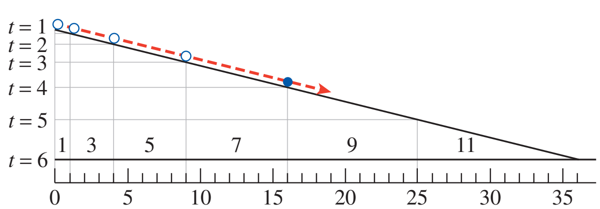
Chapter 6

Quadratic Functions

The models we have explored so far, namely, linear, exponential, logarithmic, and power, are **monotonic** functions, that is, always increasing or always decreasing on their domains. (Remember that we used power functions as models in the first quadrant only.) In this chapter, we investigate problems where the output variable may change from increasing to decreasing, or vice versa. The simplest sort of function that models this behavior is a quadratic function, one that involves the square of the variable.

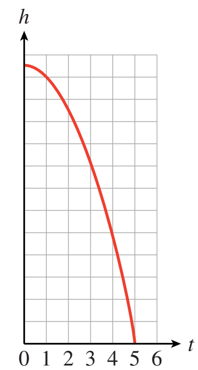
Around 1600, Galileo began to study the motion of falling objects. He used a ball rolling down an inclined plane or ramp to slow down the motion.



Galileo had no accurate way to measure time; clocks had not been invented yet. So he used water running into a jar to mark equal time intervals. After many trials, Galileo found that the ball traveled 1 unit of distance down the plane in the first time interval, 3 units in the second time interval, 5 units in the third time interval, and so on, as shown in the figure, with the distances increasing through odd units of distance as time went on.

Time	Distance traveled	Total distance
1	1	1
2	3	5
3	5	9
4	7	16
5	9	25

As you can see in the table above, the total distance traveled by the ball is proportional to the square of the time elapsed, $s = kt^2$. Galileo found that this relationship held no matter how steep he made the ramp. Plotting the height of the ball as a function of time, we obtain a portion of the graph of a quadratic function.



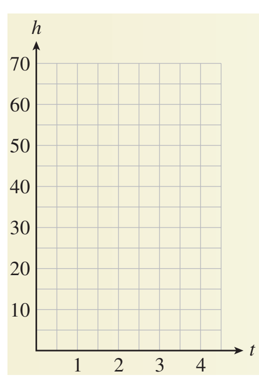
Investigation 40 Height of a Baseball. Suppose a baseball player pops up, that is, hits the baseball straight up into the air. The height, h, of the baseball t seconds after it leaves the bat can be calculated using a formula from physics. This formula takes into account the initial speed of the ball (64 feet per second) and its height when it was hit (4 feet). The formula for the height of the ball (in feet) is

$$h = -16t^2 + 64t + 4$$

1. Evaluate the formula to complete the table of values for the height of the baseball.

t	0	1	2	3	4
h					

2. Graph the height of the baseball as a function of time. Plot data points from your table, then connect the points with a smooth curve.



3. What are the coordinates of the highest point on the graph? When does the baseball reach its maximum height, and what is that height?

4. Use the formula to find the height of the baseball after $\frac{1}{2}$ second.

5. Check that your answer to part (4) corresponds to a point on your graph. Approximate from your graph another time at which the baseball is at the same height as your answer to part (4).

6. Use your graph to find two times when the baseball is at a height of 64 feet.

7. Use your graph to approximate two times when the baseball is at a height of 20 feet. Then use the formula to find the actual heights at those times.

8. Suppose the catcher catches the baseball at a height of 4 feet, before it strikes the ground. At what time was the ball caught?

9. Use your calculator to make a table of values for the equation $h = -16t^2 + 64t + 4$ with TblStart = 0 and ΔTbl = 0.5.

10. Use your calculator to graph the equation for the height of the ball, with window settings

$$\begin{aligned} &\text{Xmin} = 0, \quad \text{Xmax} = 4.5, \quad \text{Yscl} = 5 \\ &\text{Ymin} = 0, \quad \text{Ymax} = 70, \quad \text{Yscl} = 5 \end{aligned}$$

11. Use the *intersect* command to verify your answer part (7): Estimate two times when the baseball is at a height of 20 feet.

12. Use the *intersect* command to verify your answer to part (8): At what time was the ball caught if it was caught at a height of 4 feet?

6.1 Factors and x-Intercepts

In Investigation 40, p. 600, perhaps you recognized the graph of the baseball's height as a parabola. In this chapter, we shall see that the graph of any quadratic function is a parabola.

Quadratic Function.

A **quadratic function** is one that can be written in the form

$$f(x) = ax^2 + bx + c$$

where a, b, and c are constants, and a is not equal to zero.

Note 6.1.1 In the definition above, notice that if a is zero, there is no x-squared term, so the function is not quadratic.

In Investigation 40, p. 600, the height of a baseball t seconds after being hit was given by

$$h = -16t^2 + 64t + 4$$

We used a graph to find two times when the baseball was 64 feet high. Can we solve the same problem algebraically?

We are looking for values of t that produce $h = 64$ in the height equation. So, if we substitute $h = 64$ into the height equation, we would like to solve the **quadratic equation**

$$64 = -16t^2 + 64t + 4$$

This equation cannot be solved by extraction of roots, because there are two terms containing the variable t, and they cannot be combined. To solve this equation, we will appeal to a property of our number system, called the **zero-factor principle**.

6.1.1 Zero-Factor Principle

Can you multiply two numbers together and obtain a product of zero? Only if one of the two numbers happens to be zero. This property of numbers is called the **zero-factor principle**.

