(d) Graph $A(t)$ and illustrate the doubling time on your graph.

(e) Choose any point (t_1, A_1) on the graph, then find the point on the graph with vertical coordinate $2A_1$. Verify that the difference in the t-coordinates of the two points is the doubling time.

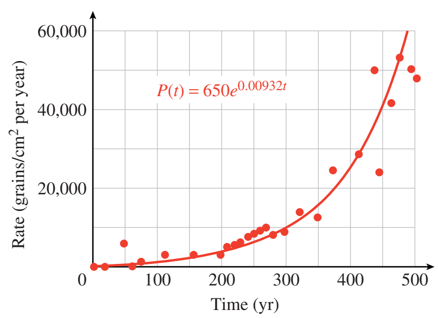
Answer.

(a) $A(t) = 500e^{0.095t}$ (b) 7.3 years (c) 7.3 years

d–e



52. The growth of plant populations can be measured by the amount of pollen they produce. The pollen from a population of pine trees that lived more than 9500 years ago in Norfolk, England, was deposited in the layers of sediment in a lake basin and dated with radiocarbon techniques.



The figure shows the rate of pollen accumulation plotted against time, and the fitted curve $P(t) = 650e^{0.00932t}$. (Source: Burton, 1998)

(a) What was the annual rate of growth in pollen accumulation?

(b) Find the doubling time for the pollen accumulation, that is, the time it took for the accumulation rate to double.

(c) By what factor did the pollen accumulation rate increase over a period of 500 years?

53. Technetium-99m (Tc-99m) is an artificially produced radionuclide used as a tracer for producing images of internal organs such as the heart, liver, and thyroid. A solution of Tc-99m with initial radioactivity of $10,000$ becquerels (Bq) decays according to the formula

$$N(t) = 10,000e^{-0.1155t}$$

where t is in hours.

(a) How long will it take the radioactivity to fall to half its initial value, or 5000 Bq?

(b) How long will it take the radioactivity to be halved again?

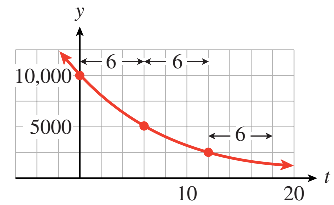
(c) Graph $N(t)$ and illustrate the half-life on your graph.

(d) Choose any point (t_1, N_1) on the graph, then find the point on the graph with vertical coordinate $0.5N_1$. Verify that the difference in the t-coordinates of the two points is the half-life.

Answer.

(a) 6 hours

(b) 6 hours

(c) 

54. All living things contain a certain amount of the isotope carbon-14. When an organism dies, the carbon-14 decays according to the formula

$$N(t) = N_0 e^{-0.000124t}$$

where t is measured in years. Scientists can estimate the age of an organic object by measuring the amount of carbon-14 remaining.

(a) When the Dead Sea scrolls were discovered in 1947, they had 78.8% of their original carbon-14. How old were the Dead Sea scrolls then?

(b) What is the half-life of carbon-14, that is, how long does it take for half of an object's carbon-14 to decay?

55. The half-life of iodine-131 is approximately 8 days.

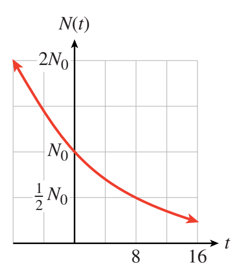
(a) If a sample initially contains N_0 grams of iodine-131, how much will it contain after 8 days? How much will it contain after 16 days? After 32 days?

(b) Use your answers to part (a) to sketch a graph of $N(t)$, the amount of iodine-131 remaining, versus time. (Choose an arbitrary height for N_0 on the vertical axis.)

(c) Calculate k, and hence find a decay law of the form $N(t) = N_0 e^{kt}$, where $k < 0$, for iodine-131.

Answer.

(a) $\frac{1}{2}N_0$, $\frac{1}{4}N_0$, $\frac{1}{16}N_0$

(b) 

(c) $N(t) = N_0 e^{-0.0866t}$

56. The half-life of hydrogen-3 is 12.5 years.

(a) If a sample initially contains N_0 grams of hydrogen-3, how much will it contain after 12.5 years? How much will it contain after 25 years?

(b) Use your answers to part (a) to sketch a graph of $N(t)$, the amount of hydrogen-3 remaining, versus time. (Choose an arbitrary height for N_0 on the vertical axis.)

(c) Calculate k, and hence find a decay law of the form $N(t) = N_0 e^{kt}$, where $k < 0$, for hydrogen-3.

57. A Geiger counter measures the amount of radioactive material present in a substance. The table shows the count rate for a sample of iodine-128 as a function of time. (Source: Hunt and Sykes, 1984)

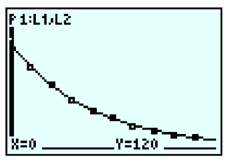
Time (min)	0	10	20	30	40	50	60	70	80	90
Counts/sec	120	90	69	54	42	33	25	19	15	13

(a) Graph the data and use your calculator's exponential regression feature to fit a curve to them.

(b) Write your equation in the form $G(t) = G_0 e^{kt}$.

(c) Calculate the half-life of iodine-128.

Answer.

(a) 

$y = 116(0.975)^t$

(b) $G(t) = 116e^{-0.025t}$

(c) 28 minutes

58. The table shows the count rate for sodium-24 registered by a Geiger counter as a function of time. (Source: Hunt and Sykes, 1984)

Time (min)	0	10	20	30	40	50	60	70	80	90
Counts/sec	180	112	71	45	28	18	11	7	4	3

(a) Graph the data and use your calculator's exponential regression feature to fit a curve to them.

(b) Write your equation in the form $G(t) = G_0 e^{kt}$.

(c) Calculate the half-life of sodium-24.

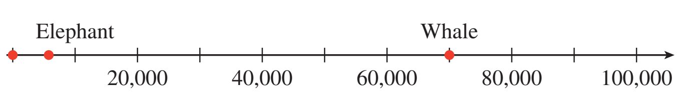
5.4 Logarithmic Scales

5.4.1 Introduction

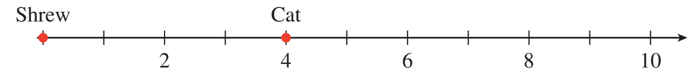
Because logarithmic functions grow very slowly, they are useful for modeling phenomena that take on a very wide range of values. For example, biologists study how metabolic functions such as heart rate are related to an animal's weight, or mass. The table shows the mass in kilograms of several mammals.

Animal	Shrew	Cat	Wolf	Horse	Elephant	Whale
Mass, kg	0.004	4	80	300	5400	70,000

Imagine trying to scale the x-axis to show all of these values. If we set tick marks at intervals of 10,000 kg, as shown below, we can plot the mass of the whale, and maybe the elephant, but the dots for the smaller animals will be indistinguishable.



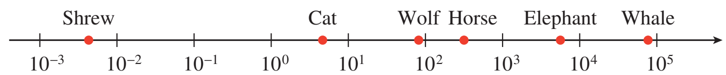
On the other hand, we can plot the mass of the cat if we set tick marks at intervals of 1 kg, but the axis will have to be extremely long to include even the wolf. We cannot show the masses of all these animals on the same scale



To get around this problem, we can plot the log of the mass, instead of the mass itself. The table below shows the base 10 log of each animal's mass, rounded to 2 decimal places.

Animal	Shrew	Cat	Wolf	Horse	Elephant	Whale
Mass, kg	0.004	4	80	300	5400	$70,000$
Log (mass)	-2.40	0.60	1.90	2.48	3.73	4.85

The logs of the masses range from -2.40 to 4.85. We can easily plot these values on a single scale, as shown below.



The scale above is called a **logarithmic scale**, or log scale. The tick marks are labeled with powers of 10, because, as you recall, a logarithm is actually an exponent. For example, the mass of the horse is 300 kg, and

$$\log(300) = 2.48 \quad \text{so} \quad 10^{2.48} = 300$$

When we plot 2.48 for the horse, we are really plotting the power of 10 that gives its mass, because $10^{2.48} = 300$ kg. The exponents on base 10 are evenly spaced on a log scale, so we plot $10^{2.48}$ about halfway between 10^2 and 10^3.