Zero-Factor Principle.

The product of two factors equals zero if and only if one or both of the factors equals zero. In symbols,

$$ab = 0 \quad \text{if and only if} \quad a = 0 \quad \text{or} \quad b = 0$$

The principle is true even if the numbers a and b are represented by algebraic expressions, such as $x - 5$ or $2x + 1$. For example, if

$$(x - 5)(2x + 1) = 0$$

then it must be true that either $x - 5 = 0$ or $2x + 1 = 0$. Thus, we can use the zero-factor principle to solve equations.

Example 6.1.2

a Solve the equation $(x - 6)(x + 2) = 0$.

b Find the x-intercepts of the graph of $f(x) = x^2 - 4x - 12$.

Solution.

a We apply the zero-factor principle to the product $(x - 6)(x + 2)$.

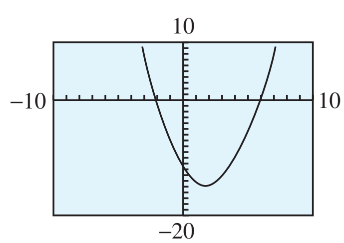
$$\begin{aligned} (x-6)(x+2) &= 0 && \text{Set each factor equal to zero.} \\ x - 6 = 0 \quad \text{or} \quad x + 2 &= 0 && \text{Solve each equation.} \\ x = 6 \quad \text{or} \quad x &= -2 \end{aligned}$$

There are two solutions, 6 and −2. (You should check that both of these values satisfy the original equation.)

b To find the x-intercepts of the graph, we set $y = 0$ and solve the equation

$$0 = x^2 - 4x - 12$$

But this is the equation we solved in part (a), because $(x - 6)(x + 2) = x^2 - 4x - 12$. The solutions of that equation were 6 and −2, so the x-intercepts of the graph are 6 and −2. You can see this by graphing the equation on your calculator, as shown below.



□

Example 6.1.2, p. 602 illustrates an important fact about the x-intercepts of a graph.

x-Intercepts of a Graph.

The x-intercepts of the graph of $y = f(x)$ are the solutions of the equation $f(x) = 0$.

Checkpoint 6.1.3 Graph the function

$$f(x) = (x - 3)(2x + 3)$$

on a calculator, and use your graph to solve the equation $f(x) = 0$. (UseXmin = −9.4, Xmax = 9.4.) Check your answer with the zero-factor principle.

Answer. $x = -\frac{3}{2},\ x = 3$

6.1.2 Solving Quadratic Equations by Factoring

Before we apply the zero-factor principle to solve a quadratic equation, we must first write the equation so that one side of the equation is zero. Let us introduce some terminology.

Forms for Quadratic Equations.

1. A quadratic equation written

$$ax^2 + bx + c = 0$$

is in **standard form**.

2. A quadratic equation written

$$a(x - r_1)(x - r_2) = 0$$

is in **factored form**.

Once we have written the equation in standard form, we factor the left side and set each variable factor equal to zero separately.

Example 6.1.4 Solve $3x(x + 1) = 2x + 2$

Solution. First, we write the equation in standard form.

$$\begin{aligned} 3x(x+1) &= 2x + 2 && \text{Apply the distributive law to the left side.} \\ 3x^2 + 3x &= 2x + 2 && \text{Subtract } \mathbf{2x + 2} \text{ from both sides.} \\ 3x^2 + x - 2 &= 0 \end{aligned}$$

Next, we factor the left side to obtain

$$(3x - 2)(x + 1) = 0$$

We then apply the zero-factor principle by setting each factor equal to zero.

$$3x - 2 = 0 \quad \text{or} \quad x + 1 = 0$$

Finally, we solve each equation to find

$$x = \frac{2}{3} \quad \text{or} \quad x = -1$$

The solutions are $\frac{2}{3}$ and -1. □

Caution 6.1.5 . When we apply the zero-factor principle, one side of the equation *must be zero*. For example, to solve the equation

$$(x-2)(x-4)=15$$

it is incorrect to set each factor equal to 15! (There are many ways that the product of two numbers can equal 15; it is not necessary that one of the numbers be 15.) We must first simplify the left side and write the equation in standard form. (The correct solutions are 7 and -1; make sure you can find these solutions.)

We summarize the factoring method for solving quadratic equations as follows.

To Solve a Quadratic Equation by Factoring.

1 Write the equation in standard form.

2 Factor the left side of the equation.

3 Apply the zero-factor principle: Set each factor equal to zero.

4 Solve each equation. There are two solutions (which may be equal).

Checkpoint 6.1.6 Solve by factoring: $(t-3)^2 = 3(9-t)$

Answer. $x=-3,\ x=6$

We can use factoring to solve the equation from Investigation 40, p. 600.

Example 6.1.7 The height, h, of a baseball t seconds after being hit is given by

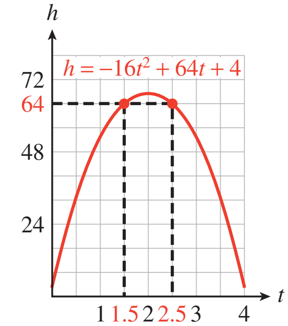
$$h=-16t^2+64t+4$$

When will the baseball reach a height of 64 feet?

Solution. We substitute 64 for h in the formula, and solve for t.

$$\begin{aligned}
64 &= -16t^2+64t+4 && \text{Write the equation in standard form.}\\
16t^2-64t+60 &= 0 && \text{Factor 4 from the left side.}\\
4(4t^2-16t+15) &= 0 && \text{Factor the quadratic expression.}\\
4(2t-3)(2t-5) &= 0 && \text{Set each variable factor equal to zero.}\\
2t-3=0 \quad\text{or}\quad 2t-5 &= 0 && \text{Solve each equation.}\\
t=\frac{3}{2} \quad\text{or}\quad t &= \frac{5}{2}
\end{aligned}$$

There are two solutions to the quadratic equation. At $t=\frac{3}{2}$ seconds, the ball reaches a height of 64 feet on the way up, and at $t=\frac{5}{2}$ seconds, the ball is 64 feet high on its way down.



□

In the solution to Example 6.1.7, p. 604, the factor 4 does not affect the solutions of the equation at all. You can understand why this is true by looking at some graphs. First, check that the two equations

$$x^2 - 4x + 3 = 0 \quad \text{and} \quad 4(x^2 - 4x + 3) = 0$$

have the same solutions, $x = 1$ and $x = 3$. Then use your graphing calculator to graph the equation

$$Y_1 = X^2 - 4X + 3$$

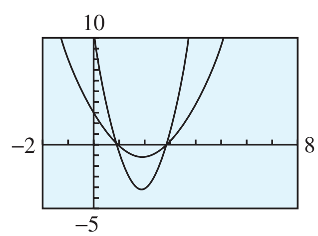
in the window

$$\text{Xmin} = -2 \quad \text{Xmax} = 8$$
$$\text{Ymin} = -5 \quad \text{Ymax} = 10$$

Notice that when $y = 0$, $x = 3$ or $x = 1$. These two points are the x-intercepts of the graph. In the same window, now graph

$$Y_2 = 4(X^2 - 4X + 3)$$

This graph has the same x-values when $y = 0$. The factor of 4 stretches the graph vertically but does not change the location of the x-intercepts.



The value of the constant factor a in the factored form of a quadratic function, $f(x) = a(x - r_1)(x - r_2)$, does not affect the location of the x-intercepts, because it does not affect the solutions of the equation $a(x - r_1)(x - r_2) = 0$.

Checkpoint 6.1.8

a Solve $f(t) = 4t - t^2 = 0$ by factoring.

b Solve $g(t) = 20t - 5t^2 = 0$ by factoring.

c Graph $y = f(t)$ and $y = g(t)$ together in the window

$$\text{Xmin} = -2 \quad \text{Xmax} = 6$$
$$\text{Ymin} = -20 \quad \text{Ymax} = 25$$

and locate the horizontal intercepts of each graph.

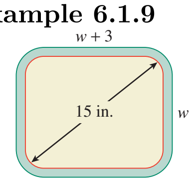
Answer.

a $t = 0, \; t = 4$ b $t = 0, \; t = 4$ c $(0, 0), \; (4, 0)$

6.1.3 Applications

Here is another example of how quadratic equations arise in applications.

Example 6.1.9



The size of a rectangular computer monitor screen is given by the length of its diagonal, as shown at left. If the length of the screen should be 3 inches greater than its width, what are the dimensions of a 15-inch monitor?

Solution. We express the two dimensions of the screen in terms of a single variable:

Width of screen: w
Length of screen: $w+3$

We can use the Pythagorean theorem to write an equation.

$$w^2 + (w+3)^2 = 15^2$$

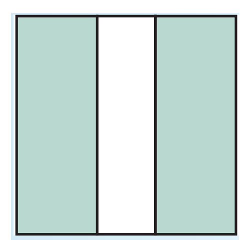
Solve the equation. Begin by simplifying the left side.

$w^2 + w^2 + 6w + 9 = 225$	Write the equation in standard form.
$2w^2 + 6w - 216 = 0$	Factor 2 from the left side.
$2(w^2 + 3w - 108) = 0$	Factor the quadratic expression.
$2(w-9)(w+12) = 0$	Set each variable factor equal to zero.
$w - 9 = 0 \quad \text{or} \quad w + 12 = 0$	Solve each equation.
$w = 9 \quad \text{or} \quad w = -12$	

Because the width of the screen cannot be a negative number, we discard the solution $w = -12$. Thus, the width is $w = 9$ inches, and the length is $w+3 = 12$ inches. □

Checkpoint 6.1.10

Francine is designing the layout for a botanical garden. The plan includes a square herb garden, with a path 5 feet wide through the center of the garden, as shown at right. To include all the species of herbs, the planted area must be 300 square feet. Find the dimensions of the herb garden.



Answer. 20 feet by 20 feet

6.1.4 Solutions of Quadratic Equations

We have seen that the solutions of the quadratic equation

$$a(x-r_1)(x-r_2) = 0$$

are r_1 and r_2. Thus, if we know the two solutions of a quadratic equation, we can work backward and reconstruct the equation, starting from its factored form. We can then write the equation in standard form by multiplying together the factors.

Example 6.1.11 Find a quadratic equation whose solutions are $\frac{1}{2}$ and -3.