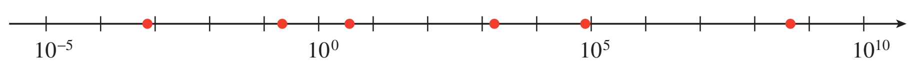
Example 5.4.1 Plot the values on a log scale.

x	0.0007	0.2	3.5	1600	$72,000$	4×10^8

Solution. We actually plot the logs of the values, so we first compute the base 10 logarithm of each number.

x	0.0007	0.2	3.5	1600	$72,000$	4×10^8
$\log(x)$	-3.15	-0.70	0.54	3.20	4.86	8.60

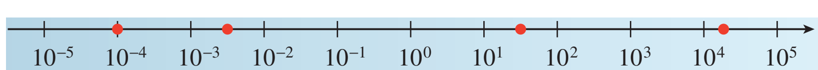
Then we plot each logarithm, estimating its position between integer exponents. For example, we plot the first value, -3.15, closer to -3 than to -4. The finished plot is shown below.



□

Checkpoint 5.4.2 Complete the table by estimating the logarithm of each point plotted on the log scale below. Then give a decimal value for each point.

$\log(x)$				
x				



Answer.

$\log(x)$	-4	-2.5	1.5	4.25
x	0.0001	0.0032	31.6	$17,782.8$

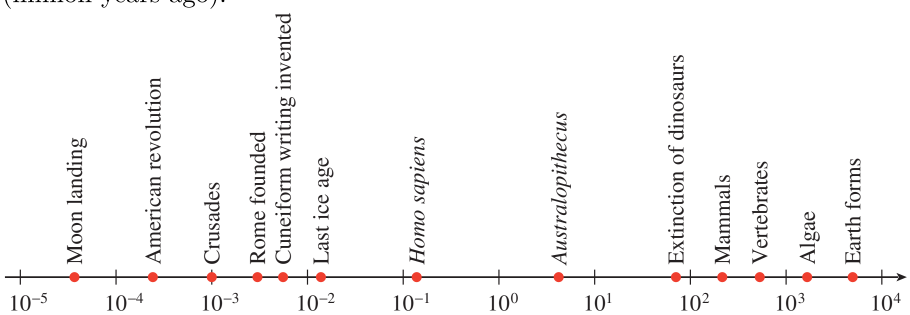
5.4.2 Using Log Scales

By now, you have noticed that the values represented by points on a log scale increase rapidly as we move to the right along the scale. Also notice that $10^0 = 1$, so the "middle" of a log scale represents 1 (not zero, as on a linear scale).

Points to the left of 10^0 represent fractions between 0 and 1, because powers of 10 with negative exponents are numbers less than 1. Their values decrease toward 0 as we move to the left, but they never become negative.

We cannot plot negative numbers or zero on a log scale, because the log of a negative number or zero is undefined.

Example 5.4.3 The figure shows a timeline for life on Earth, in units of Mya (million years ago).



Approximately how long ago did each of the following events occur?

a Formation of Earth

b Dinosaurs became extinct

c The last ice age

d The Crusades

Solution.

a We read from the timeline that the Earth was formed between 10^3 and 10^4, or between 1000 and $10,000$ million years ago. We estimate that Earth formed 5000 million years ago.

b The extinction of the dinosaurs is plotted between 10^1 and 10^2, or between 10 and 100 million years ago. Because the point is closer to 10^2, we estimate their extinction at 70 million years ago.

c The last ice age is plotted just after 10^{-2}, or 0.01 million years ago. One-hundredth of a million is $10,000$, so we estimate that the ice age occurred a little more than $10,000$ years ago.

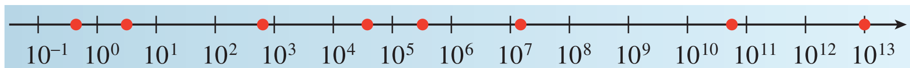
d The Crusades occurred about 10^{-3}, or about 0.001 million years ago. One-thousandth of a million is 1000, so the Crusades occurred about 1000 years ago, or about 1000 A.D.

□

Checkpoint 5.4.4 Plot the following dollar values on a log scale.

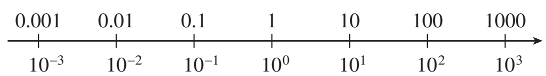
Postage stamp	0.47
Medium cappuccino	3.65
Notebook computer	679
One year at Harvard	88,600
2016 Lamborghini	530,075
Kobe Bryant's salary	25,000,000
Bill Gates's financial worth	79,400,000,000
U.S. National debt	19,341,810,000

Answer.

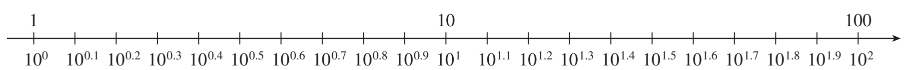


5.4.3 Equal Increments on a Log Scale

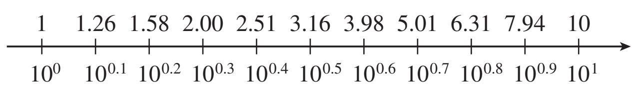
Log scales allow us to plot a wide range of values, but there is a trade-off. Equal increments on a log scale do not correspond to equal differences in value, as they do on a linear scale. You can see why in the figure below: The difference between 10^1 and 10^0 is $10 - 1 = 9$, but the difference between 10^2 and 10^1 is $100 - 10 = 90$.



If we include tick marks for intermediate values on the log scale, they look like this.



Once again, the difference between, say, $10^{0.1}$ and $10^{0.2}$ is not the same as the difference between $10^{0.2}$ and $10^{0.3}$. The decimal values of the powers $10^{0.1}$ through $10^{0.9}$, rounded to two places, are shown below.



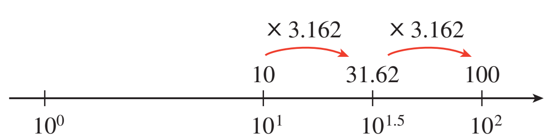
As we move from left to right on this scale, we *multiply* the value at the previous tick mark by $10^{0.1}$, or about 1.258. For example,

$$10^{0.2} = 1.258 \times 10^{0.1} = 1.585$$
$$10^{0.3} = 1.585 \times 10^{0.1} = 1.995$$

and so on. Moving up by equal increments on a log scale does not add equal amounts to the values plotted; it *multiplies* the values by equal *factors*.

Example 5.4.5 What number is halfway between 10 and 100 on a log scale?

Solution. On a log scale, the number $10^{1.5}$ is halfway between 10^1 and 10^2, as shown below.



Now, $10^{1.5} = 10\sqrt{10}$, or approximately 31.62. Note how equal increments of

0.5 on the log scale correspond to equal factors of $10^{0.5}$ in the values plotted:

$$10 \times 3.162 = 31.62 \qquad \text{and} \qquad 31.62 \times 3.162 = 100$$
$$10^1 \times 10^{0.5} = 101.5 \qquad \text{and} \qquad 101.5 \times 10^{0.5} = 10^2$$

□

Checkpoint 5.4.6 What number is halfway between $10^{1.5}$ and 10^2 on a log scale?

Answer. 56.23

If we would like to label the log scale with integers, we get a very different-looking scale, one in which the tick marks are not evenly spaced.

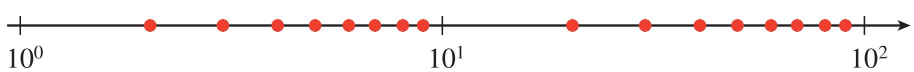
Example 5.4.7 Plot the integer values 2 through 9 and 20 through 90 on a log scale.

Solution. We compute the logarithm of each integer value.

x	2	3	4	5	6	7	8	9
$\log(x)$	0.301	0.477	0.602	0.699	0.778	0.845	0.903	0.954

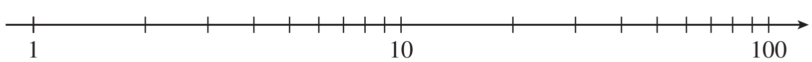
x	20	30	40	50	60	70	80	90
$\log(x)$	1.301	1.477	1.602	1.699	1.778	1.845	1.903	1.954

We plot on a log scale, as shown below.