Solution. The quadratic equation is

$$\left(x - \frac{1}{2}\right)\left[x - (-3)\right] = 0$$
$$\left(x - \frac{1}{2}\right)(x + 3) = 0$$

To write the equation in standard form, we multiply the factors together.

$$x^2 + \frac{5}{2}x - \frac{3}{2} = 0$$

We can also find an equation with integer coefficients if we clear the equation of fractions. Multiply both sides by 2:

$$2\left(x^2 + \frac{5}{2}x - \frac{3}{2}\right) = 2(0)$$
$$2x^2 + 5x - 3 = 0$$

You can check that the solutions of this last equation are in fact $\frac{1}{2}$ and -3. Multiplying both sides of an equation by a constant factor does not change its solutions. □

Checkpoint 6.1.12 Find a quadratic equation with integer coefficients whose solutions are $\frac{2}{3}$ and -5.

Answer. $3x^2 + 13x - 10 = 0$

Note 6.1.13 A quadratic equation in one variable always has two solutions. However, in some cases, the solutions may be equal. For example, the equation $x^2 - 2x + 1 = 0$ can be solved by factoring as follows:

$$(x - 1)(x - 1) = 0 \quad \text{Apply the zero-factor principle.}$$
$$x - 1 = 0 \quad \text{or} \quad x - 1 = 0$$

Both of these equations have solution 1. We say that 1 is a solution of **multiplicity** two, meaning that it occurs twice as a solution of the quadratic equation.

6.1.5 Equations Quadratic in Form

The equation
$$x^6 - 4x^3 - 5 = 0$$
is not quadratic, but if we make the substitution $u = x^3$, the equation becomes
$$u^2 - 4u - 5 = 0$$

An equation is called **quadratic in form** if we can use a substitution to write it as
$$au^2 + bu + c = 0$$
where u stands for an algebraic expression. Such equations can be solved by the same techniques we use to solve quadratic equations.

Example 6.1.14 Use the substitution $u = x^3$ to solve the equation
$$x^6 - 4x^3 - 5 = 0$$

Solution. We set $u = x^3$, so that $u^2 = \left(x^3\right)^2 = x^6$. The original equation then becomes a quadratic equation in the variable u, which we can solve by factoring.

$$\begin{aligned} u^2 - 4u - 5 &= 0 && \text{Factor the left side.} \\ (u+1)(u-5) &= 0 && \text{Apply the zero-factor principle.} \\ u+1 = 0 \quad \text{or} \quad u-5 &= 0 && \text{Solve each equation for } \boldsymbol{u}. \\ u = -1 \quad \text{or} \quad u &= 5 \end{aligned}$$

Finally, we replace u by x^3 and solve for x.

$$\begin{aligned} x^3 = -1 \qquad &\text{or} \qquad x^3 = 5 && \text{Take cube roots.} \\ x = \sqrt[3]{-1} = -1 \quad &\text{or} \quad x = \sqrt[3]{5} \end{aligned}$$

You can verify that the solutions of the original equation are -1 and $\sqrt[3]{5}$. □

We say that the equation in Example 6.1.14, p. 607, $x^6 - 4x^3 - 5 = 0$, is **quadratic in** x^3. We chose the substitution $u = x^3$ because $x^6 = u^2$.

Checkpoint 6.1.15 Use the substitution $u = x^2$ to solve the equation $x^4 - 5x^2 + 6 = 0$.

Answer. $x = \pm\sqrt{2}, \;\; x = \pm\sqrt{3}$

Usually, you can choose the simpler variable term in the equation for the u-substitution. For example, in Checkpoint 6.1.15, p. 608 we chose $u = x^2$ because $u^2 = \left(x^2\right)^2 = x^4$, which is the first term of the equation. Once you have chosen the u-substitution, you should check that the other variable term is then a multiple of u^2; otherwise, the equation is not quadratic in form.

Example 6.1.16 Solve the equation $e^{2x} - 7e^x + 12 = 0$.

Solution. We use the substitution $u = e^x$, because $u^2 = (e^x)^2 = e^{2x}$. The original equation then becomes

$$\begin{aligned} u^2 - 7u + 12 &= 0 && \text{Factor the left side.} \\ (u-3)(u-4) &= 0 && \text{Apply the zero-factor principle.} \\ u-3 = 0 \quad \text{or} \quad u-4 &= 0 && \text{Solve each equation for} \boldsymbol{u}. \\ u = 3 \quad \text{or} \quad u &= 4 \end{aligned}$$

Finally, we replace u by e^x and solve for x.

$$\begin{aligned} e^x = 3 \qquad &\text{or} \qquad e^x = 4 \\ x = \ln(3) \quad &\text{or} \quad x = \ln(4) \end{aligned}$$

You should verify that the solutions of the original equation are $\ln(3)$ and $\ln(4)$. □

Checkpoint 6.1.17 Solve the equation $10^{2x} - 3 \cdot 10^x + 2 = 0$, and check the solutions.

Answer. $x = 0, \;\; x = \log(2)$

6.1.6 Section Summary

6.1.6.1 Vocabulary

Look up the definitions of new terms in the Glossary.

- Quadratic function
- Zero-factor principle
- Standard form
- Factored form
- Multiplicity
- Monotonic

6.1.6.2 CONCEPTS

1 Quadratic Function.

A **quadratic function** is one that can be written in the form

$$f(x) = ax^2 + bx + c$$

where a, b, and c are constants, and a is not equal to zero.

2 Zero-Factor Principle.

The product of two factors equals zero if and only if one or both of the factors equals zero. In symbols,

$$ab = 0 \quad \text{if and only if} \quad a = 0 \quad \text{or} \quad b = 0$$

3 x-Intercepts of a Graph.

The x-intercepts of the graph of $y = f(x)$ are the solutions of the equation $f(x) = 0$.

4 A quadratic equation written as $ax^2 + bx + c = 0$ is in **standard form**.

A quadratic equation written as $a(x - r_1)(x - r_2) = 0$ is in **factored form**.

5 To Solve a Quadratic Equation by Factoring.

1 Write the equation in standard form.

2 Factor the left side of the equation.

3 Apply the zero-factor principle: Set each factor equal to zero.

4 Solve each equation. There are two solutions (which may be equal).

6 Every quadratic equation has two solutions, which may be the same.

7 The value of the constant a in the factored form of a quadratic equation does not affect the solutions.

8 Each solution of a quadratic equation corresponds to a factor in the factored form.

9 An equation is called **quadratic in form** if we can use a substitution to write it as $au^2 + bu + c = 0$, where u stands for an algebraic expression.

6.1.6.3 STUDY QUESTIONS

1. a Find a pair of numbers whose product is 6. Now find a different pair of numbers whose product is 6. Can you find more such pairs?

   b Find a pair of numbers whose product is 0. What is true about any such pair?

2. Before you begin factoring to solve a quadratic equation, what should you do?

3. How can you find the x-intercepts of the graph of $y = f(x)$ without looking at the graph?

4. How many solutions does a quadratic equation have?

5. a Write a linear equation whose only solution is $x = 3$.

   b Write a quadratic equation whose only solution is $x = 3$.

6. If you know the solutions of $ax^2 + bx + c = 0$, how can you find the solutions of $5(ax^2 + bx + c) = 0$?

7. Is the equation $x^9 - 6x^3 + 8 = 0$ quadratic in form? Why or why not?

8. Delbert says that he can solve the equation $x(x + 5) = 2(x + 5)$ by canceling the factor $(x + 5)$ to get $x = 2$. Comment on his method

6.1.6.4 SKILLS

Practice each skill in the Homework 6.1.7, p. 610 problems listed.

1. Use the zero-factor principle and find x-intercepts: #3–10
2. Solve quadratic equations by factoring: #11–24
3. Use the x-intercepts of the graph to factor a quadratic equation: #25–28, 37–40
4. Write a quadratic equation with given solutions: #29–36
5. Solve applied problems involving quadratic equations: #41–50
6. Solve equations that are quadratic in form: #51–62

6.1.7 Factors and x-intercepts (Homework 6.1)

1. Delbert stands at the top of a 300-foot cliff and throws his algebra book directly upward with a velocity of 20 feet per second. The height of his book above the ground t seconds later is given by the equation

$$h = -16t^2 + 20t + 300$$

where h is in feet.

(a) Use your calculator to make a table of values for the height equation, with increments of 0.5 second.

(b) Graph the height equation on your calculator. Use your table of values to help you choose appropriate window settings.

(c) What is the highest altitude Delbert's book reaches? When does it reach that height? Use the TRACE feature to find approximate answers first. Then use the **Table** feature to improve your estimate.

(d) When does Delbert's book pass him on its way down? (Delbert is standing at a height of 300 feet.) Use the **intersect** command.

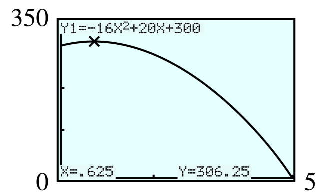
(e) How long will it take Delbert's book to hit the ground at the bottom of the cliff?

Answer.

(a)

t	0	0.5	1	1.5	2	2.5	3	3.5	4	4.5	5
h	300	306	304	294	276	250	216	174	124	66	0

(b)



(c) 306.25 ft at 0.625 sec

(d) 1.25 sec

(e) 5 sec

2. James Bond stands on top of a 240-foot building and throws a film canister upward to a fellow agent in a helicopter 16 feet above the building. The height of the film above the ground t seconds later is given by the formula

$$h = -16t^2 + 32t + 240$$

where h is in feet.

(a) Use your calculator to make a table of values for the height formula, with increments of 0.5 second.

(b) Graph the height formula on your calculator. Use your table of values to help you choose appropriate window settings.

(c) How long will it take the film canister to reach the agent in the helicopter? (What is the agent's altitude?) Use the `TRACE` feature to find approximate answers first. Then use the **Table** feature to improve your estimate.

(d) If the agent misses the canister, when will it pass James Bond on the way down? Use the intersect command.

(e) How long will it take it to hit the ground?

In Problems 3–10, use a graph to solve the equation $y = 0$. (Use Xmin $= -9.4$, Xmax $= 9.4$.) Check your answers with the zero-factor principle.

3. $y = (2x + 5)(x - 2)$

Answer. $\dfrac{-5}{2}$, 2

4. $y = (x + 1)(4x - 1)$

5. $y = x(3x + 10)$

Answer. 0, $\dfrac{-10}{3}$

6. $y = x(3x - 7)$

7. $y = (4x + 3)(x + 8)$

Answer. $\dfrac{-3}{4}$, -8

8. $y = (x - 2)(x - 9)$

9. $y = (x-4)^2$

Answer. 4

10. $y = (x+6)^2$

For Problems 11-24, solve by factoring. (See Algebra Skills Refresher Appendix A.8, p. 912 to review factoring.)