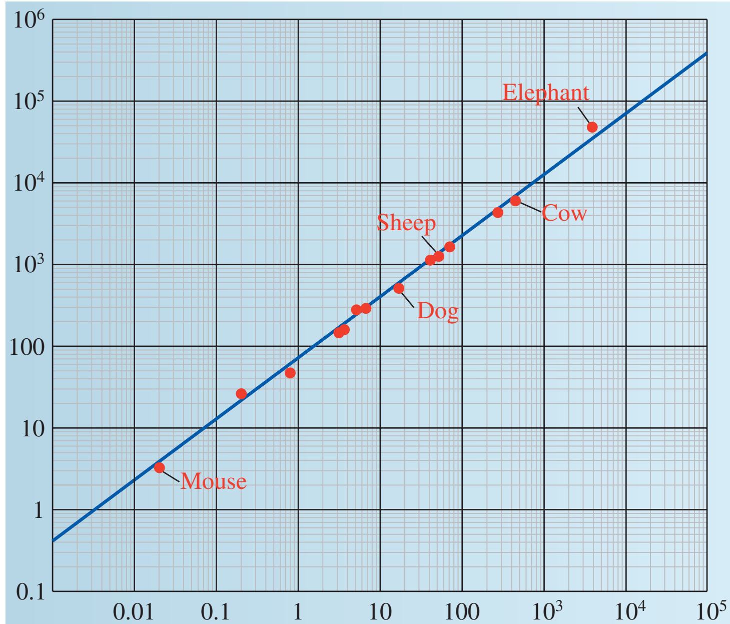
□

On the log scale in Example 5.4.7, p. 575, notice how the integer values are spaced: They get closer together as they approach the next power of 10. You will often see log scales labeled not with powers of 10, but with integer values, like this:



In fact, **log-log graph paper** scales both axes with logarithmic scales.

Checkpoint 5.4.8 The opening page of Chapter 3, p. 289 shows the "mouse-to-elephant" curve, a graph of the metabolic rate of mammals as a function of their mass. (The elephant does not appear on that graph, because its mass is too big.) The figure below shows the same function, graphed on log-log paper.



Use this graph to estimate the mass and metabolic rate for the following animals, labeled on the graph.

Animal	Mouse	Dog	Sheep	Cow	Elephant
Mass (kg)					
Metabolic rate (kcal/day)					

Answer.

Animal	Mouse	Dog	Sheep	Cow	Elephant
Mass (kg)	0.02	15	50	500	4000
Metabolic rate (kcal/day)	3.5	500	1500	6000	$50,000$

5.4.4 Acidity and the pH Scale

You may have already encountered log scales in some everyday applications. A simple example is the **pH scale**, used by chemists to measure the acidity of a substance or chemical compound. This scale is based on the concentration of hydrogen ions in the substance, denoted by $[H^+]$. The pH value is defined by the formula

$$\text{pH} = -\log\left([H^+]\right)$$

Values for pH fall between 0 and 14, with 7 indicating a neutral solution. The lower the pH value, the more acidic the substance. Some common substances and their pH values are shown in the table.

Substance	pH	$[H^+]$
Battery acid	1	0.1
Lemon juice	2	0.01
Vinegar	3	0.001
Milk	6.4	$10^{-6.4}$
Baking soda	8.4	$10^{-8.4}$
Milk of magnesia	10.5	$10^{-10.5}$
Lye	13	10^{-13}

Example 5.4.9

a Calculate the pH of a solution with a hydrogen ion concentration of 3.98×10^{-5}.

b The water in a swimming pool should be maintained at a pH of 7.5. What is the hydrogen ion concentration of the water?

Solution.

a We use a calculator to evaluate the pH formula with $[H^+] = 3.98 \times 10^{-5}$.

$$\text{pH} = -\log(3.98 \times 10^{-5}) \approx 4.4$$

b We solve the equation

$$7.5 = -\log\left([H^+]\right)$$

for $[H^+]$. First, we write

$$-7.5 = \log\left([H^+]\right)$$

Then we convert the equation to exponential form to get

$$[H^+] = 10^{-7.5} \approx 3.2 \times 10^{-8}$$

The hydrogen ion concentration of the water is 3.2×10^{-8}.

□

Checkpoint 5.4.10 The pH of the water in a tide pool is 8.3. What is the hydrogen ion concentration of the water?

Answer. 5.01×10^{-9}

A decrease of 1 on the pH scale corresponds to an increase in acidity by a factor of 10. Thus, lemon juice is 10 times more acidic than vinegar, and battery acid is 100 times more acidic than vinegar.

5.4.5 Decibels

The **decibel scale**, used to measure the loudness or intensity of a sound, is another example of a logarithmic scale. The loudness of a sound is measured in decibels, D, by

$$D = 10\log\left(\frac{I}{10^{-12}}\right)$$

where I is the intensity of its sound waves (in watts per square meter). The table below shows the intensity of some common sounds, measured in watts per square meter.

Sound	Intensity (watts/m^2)	Decibels
Whisper	10^{-10}	20
Background music	10^{-8}	40
Loud conversation	10^{-6}	60
Heavy traffic	10^{-4}	80
Jet airplane	10^{-2}	100
Thunder	10^{-1}	110

Consider the ratio of the intensity of thunder to that of a whisper:

$$\frac{\text{Intensity of thunder}}{\text{Intensity of a whisper}} = \frac{10^{-1}}{10^{-10}} = 10^9$$

Thunder is 10^9, or one billion times more intense than a whisper. It would be impossible to show such a wide range of values on a graph and still maintain reasonable precision. When we use a log scale, however, there is a difference of only 90 decibels between a whisper and thunder.

Example 5.4.11

a Normal breathing generates about 10^{-11} watts per square meter at a distance of 3 feet. Find the number of decibels for a breath 3 feet away.

b Normal conversation registers at about 40 decibels. How many times more intense than breathing is normal conversation?

Solution.

a We evaluate the decibel formula with $I = \mathbf{10^{-11}}$ to find

$$\begin{aligned} D &= 10\log\left(\frac{\mathbf{10^{-11}}}{10^{-12}}\right) = 10\log(10^1) \\ &= 10(1) = 10 \text{ decibels} \end{aligned}$$

b We let I_b stand for the sound intensity of breathing, and I_c stand for the intensity of normal conversation. We are looking for the ratio I_c/I_b. From part (a), we know that

$$I_w = 10^{-11}$$

and from the formula for decibels, we have

$$40 = 10\log\left(\frac{I_c}{10^{-12}}\right)$$

which we can solve for I_c. Dividing both sides of the equation by 10 and rewriting in exponential form, we have

$$\begin{aligned} \frac{I_c}{10^{-12}} &= 10^4 && \text{Multiply both sides by } \mathbf{10^{-12}}\textbf{.} \\ I_c &= 10^4(10^{-12}) = 10^{-8} \end{aligned}$$

Finally, we compute the ratio $\dfrac{I_c}{I_b}$:

$$\frac{I_c}{I_b} = \frac{10^{-8}}{10^{-11}} = 10^3$$

Normal conversation is 1000 times more intense than breathing.

□

Checkpoint 5.4.12 The noise of city traffic registers at about 70 decibels.

a What is the intensity of traffic noise, in watts per square meter?

b How many times more intense is traffic noise than conversation?

Answer.

a $I = 10^{-5}$ watts/m^2

b 1000

Caution 5.4.13 Both the decibel model and the Richter scale in the next example use expressions of the form $\log\left(\dfrac{a}{b}\right)$. Be careful to follow the order of operations when using these models. We must compute the quotient $\dfrac{a}{b}$ before taking a logarithm. In particular, it is *not* true that $\log\left(\dfrac{a}{b}\right)$ can be simplified to $\dfrac{\log(a)}{\log(b)}$.

5.4.6 The Richter Scale

One method for measuring the magnitude of an earthquake compares the amplitude A of its seismographic trace with the amplitude A_0 of the smallest detectable earthquake. The log of their ratio is the Richter magnitude, M. Thus,

$$M = \log\left(\frac{A}{A_0}\right)$$

Example 5.4.14

a The Northridge earthquake of January 1994 registered 6.9 on the Richter scale. What would be the magnitude of an earthquake 100 times as powerful as the Northridge quake?

b How many times more powerful than the Northridge quake was the San Francisco earthquake of 1989, which registered 7.1 on the Richter scale?