11. $2a^2 + 5a - 3 = 0$

Answer. $\frac{1}{2}, \ -3$

12. $3b^2 - 4b - 4 = 0$

13. $2x^2 = 6x$

Answer. 0, 3

14. $5z^2 = 5z$

15. $3y^2 - 6y = -3$

Answer. 1

16. $4y^2 + 4y = 8$

17. $x(2x-3) = -1$

Answer. $\frac{1}{2}, \ 1$

18. $2x(x-2) = x+3$

19. $t(t-3) = 2(t-3)$

Answer. 2, 3

20. $5(t+2) = t(t+2)$

21. $z(3z+2) = (z+2)^2$

Answer. $-1, \ 2$

22. $(z-1)^2 = 2z^2 + 3z - 5$

23. $(v+2)(v-5) = 8$

Answer. $-3, \ 6$

24. $(w+1)(2w-3) = 3$

In Problems 25–28, graph each set of functions in the standard window.What do you notice about the x-intercepts? Generalize your observation, and test your idea with examples.

25.

(a) $f(x) = x^2 - x - 20$

(b) $g(x) = 2(x^2 - x - 20)$

(c) $h(x) = 0.5(x^2 - x - 20)$

Answer. The 3 graphs have the same x-intercepts. In general, the graph of $y = ax^2 + bx + c$ has the same x-intercepts as the graph of $y = k(ax^2 + bx + c)$.

26.

(a) $f(x) = x^2 + 2x - 15$

(b) $g(x) = 3(x^2 + 2x - 15)$

(c) $h(x) = 0.2(x^2 + 2x - 15)$

27.

(a) $f(x) = x^2 + 6x - 16$

(b) $g(x) = -2(x^2 + 6x - 16)$

(c) $h(x) = -0.1(x^2+6x-16)$

Answer. The 3 graphs have the same x-intercepts. In general, the graph of $y = ax^2 + bx + c$ has the same x-intercepts as the graph of $y = k(ax^2 + bx + c)$.

28.

(a) $f(x) = x^2 - 16$

(b) $g(x) = -1.5(x^2 - 16)$

(c) $h(x) = -0.4(x^2 - 16)$

In Problems 29–36, write a quadratic equation whose solutions are given. The equation should be in standard form with integer coefficients.

29. -2 and 1

Answer.
$x^2 + x - 2 = 0$

30. -4 and 3

31. 0 and -5

Answer.
$x^2 + 5x = 0$

32. 0 and 5

33. -3 and $\dfrac{1}{2}$

Answer.
$2x^2 + 5x - 3 = 0$

34. $\dfrac{-2}{3}$ and 4

35. $\dfrac{-1}{4}$ and $\dfrac{3}{2}$

Answer.
$8x^2 - 10x - 3 = 0$

36. $\dfrac{-1}{3}$ and $\dfrac{-1}{2}$

For problems 37-40, graph the function in the **ZInteger** window, and locate the x-intercepts of the graph. Use the x-intercepts to write the quadratic expression in factored form.

37. $f(x) = 0.1(x^2 - 3x - 270)$

Answer.
$f(x) = 0.1(x - 18)(x + 15)$

38. $h(x) = 0.1(x^2 + 9x - 360)$

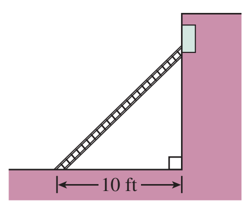
39. $g(x) = -0.08(x^2 + 14x - 576)$

Answer.
$g(x) = -0.08(x - 18)(x + 32)$

40. $F(x) = -0.06(x^2 - 22x - 504)$

Use the Pythagorean theorem to solve Problems 41 and 42. (See Algebra Skills Refresher Appendix A.11, p. 944 to review the Pythagorean theorem.)

41. One end of a ladder is 10 feet from the base of a wall, and the other end reaches a window in the wall. The ladder is 2 feet longer than the height of the window.



(a) Write a quadratic equation about the height of the window.

(b) Solve your equation to find the height of the window.

Answer.

(a) $10^2 + h^2 = (h + 2)^2$ (b) 24 ft

42. The diagonal of a rectangle is 20 inches. One side of the rectangle is 4 inches shorter than the other side.

(a) Write a quadratic equation about the length of the rectangle.

(b) Solve your equation to find the dimensions of the rectangle.

Use the following formula to answer Problems 43 and 44. If an object is thrown into the air from a height s_0 above the ground with an initial velocity v_0, its height t seconds later is given by the formula

$$h = -\frac{1}{2}gt^2 + v_0t + s_0$$

where g is a constant that measures the force of gravity.

43. A tennis ball is thrown into the air with an initial velocity of 16 feet per second from a height of 8 feet. The value of g is 32.

(a) Write a quadratic equation that gives the height of the tennis ball at time t.

(b) Find the height of the tennis ball at $t = \frac{1}{2}$ second and at $t = 1$ second.

(c) Write and solve an equation to answer the question: At what time is the tennis ball 11 feet high?

(d) Use the Table feature on your calculator to verify your answers to parts (b) and (c). (What value of ΔTbl is useful for this problem?)

(e) Graph your equation from part (a) on your calculator. Use your table to help you choose an appropriate window.

(f) If nobody hits the tennis ball, approximately how long will it be in the air?

Answer.

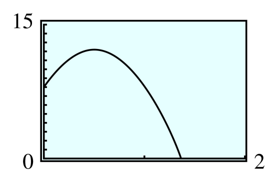
(a) $h = -16t^2 + 16t + 8$

(b) 12 ft; 8 ft

(c) $11 = -16t^2 + 16t + 8$; at $\frac{1}{4}$ sec and $\frac{3}{4}$ sec

(d) $\Delta\text{Tbl} = 0.25$

(e)



(f) 1.37 sec

44. A mountain climber stands on a ledge 80 feet above the ground and tosses a rope down to a companion clinging to the rock face below the ledge. The initial velocity of the rope is −8 feet per second, and the value of g is 32.