Solution.

a We let A_N represent the amplitude of the Northridge quake and A_H represent the amplitude of a quake 100 times more powerful. From the Richter model, we have

$$6.9 = \log\left(\frac{A_N}{A_0}\right)$$

or, rewriting in exponential form,

$$\frac{A_N}{A_0} = 10^{6.9}$$

Now, $A_H = 100A_N$, so

$$\begin{aligned}\frac{A_H}{A_0} &= \frac{100A_N}{A_0}\\ &= 100\left(\frac{A_N}{A_0}\right) = 10^2(10^{6.9})\\ &= 10^{8.9}\end{aligned}$$

Thus, the magnitude of the more powerful quake is

$$\begin{aligned}\log\left(\frac{A_H}{A_0}\right) &= \log(10^{8.9})\\ &= 8.9\end{aligned}$$

b We let A_S stand for the amplitude of the San Francisco earthquake. We are looking for the ratio A_S/A_N. First, we use the Richter formula to compute values for A_S and A_N.

$$6.9 = \log\left(\frac{A_N}{A_0}\right) \quad \text{and} \quad 7.1 = \log\left(\frac{A_s}{A_0}\right)$$

Rewriting each equation in exponential form, we have

$$\frac{A_N}{A_0} = 10^{6.9} \quad \text{and} \quad \frac{A_S}{A_0} = 10^{7.1}$$

or

$$A_N = 10^{6.9}A_0 \quad \text{and} \quad A_S = 10^{7.1}A_0$$

Now we can compute the ratio we want:

$$\frac{A_S}{A_N} = \frac{10^{7.1}A_0}{10^{6.9}A_0} = 10^{0.2}$$

The San Francisco earthquake was $10^{0.2}$, or approximately 1.58 times as powerful as the Northridge quake.

□

Checkpoint 5.4.15 In October 2005, a magnitude 7.6 earthquake struck Pakistan. How much more powerful was this earthquake than the 1989 San Francisco earthquake of magnitude 7.1?

Answer. 3.16

Note 5.4.16 An earthquake 100, or 10^2, times as strong is only two units greater in magnitude on the Richter scale. In general, a difference of K units on the Richter scale (or any logarithmic scale) corresponds to a factor of 10^K units in the intensity of the quake.

Example 5.4.17 On a log scale, the weights of two animals differ by 1.6 units. What is the ratio of their actual weights?

Solution. A difference of 1.6 on a log scale corresponds to a factor of $10^{1.6}$ in the actual weights. Thus, the heavier animal is $10^{1.6}$, or 39.8 times as heavy as the lighter animal. □

Checkpoint 5.4.18 Two points, labeled A and B, differ by 2.5 units on a log scale. What is the ratio of their decimal values?

Answer. 316.2

5.4.7 Section Summary

5.4.7.1 Vocabulary

Look up the definitions of new terms in the Glossary.

- Log scale
- Log-log paper

5.4.7.2 CONCEPTS

1 A **log scale** is useful for plotting values that vary greatly in magnitude. We plot the log of the variable instead of the variable itself.

2 A log scale is a **multiplicative scale**: Each increment of equal length on the scale indicates that the value is multiplied by an equal amount.

3 The pH value of a substance is defined by the formula

$$\text{pH} = -\log\left([H^+]\right)$$

where $[H^+]$ denotes the concentration of hydrogen ions in the substance.

4 The loudness of a sound is measured in decibels, D, by

$$D = 10\log\left(\frac{I}{10^{-12}}\right)$$

where I is the intensity of its sound waves (in watts per square meter).

5 The Richter magnitude, M, of an earthquake is given by

$$M = \log\left(\frac{A}{A_0}\right)$$

where A is the amplitude of its seismographic trace and A_0 is the amplitude of the smallest detectable earthquake.

6 A *difference* of K units on a logarithmic scale corresponds to a *factor* of 10^K units in the value of the variable.

5.4.7.3 STUDY QUESTIONS

1 What numbers are used to label the axis on a log scale?

2 What does it mean to say that a log scale is a multiplicative scale?

3 Delbert says that 80 decibels is twice as loud as 40 decibels. Is he correct? Why or why not?

4 Which is farther on a log scale, the distance between 5 and 15, or the distance between 0.5 and 1.5?

5.4.7.4 SKILLS

Practice each skill in the Homework 5.4.8, p. 581 problems listed.

1 Plot values on a log scale: #1–4, 9 and 10

2 Read values from a log scale: #5–8, 11–14, 19 and 20

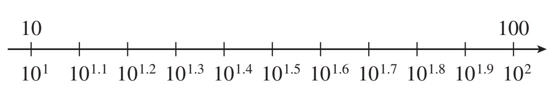
3 Compare values on a log scale: #15–18

4 Use log scales in applications: #21–40

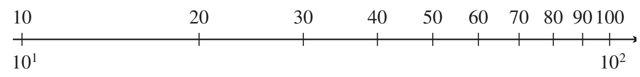
5.4.8 Logarithmic Scales (Homework 5.4)

1.

(a) The log scale is labeled with powers of 10. Finish labeling the tick marks in the figure with their corresponding decimal values.

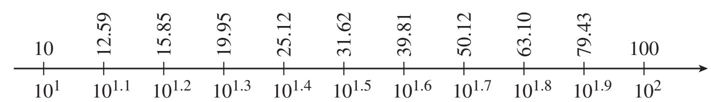


(b) The log scale is labeled with integer values. Label the tick marks in the figure with the corresponding powers of 10.

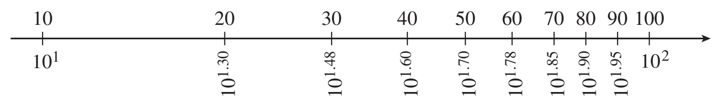


Answer.

(a)

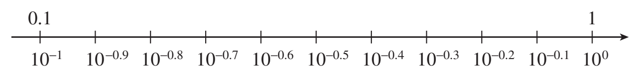


(b)

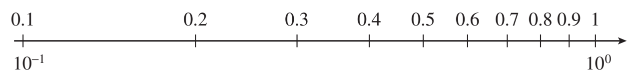


2.

(a) The log scale is labeled with powers of 10. Finish labeling the tick marks in the figure with their corresponding decimal values.



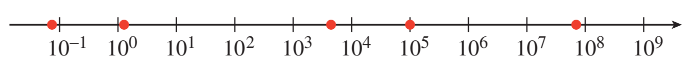
(b) The log scale is labeled with integer values. Label the tick marks in the figure with the corresponding powers of 10.



3. Plot the values on a log scale.

x	0.075	1.3	4200	$87,000$	6.5×10^7

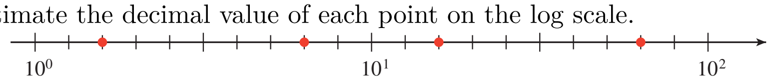
Answer.



4. Plot the values on a log scale.

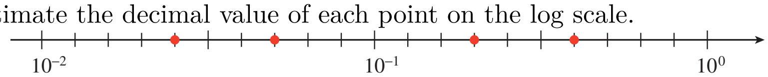
x	4×10^{-4}	0.008	0.9	27	90

5. Estimate the decimal value of each point on the log scale.

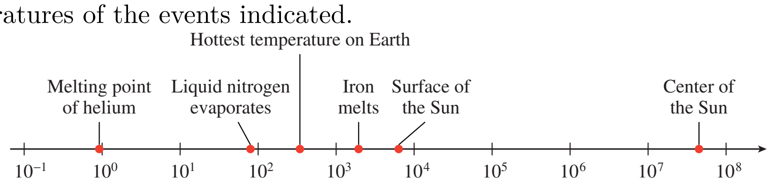


Answer. 1.58, 6.31, 15.8, 63.1

6. Estimate the decimal value of each point on the log scale.

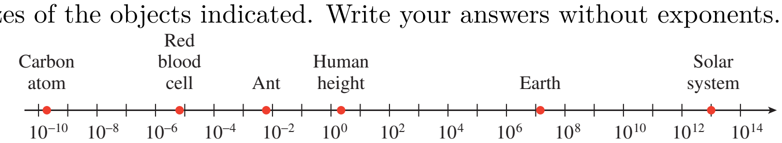


7. The log scale shows various temperatures in Kelvins. Estimate the temperatures of the events indicated.



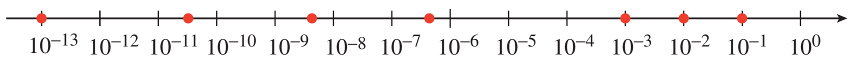
Answer. 1, 80, 330, 1600, 7000, 4×10^7

8. The log scale shows the size of various objects, in meters. Estimate the sizes of the objects indicated. Write your answers without exponents.



9. Plot the values of $[H^+]$ in the section "Acidity and the pH Scale" on a log scale.

Answer.

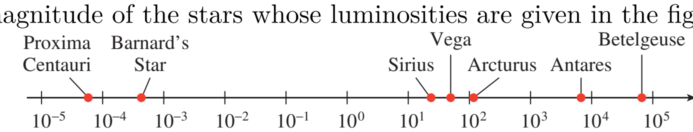


10. Plot the values of sound intensity in the section "Decibels" on a log scale.

11. The magnitude of a star is a measure of its brightness. It is given by the formula

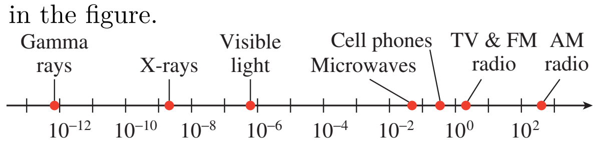
$$m = 4.83 - 2.5\log(L)$$

where L is the luminosity of the star, measured in solar units. Calculate the magnitude of the stars whose luminosities are given in the figure.



Answer. Proxima Centauri: 15.5; Barnard: 13.2; Sirius: 1.4; Vega: 0.6; Arcturus: −0.4; Antares: −4.7; Betelgeuse: −7.2

12. Estimate the wavelength, in meters, of the types of electromagnetic radiation shown in the figure.



13. The **risk magnitude** of an event is defined by $R = 10 + \log(p)$, where p is the probability of the event occurring. Calculate the probability of each event.

(a) The sun will rise tomorrow, $R = 10$.

(b) The next child born in Arizona will be a boy, $R = 9.7$.

(c) A major hurricane will strike North Carolina this year, $R = 9.1$.

(d) A 100-meter asteroid will collide with Earth this year, $R = 8.0$.

(e) You will be involved in an automobile accident during a 10-mile trip, $R = 5.9$.

(f) A comet will collide with Earth this year, $R = 3.5$.

(g) You will die in an automobile accident on a 1000-mile trip, $R = 2.3$

(h) You will die in a plane crash on a 1000-mile trip, $R = 0.9$.

Answer.

(a) 1

(b) 0.5012

(c) 0.1259

(d) 0.01

(e) 0.000079

(f) 3.2×10^{-7}

(g) 2×10^{-8}

(h) 8×10^{-10}

14. Have you ever wondered why time seems to pass more quickly as we grow older? One theory suggests that the human mind judges the length of a long period of time by comparing it with its current age. For example, a year is 20% of a 5-year-old's lifetime, but only 5% of a 20-year-old's, so a year feels longer to a 5-year-old. Thus, psychological time follows a log scale, like the one shown in the figure.

(a) Label the tick marks with their base 10 logarithms, rounded to 3 decimal places. What do you notice about the values?

(b) By computing their logs, locate 18 and 22 on the scale

(c) Four years of college seems like a long time to an 18-year-old. What length of time feels the same to a 40-year-old?

(d) How long will the rest of your life feel? Let A be your current age, and let L be the age to which you think you will live. Compute the difference of their logs. Now move backward on the log scale an equal distance from your current age. What is the age at that spot? Call that age B. The rest of your life will feel the same as your life from age B until now.

(e) Compute B using a proportion instead of logs.

15.

(a) What number is halfway between $10^{1.5}$ and 10^2 on a log scale?

(b) What number is halfway between 20 and 30 on a log scale?

Answer.

(a) $10^{1.75} \approx 56.2341$ (b) $10^{(\log(600))/2} \approx 24.4949$

16.

(a) What number is halfway between $10^{3.0}$ and $10^{3.5}$ on a log scale?

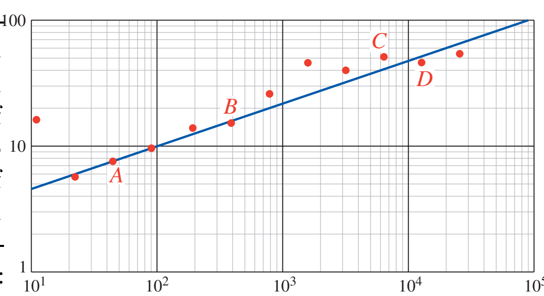
(b) Plot 500 and 600 on a log scale. What is halfway between them on this scale

17. The distances to two stars are separated by 3.4 units on a log scale. What is the ratio of their distances?

Answer. $10^{3.4} \approx 2512$

18. The populations of two cities are separated by 2.8 units on a log scale. What is the ratio of their populations?

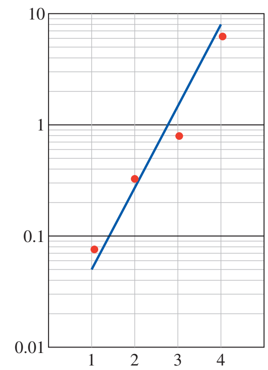
19. The probability of discovering an oil field increases with its diameter, defined to be the square root of its area. Use the graph to estimate the diameter of the oil fields at the labeled points, and their probability of discovery. (Source: Deffeyes, 2001)



Answer. A: $a \approx 45$, $p \approx 7.4\%$; B: $a \approx 400$, $p \approx 15\%$; C: $a \approx 6000$, $p \approx 50\%$; D: $a \approx 13000$, $p \approx 45\%$

20.

The **order** of a stream is a measure of its size. Use the graph to estimate the drainage area, in square miles, for streams of orders 1 through 4. (Source: Leopold, Wolman, and Miller)



In Problems 21–40, use the appropriate formulas for logarithmic models.

21. The hydrogen ion concentration of vinegar is about 6.3×10^{-4}. Calculate the pH of vinegar.

Answer. 3.2

22. The hydrogen ion concentration of spinach is about 3.2×10^{-6}. Calculate the pH of spinach.

23. The pH of lime juice is 1.9. Calculate its hydrogen ion concentration.

Answer. 0.0126

24. The pH of ammonia is 9.8. Calculate its hydrogen ion concentration.

25. A lawn mower generates a noise of intensity 10^{-2} watts per square meter. Find the decibel level of the sound of a lawn mower.

Answer. 100

26. A jet airplane generates 100 watts per square meter at a distance of 100 feet. Find the decibel level for a jet airplane.

27. The loudest sound emitted by any living source is made by the blue whale. Its whistles have been measured at 188 decibels and are detectable 500 miles away. Find the intensity of the blue whale's whistle in watts per square meter.

Answer. $6,309,573$ watts per square meter

28. The loudest sound created in a laboratory registered at 210 decibels. The energy from such a sound is sufficientto bore holes in solid material. Find the intensity of a210-decibel sound.

29. At a concert by The Who in 1976, the sound level 50 meters from the stage registered 120 decibels. How many times more intense was this than a 90-decibel sound (the threshold of pain for the human ear)?

Answer. 1000

30. The loudest scientifically measured shouting by a human being registered 123.2 decibels. How many times more intense was this than normal conversation at 40 decibels?

31. The pH of normal rain is 5.6. Some areas of Ontario have experienced acid rain with a pH of 4.5. How many times more acidic is acid rain than normal rain?

Answer. 12.6

32. The pH of normal hair is about 5, the average pH of shampoo is 8, and 4 for conditioner. Compare the acidityof normal hair, shampoo, and conditioner.