(a) Write a quadratic equation that gives the height of the rope at time t.

(b) What is the height of the rope after $\frac{1}{2}$ second? After 1 second?

(c) Write and solve an equation to answer the question: How long does it take the rope to reach the second climber, who is 17 feet above the ground?

(d) Use the Table feature on your calculator to verify your answers to parts (b) and (c). (What value of ΔTbl is useful for this problem?)

(e) Graph your equation from part (a) on your calculator. Use your table to help you choose an appropriate window.

(f) If the second climber misses the rope, approximately how long will the rope take to reach the ground?

For Problems 45 and 46, you may want to review Investigation 9, p. 151, Perimeter and Area, in Chapter 2, p. 149.

45. A rancher has 360 yards of fence to enclose a rectangular pasture. If the pasture should be 8000 square yards in area, what should its dimensions be? We will use 3 methods to solve this problem: a table of values, a graph, and an algebraic equation.

(a) Make a table by hand that shows the areas of pastures of various widths, as shown here.

Width	Length	Area
10	170	1700
⋮	⋮	⋮

(To find the length of each pasture, ask yourself, What is the sum of the length plus the width if there are 360 yards of fence?) Continue the table until you find the pasture whose area is 8000 square yards.

(b) Write an expression for the length of the pasture if its width is x. Next, write an expression for the area, A, of the pasture if its width is x. Graph the equation for A on your calculator, and use the graph to find the pasture of area 8000 square yards.

(c) Write an equation for the area, A, of the pasture in terms of its width x. Solve your equation algebraically for $A = 8000$. Explain why there are two solutions.

Answer.

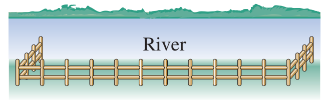
(a)

Width	Length	Area
10	170	1700
20	160	3200
30	150	4500
40	140	5600
50	130	6500
60	120	7200
70	110	7700
80	100	8000

(b) $l = 180 - x$, $A = 180x - x^2$; 80 yd by 100 yd

(c) $180x - x^2 = 8000$, 80 yd by 100 yd, or 100 yd by 80 yd. There are two solutions because the pasture can be oriented in two directions.

46. If the rancher in Problem 45 uses a riverbank to border one side of the pasture as shown in the figure, he can enclose 16, 000 square yards with 360 yards of fence. What will the dimensions of the pasture be then? We will use three methods to solve this problem: a table of values, a graph, and an algebraic equation.



(a) Make a table by hand that shows the areas of pastures of various widths, as shown here.

Width	Length	Area
10	340	3400
20	320	6400
⋮	⋮	⋮

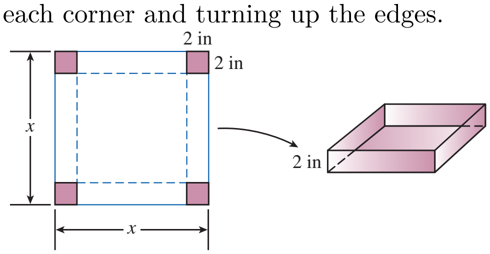
(Be careful computing the length of the pasture: Remember that one side of the pasture does not need any fence!) Continue the table until you find the pasture whose area is 16,000 square yards.

(b) Write an expression for the length of the pasture if its width is x. Next, write an expression for the area, A, of the pasture if its width is x. Graph the equation for A, and use the graph to find the pasture of area 16,000 square yards.

(c) Write an equation for the area, A, of the pasture in terms of its width x. Solve your equation algebraically for $A = 16,000$.

For Problems 47 and 48, you will need the formula for the volume of a box.

47. A box is made from a square piece of cardboard by cutting 2-inch squares from each corner and turning up the edges.



(a) If the piece of cardboard is x inches square, write expressions for the length, width, and height of the box. Then write an expression for the volume, V, of the box in terms of x.

(b) Use your calculator to make a table of values showing the volumes of boxes made from cardboard squares of side 4 inches, 5 inches, and so on.

(c) Graph your expression for the volume on your calculator. What happens to V as x increases?

(d) Use your table or your graph to find what size cardboard you need to make a box with volume 50 cubic inches.

(e) Write and solve a quadratic equation to answer part (d).

Answer.

(a) $l = x - 4, \quad w = x - 4, \quad h = 2, \quad V = 2(x-4)^2$

(b)

x	4	5	6	7	8	9	10
V	0	2	8	18	32	50	72

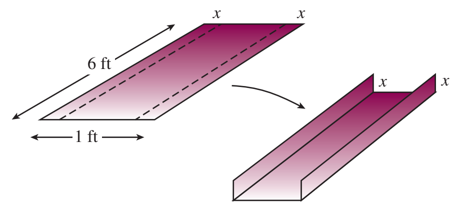
(c) As x increases, V increases.

(d) 9 inches by 9 inches.

(e) $2(x-4)^2 = 50,\ \ x = 9$

48. A length of rain gutter is made from a piece of aluminum 6 feet long and 1 foot wide.

(a) If a strip of width x is turned up along each long edge, write expressions for the length, width, and height of the gutter. Then write an expression for the volume, V, of the gutter in terms of x.



(b) Use your calculator to make a table of values showing the volumes of various rain gutters formed by turning up edges of 0.1 foot, 0.2 foot, and so on.

(c) Graph your expression for the volume. What happens to V as x increases?