33. How much more acidic is milk than baking soda? (Refer to the table in this section..)

Answer. 100

34. Compare the acidity of lye and milk of magnesia. (Refer to the table in this section..)

35. In 1964, an earthquake in Alaska measured 8.4 on the Richter scale. An earthquake measuring 4.0 is consideredsmall and causes little damage. How many times stronger was the Alaska quake than one measuring 4.0?

Answer. $\approx 25,000$

36. On April 30, 1986, an earthquake in Mexico City measured 7.0 on the Richter scale. On September 21, a second earthquake occurred, this one measuring 8.1, hit Mexico City. How many times stronger was the September quake than the one in April?

37. A small earthquake measured 4.2 on the Richter scale. What is the magnitude of an earthquake three times as strong?

Answer. 4.7

38. Earthquakes measuring 3.0 on the Richter scale often go unnoticed. What is the magnitude of a quake 200 times as strong as a 3.0 quake?

39. The sound of rainfall registers at 50 decibels. What is the decibel level of a sound twice as loud?

Answer. 53

40. The magnitude, m, of a star is a function of its luminosity, L, given by

$$m = 4.83 - 2.5\log(L)$$

If one star is 10 times as luminous as another star, what is the difference in their magnitudes?

5.5 Chapter Summary and Review

5.5.1 Key Concepts

1 **Inverse Functions.**

If the inverse of a function f is also a function, then the inverse is denoted by the symbol f^{-1}, and

$$f^{-1}(b) = a \quad \text{if and only if} \quad f(a) = b$$

2 We can make a table of values for the inverse function, f^{-1}, by interchanging the columns of a table for f.

3 If a function is defined by a formula in the form $y = f(x)$, we can find a formula for its inverse function by solving the equation for x to get $x = f^{-1}(y)$.

4 The inverse function f^{-1} undoes the effect of the function f, that is, if we apply the inverse function to the output of f, we return to the original input value.

5 If f^{-1} is the inverse function for f, then f is also the inverse function for f^{-1}.

6 The graphs of f and its inverse function are **symmetric about the line** $y = x$.

7 **Horizontal line test**: If no horizontal line intersects the graph of a function more than once, then the inverse is also a function.

8 A function that passes the horizontal line test is called **one-to-one**.

9 The inverse of a function f is also a function if and only if f is one-to-one.

10 We define the logarithmic function, $g(x) = \log_b(x)$, which takes the log base b of its input values. The log function $g(x) = \log_b(x)$ is the inverse of the exponential function $f(x) = b^x$.

11

Because $f(x) = b^x$ and $g(x) = \log_b(x)$ are inverse functions for $b > 0,\ b \neq 1$,

$$\log_b(b^x) = x \quad \text{, for all } x \quad \text{and} \quad b^{\log_b(x)} = x \quad \text{, for } x > 0$$

12 **Logarithmic Functions** $y = \log_b(x)$**.**

1 Domain: all positive real numbers

2 Range: all real numbers

3 x-intercept: $(1, 0)$

4 y-intercept: none

5 Vertical asymptote at $x = 0$

6 The graphs of $y = \log_b(x)$ and $y = b^x$ are symmetric about the line $y = x$.

13 A **logarithmic equation** is one where the variable appears inside of a logarithm. We can solve logarithmic equations by converting to exponential form.

14 **Steps for Solving Logarithmic Equations.**

1 Use the properties of logarithms to combine all logs into one log.

2 Isolate the log on one side of the equation.

3 Convert the equation to exponential form.

4 Solve for the variable.

5 Check for extraneous solutions.

15 The natural base is an irrational number called e, where

$$e \approx 2.71828182845$$

16 The **natural exponential function** is the function $f(x) = e^x$. The **natural log function** is the function $g(x) = \ln(x) = \log_e(x)$.

17 **Conversion Formulas for Natural Logs.**

$$\boldsymbol{y = \ln(x)} \quad \text{if and only if} \quad \boldsymbol{e^y = x}$$

18 **Properties of Natural Logarithms.**

If $x, y > 0$, then

1 $\ln(xy) = \ln(x) + \ln(y)$

2 $\ln\left(\dfrac{x}{y}\right) = \ln(x) - \ln(y)$

3 $\ln(x^k) = k\ln(x)$

also

$$\ln(e^x) = x, \text{ for all } x \text{ and } e^{\ln(x)} = x, \text{ for } x > 0$$

19 We use the natural logarithm to solve exponential equations with base e.

20 **Exponential Growth and Decay.**

The function

$$P(t) = P_0 e^{kt}$$

describes exponential growth if $k > 0$, and exponential decay if $k < 0$.

21 **Continuous compounding**: The amount accumulated in an account after t years at interest rate r compounded continuously is given by

$$A(t) = Pe^{rt}$$

where P is the principal invested.

22 A **log scale** is useful for plotting values that vary greatly in magnitude. We plot the log of the variable, instead of the variable itself.

23 A log scale is a **multiplicative scale**: Each increment of equal length on the scale indicates that the value is multiplied by an equal amount.

24 The pH value of a substance is defined by the formula

$$\text{pH} = -\log\left([H^+]\right)$$

where $[H^+]$ denotes the concentration of hydrogen ions in the substance.

25 The loudness of a sound is measured in decibels, D, by

$$D = 10\log\left(\frac{I}{10^{-12}}\right)$$

where I is the intensity of its sound waves (in watts per square meter).

26 The Richter magnitude, M, of an earthquake is given by

$$M = \log\left(\frac{A}{A_0}\right)$$

where A is the amplitude of its seismographic trace and A_0 is the amplitude of the smallest detectable earthquake.

27 A *difference* of K units on a logarithmic scale corresponds to a *factor* of 10^K units in the value of the variable.

.

5.5.2 Chapter 5 Review Problems

For Problems 1-4, make a table of values for the inverse function.

1. $f(x) = x^3 + x + 1$

Answer.

y	-1	1	3	11
$x = f^{-1}(y)$	-1	0	1	2

2. $g(x) = x + 6\sqrt[3]{x}$

3. $g(w) = \dfrac{1+w}{w-3}$

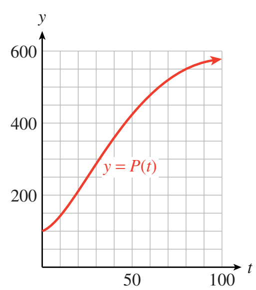
Answer.

y	0	$\frac{-1}{3}$	-1	-3
$w = g^{-1}(y)$	-1	0	1	2

4. $f(n) = \dfrac{n}{1+n}$

For Problems 5-6, use the graph to find the function values.

5.

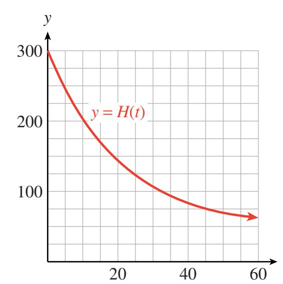


(a) $P^{-1}(350)$ (b) $P^{-1}(100)$

Answer.

(a) $P^{-1}(350) = 40$ (b) $P^{-1}(100) = 0$

6.



(a) $H^{-1}(200)$ (b) $H^{-1}(75)$

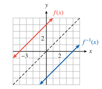
For Problems 7–12,

a Find a formula for the inverse f^{-1} of each function.

b Graph the function and its inverse on the same set of axes, along with the graph of $y = x$.

7. $f(x) = x + 4$

Answer.

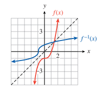
(a) $f^{-1}(x) = x - 4$ (b) 

8. $f(x) = \dfrac{x-2}{4}$

9. $f(x) = x^3 - 1$

Answer.

(a) $f^{-1}(x) = \sqrt[3]{x+1}$

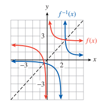
(b) 

10. $f(x) = \dfrac{1}{x+2}$

11. $f(x) = \dfrac{1}{x} + 2$

Answer.

(a) $f^{-1}(x) = \dfrac{1}{x-2}$

(b) 

12. $f(x) = \sqrt[3]{x} - 2$

13. If $F(t) = \dfrac{3}{4}t + 2$, find $F^{-1}(2)$.

Answer. 0

14. If $G(x) = \dfrac{1}{x} - 4$, find $G^{-1}(3)$.

15. The table shows the revenue, R, from sales of the Miracle Mop as a function of the number of dollars spent on advertising, A. Let f be the name of the function defined by the table, so $R = f(A)$.

A (thousands of dollars)	100	150	200	250	300
R (thousands of dollars)	250	280	300	310	315

(a) Evaluate $f^{-1}(300)$. Explain its meaning in this context.

(b) Write two equations to answer the following question, one using f and one using f^{-1}: How much should we spend on advertising to generate revenue of \$250,000?

Answer.

(a) $f^{-1}(300) = 200$: \$200,000 in advertising results in \$300,000 in revenue.

(b) $f(A) = 250$ or $A = f^{-1}(250)$

16. The table shows the systolic blood pressure, S, of a patient as a function of the dosage, d, of medication he receives. Let g be the name of the function defined by the table, so $S = g(d)$.

d (mg)	190	195	200	210	220
S (mm Hg)	220	200	190	185	183

(a) Evaluate $g^{-1}(200)$. Explain its meaning in this context.

(b) Write two equations to answer the following question, one using g and one using g^{-1}: What dosage results in systolic blood pressure of 220?

For Problems 17-24, write the equation in exponential form.

17. $\log(0.001) = z$

Answer. $10^z = 0.001$

18. $\log_3(20) = t$

19. $\log_2(3) = x - 2$

Answer. $2^{x-2} = 3$

20. $\log_5(3) = 6 - 2p$

21. $\log_b(3x+1)=3$

Answer. $b^3=3x+1$

22. $\log_m(8)=4t$

23. $\log_n(q)=p-1$

Answer. $n^{p-1}=q$

24. $\log_q(p+2)=w$

For Problems 25-28, simplify.

25. $10^{\log(6n)}$

Answer. $6n$

26. $\log(100^x)$

27. $\log_2\left(4^{x+3}\right)$

Answer. $2x+6$

28. $3^{2\log_3(t)}$

For Problems 29-36, solve.

29. $\log_3\left(\dfrac{1}{3}\right)=y$

Answer. -1

30. $\log_3(x)=4$

31. $\log_2(y)=-1$

Answer. $\dfrac{1}{2}$

32. $\log_5(y)=-2$

33. $\log_b(16)=2$

Answer. 4

34. $\log_b(9)=\dfrac{1}{2}$

35. $\log_4\left(\dfrac{1}{2}t+1\right)=-2$

Answer. $\dfrac{-15}{8}$

36. $\log_2(3x-1)=3$

For Problems 37-40, solve.

37. $\log_3(x)+\log_3(4)=2$

Answer. $\dfrac{9}{4}$

38. $\log_2(x+2)-\log_2(3)=6$

39. $\log(x-1)+\log(x+2)=1$

Answer. 3

40. $\log(x+2)-\log(x-3)=1$

For Problems 41-46, solve.

41. $e^x=4.7$

Answer. $x\approx 1.548$

42. $e^x=0.5$

43. $\ln(x)=6.02$

Answer. $x\approx 411.58$

44. $\ln(x)=-1.4$

45. $4.73=1.2e^{0.6x}$

Answer. $x\approx 2.286$

46. $1.75=0.3e^{-1.2x}$

For Problems 47-50, simplify.

47. $e^{(\ln(x))/2}$

Answer. $\sqrt{x}$

48. $\ln\left(\dfrac{1}{e}\right)^{2n}$

49. $\ln\left(\dfrac{e^k}{e^3}\right)$

Answer. $k-3$

50. $e^{\ln(e+x)}$

51. In 1970, the population of New York City was $7,894,862$. In 1980, the population had fallen to $7,071,639$.

(a) Write an exponential function using base e for the population of New York over that decade.

(b) By what percent did the population decline annually?

Answer.

(a) $P = 7,894,862e^{-0.011t}$

(b) 1.095%

52. In 1990, the population of New York City was $7,322,564$. In 2000, the population was $8,008,278$.

(a) Write an exponential function using base e for the population of New York over that decade.

(b) By what percent did the population increase annually?

53. You deposit \$1000 in a savings account paying 5% interest compounded continuously.

(a) Find the amount in the account after 7 years.

(b) How long will it take for the original principal to double?

(c) Find a formula for the time t required for the amount to reach A.

Answer.

(a) \$1419.07

(b) 13.9 years

(c) $t = 20\ln\left(\dfrac{A}{1000}\right)$

54. The voltage, V, across a capacitor in a certain circuit is given by the function

$$V(t) = 100(1 - e^{-0.5t})$$

where t is the time in seconds.

(a) Make a table of values and graph $V(t)$ for $t = 0$ to $t = 10$.

(b) Describe the graph. What happens to the voltage in the long run?

(c) How much time must elapse (to the nearest hundredth of a second) for the voltage to reach 75 volts?

55. Solve for t: $y = 12e^{-kt} + 6$

Answer. $t = \dfrac{-1}{k}\ln\left(\dfrac{y-6}{12}\right)$

56. Solve for k: $N = N_0 + 4\ln(k + 10)$

57. Solve for M: $Q = \dfrac{1}{t}\left(\dfrac{\log(M)}{\log(N)}\right)$

Answer. $M = N^{Qt}$

58. Solve for t: $C_H = C_L \cdot 10^k t$

59. Express $P(t) = 750e^{0.32t}$ in the form $P(t) = P_0b^t$.

Answer. $P(t) = 750(1.3771)^t$

60. Express $P(t) = 80e^{-0.6t}$ in the form $P(t) = P_0b^t$.

61. Express $N(t) = 600(0.4)^t$ in the form $N(t) = N_0e^{kt}$.

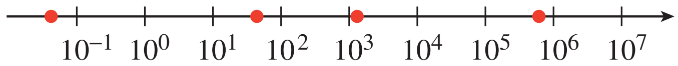
Answer. $N(t) = 600e^{-0.9163t}$

62. Express $N(t) = 100(1.06)^t$ in the form $N(t) = N_0e^{kt}$.

63. Plot the values on a log scale.

x	0.04	45	1200	560,000

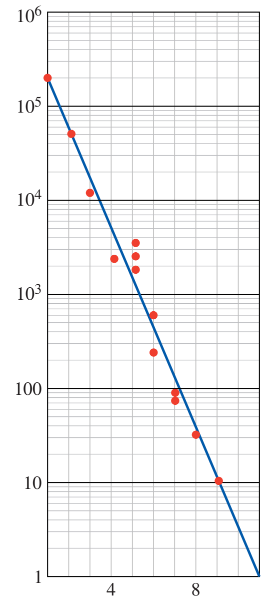
Answer.



64. Plot the values on a log scale.

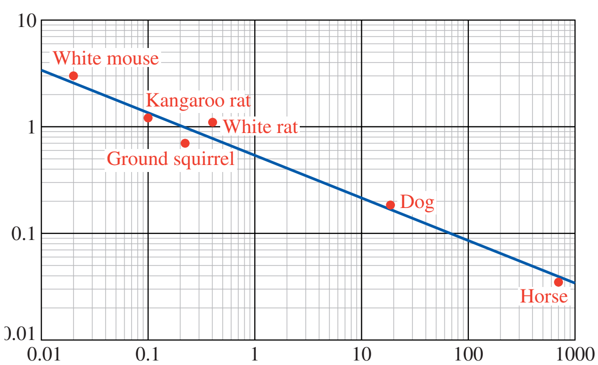
x	0.0007	0.8	3.2	2500

65. The graph describes a network of streams near Santa Fe, New Mexico. It shows the number of streams of a given order, which is a measure of their size. Use the graph to estimate the number of streams of orders 3, 4, 8, and 9. (Source: Leopold, Wolman, and Miller)



Answer. Order 3: $17,000$; Order 4: 5000; Order 8: 40; Order 9: 11

66. Large animals use oxygen more efficiently when running than small animals do. The graph shows the amount of oxygen various animals use, per gram of their body weight, to run 1 kilometer. Estimate the body mass and oxygen use for a kangaroo rat, a dog, and a horse. (Source: Schmidt-Neilsen, 1972)



67. The pH of an unknown substance is 6.3. What is its hydrogen ion concentration?

Answer. 5×10^{-7}

68. The noise of a leaf blower was measured at 110 decibels. What was the intensity of the sound waves?

69. A refrigerator produces 50 decibels of noise, and a vacuum cleaner produces 85 decibels. How much more intense are the sound waves from a vacuum cleaner than those from a refrigerator?

Answer. 3160

70. In 2004, a magnitude 9.0 earthquake struck Sumatra in Indonesia. How much more powerful was this quake than the 1906 San Francisco earthquake of magnitude 8.3?

5.6 Projects for Chapter 5

Project 34 The Logistic Function. In this project, we investigate the graph of the logistic function.

a Graph the **sigmoid function**, $s(t) = \dfrac{1}{1+e^{-t}}$, in the window

$$\text{Xmin} = -4 \qquad \text{Xmax} = 4$$
$$\text{Ymin} = -1 \qquad \text{Ymax} = 2$$

What are the domain and range of the function? List the intercepts of the graph, as well as any horizontal or vertical asymptotes. Estimate the coordinates of the **inflection point**, where the graph changes concavity.

b Graph the two functions $Y_1(t) = \dfrac{5}{1+4e^{-t}}$ and $Y_2 = \dfrac{10}{1+9e^{-t}}$ in the window

$$\text{Xmin} = -2 \qquad \text{Xmax} = 10$$
$$\text{Ymin} = -1 \qquad \text{Ymax} = 11$$

How do the graphs of these functions differ from the sigmoid function? State the domain and range, intercepts, and asymptotes of Y_1 and Y_2. Estimate the coordinates of their inflection points.

c The function $P(t) = \dfrac{KP_0}{P_0 + (K - P_0)e^{-rt}}$ is called a **logistic function**. It is used to model population growth, among other things. It has three parameters, K, P_0, and r. The parameter K is called the **carrying capacity**. The functions Y_1 and Y_2 in part (b) are logistic functions with $P_0 = 1$ and $r = 1$. What does the value of K tell you about the graph? What do you notice about the vertical coordinate of the inflection point?

d Graph the function $P(t) = \dfrac{10P_0}{P_0 + (10 - P_0)e^{-t}}$ for $P_0 = 3$, 4, and 5. What does the value of P_0 tell you about the graph?

e Graph the function $P(t) = \dfrac{20}{2+8e^{-rt}}$ for $r = 0.5$, 1, and 2. What does the value of r tell you about the graph?

Project 35 Bell-shaped Curve. In this project, we investigate the normal or bell-shaped curve.

a Graph the function $f(x) = e^{-x^2}$, in the window

$$\text{Xmin} = -2 \qquad \text{Xmax} = 2$$
$$\text{Ymin} = -1 \qquad \text{Ymax} = 2$$

What are the domain and range of the function? List the intercepts of the graph, as well as any horizontal or vertical asymptotes. Estimate the coordinates of the **inflection point**, where the graph changes concavity.

b Graph the function $f(x) = e^{-(x-m)^2}$ for $m = -1$, 0, 1, and 2. How does the value of m affect the graph?

c The function

$$N(x) = \frac{1}{s\sqrt{2\pi}} e^{-(x-m)^2/2s^2}$$

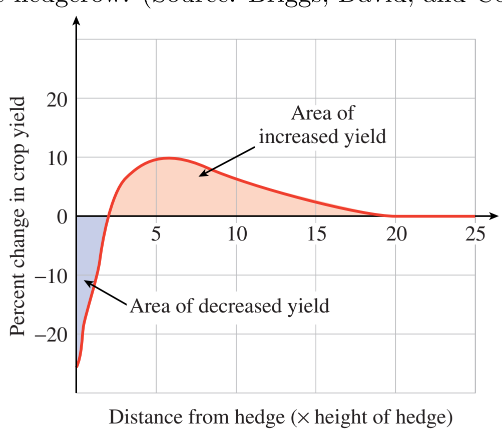
is called the **normal** curve. It is used in statistics to describe the distribution of a variable, such as height, among a population. The parameter m gives the **mean** of the distribution, and s gives the **standard deviation**. For example, the distribution of height among American women has a mean of 64 inches and a standard deviation of 2.5 inches. Graph $N(x)$ for these values.

d Graph the function

$$N(x) = \frac{1}{s\sqrt{2\pi}} e^{-(x-m)^2/2s^2}$$

for $s = 0.5$, 0.8, 1, and 1.2. (You may have to adjust the window to get a good graph.) How does the value of s affect the graph?

Project 36 Do hedgerows planted at the boundaries of a field have a good or bad effect on crop yields? Hedges provide some shelter for the crops and retain moisture, but they may compete for nutrients or create too much shade. Results of studies on the microclimates produced by hedges are summarized in the figure, which shows how crop yields increase or decrease as a function of distance from the hedgerow. (Source: Briggs, David, and Courtney, 1985)



a We will use trial-and-improvement to fit a curve to the graph. First, graph $y_1 = xe^{-x}$ in the window Xmin $= -2$, Xmax $= 5$, Ymin $= -1$, Ymax $= 1$ to see that it has the right shape.

b Graph $y_2 = (x-2)e^{-(x-2)}$ on the same axes. How is the graph of y_2 different from the graph of y_1?

c Next we'll find the correct scale by trying functions of the form $y = a(x-2)e^{-(x-2)/b}$. Experiment with different whole number values of a and b. How do the values of a and b affect the curve?

d Graph $y = 5(x-2)e^{-(x-2)/4}$ in the window Xmin $= -5$, Xmax $= 25$, Ymin $= -20$, Ymax $= 25$. This function is a reasonable approximation for the curve in the figure. Compare the area of decreased yield (below the x-axis) with the area of increased yield (above the x-axis). Which area is larger? Is the overall effect of hedgerows on crop yield good or bad?

e About how far from the hedgerow do the beneficial effects extend? If the average hedgerow is about 2.5 meters tall, how large should the field be to exploit their advantages?

Project 37 Carbon Content. Organic matter in the ground decomposes over time, and if the soil is cultivated properly, the fraction of its original organic carbon content is given by

$$C(t) = \frac{a}{b} - \frac{a-b}{b}e^{-bt}$$

where t is in years, and a and b are constants. (Source: Briggs, David, and Courtney, 1985)

a Write and simplify the formula for $C(t)$ if $a = 0.01$, $b = 0.028$.

b Graph $C(t)$ in the window Xmin = 0, Xmax = 200, Ymin = 0, Ymax = 1.5.

c What value does $C(t)$ approach as t increases? Compare this value to $\frac{a}{b}$.

d The half-life of this function is the amount of time until $C(t)$ declines halfway to its limiting value, $\frac{a}{b}$. What is the half-life?

Project 38 Change of Base. This project derives the **change of base** formula.

a Follow the steps below to calculate $\log_8 20$.

Step 1 Let $x = \log_8 20$. Write the equation in exponential form.

Step 2 Take the logarithm base 10 of both sides of your new equation.

Step 3 Simplify and solve for x.

b Follow the steps in part (a) to calculate $\log_8 5$.

c Use part (a) to find a formula for calculating $\log_8 Q$, where Q is any positive number.

d Find a formula for calculating $\log_b Q$, where $b > 1$ and Q is any positive number.

e Find a formula for calculating $\ln Q$ in terms of $\log_{10} Q$.

f Find a formula for calculating $\log_{10} Q$ in terms of $\ln Q$.

Project 39 Log Equations. In this project, we solve logarithm equations with a graphing calculator. We have already used the **Intersect** feature to find approximate solutions for linear, exponential, and other types of equations in one variable. The same technique works for equations that involve common or natural logarithms.

a Solve $\log_{10}(x+1) + \log_{10}(x-2) = 1$ using the **Intersect** feature by setting $Y_1 = \log(x+1) + \log(x-2)$ and $Y_2 = 1$. What about logarithmic equations with other bases? The calculator typically does not have a log key for bases other than 10 or e. However, by using the change of base formula from Project 5, we can rewrite any logarithm in terms of a common or natural logarithm.

b Use the change of base formula to write $y = \log_2 x$ and $y = \log_2(x-2)$ in terms of common logarithms.

c Solve $\log_2 x + \log_2(x-2) = 3$ by using the **Intersect** feature on your calculator.

d Solve $\log_3(x-2) - \log_3(x+1) = 3$ by using the **Intersect** feature on your calculator.