(d) Use your table or your graph to discover how much metal should be turned up along each long edge so that the gutter has a capacity of $\frac{3}{4}$ cubic foot of rainwater.

(e) Write and solve a quadratic equation to answer part (d).

Problems 49 and 50 deal with wildlife management. The annual increase, I, in a population often depends on the size x of the population, according to the formula

$$I = kCx - kx^2$$

where k and C are constants related to the fertility of the population and the availability of food.

49. The annual increase, $f(x)$, in the deer population in a national park is given by

$$f(x) = 1.2x - 0.0002x^2$$

where x is the size of the population that year.

(a) Make a table of values for $f(x)$ for $0 \le x \le 7000$. Use increments of 500 in x.

(b) How much will a population of 2000 deer increase? A population of 5000 deer? A population of 7000 deer?

(c) Use your calculator to graph the annual increase, $f(x)$, versus the size of the population, x, for $0 \le x \le 7000$.

(d) What do the x-intercepts tell us about the deer population?

(e) Estimate the population size that results in the largest annual increase. What is that increase?

Answer.

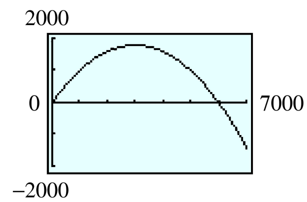
(a)

x	0	500	1000	1500	2000	2500	3000	3500
I	0	550	1000	1350	1600	1750	1800	1750

x	4000	4500	5000	5500	6000	6500	7000
I	1600	1350	1000	550	0	-650	-1400

(b) 1600, 1000, -1400

(c)



(d) No increase

(e) 3000; 1800

50. Commercial fishermen rely on a steady supply of fish in their area. To avoid overfishing, they adjust their harvest to the size of the population. The function

$$g(x) = 0.4x - 0.0001x^2$$

gives the annual rate of growth, in tons per year, of a fish population of biomass x tons.

(a) Make a table of values for $g(x)$ for $0 \le x \le 5000$. Use increments of 500 in x.

(b) How much will a population of 1000 tons increase? A population of 3000 tons? A population of 5000 tons?

(c) Use your calculator to graph the annual increase, $g(x)$, versus the size of the population, x, for $0 \le x \le 5000$.

(d) What do the x-intercepts tell us about the fish population?

(e) Estimate the population size that results in the largest annual increase. What is that increase?

For Problems 51-62, use a substitution to solve the equation.

51. $a^4 + a^2 - 2 = 0$

Answer. ± 1

52. $t^6 - t^3 - 6 = 0$

53. $4b^6 - 3 = b^3$

Answer. $\sqrt[3]{-3/4}$, 1

54. $3x^4 + 1 = 4x^2$

55. $c^{2/3} + 2c^{1/3} - 3 = 0$

Answer. -27, 1

56. $y^{1/2} - 3y^{1/4} - 4 = 0$

57. $10^{2w} - 5 \cdot 10^w + 6 = 0$

Answer. $\log(2)$, $\log(3)$

58. $e^{2x} - 5e^x + 4 = 0$

59. $5^{2t} - 30 \cdot 5^t + 125 = 0$

Answer. 1, 2

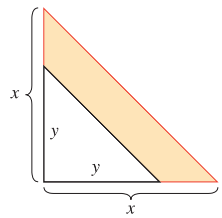
60. $e^{4r} - 3e^{2r} + 2 = 0$

61. $\dfrac{1}{m^2} + \dfrac{5}{m} - 6 = 0$

Answer. $\dfrac{-1}{6}$, 1

62. $\dfrac{1}{s^2} + \dfrac{4}{s} - 5 = 0$

63. The sail in the figure is a right triangle of base and height x. It has a colored stripe along the hypotenuse and a white triangle of base and height y in the lower corner.



(a) Write an expression for the area of the colored stripe.

(b) Express the area of the stripe in factored form.

(c) If the sail is $7\frac{1}{2}$ feet high and the white strip is $4\frac{1}{2}$ feet high, use your answer to (b) to calculate mentally the area of the stripe.

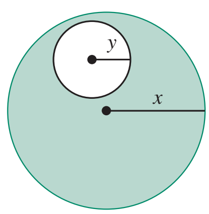
Answer.

(a) $A = \dfrac{1}{2}(x^2 - y^2)$

(b) $A = \dfrac{1}{2}(x - y)(x + y)$

(c) 18 sq ft

64. An hors d'oeuvres tray has radius x, and the dip container has radius y, as shown in the figure.



(a) Write an expression for the area for the chips (shaded region).

(b) Express the area in factored form.

(c) If the tray has radius $8\frac{1}{2}$ inches and the space for the dip has radius $2\frac{1}{2}$ inches, use your answer to part (b) to calculate mentally the area for chips. (Express your answer as a multiple of π.)

6.2 Solving Quadratic Equations

Not every quadratic equation can be solved by factoring or by extraction of roots. For example, the expression x^2+x-1 cannot be factored, so the equation $x^2 + x - 1 = 0$ cannot be solved by factoring. For other equations, factoring may be difficult. In this section we learn two methods that can be used to solve any quadratic equation.

6.2.1 Quadratic Formula

Instead of completing the square every time we solve a new quadratic equation, we can complete the square on the general quadratic equation,

$$ax^2 + bx + c = 0, \quad a \neq 0$$

and obtain a formula for the solutions of any quadratic equation.

The Quadratic Formula.

The solutions of the equation $ax^2 + bx + c = 0, \quad a \neq 0$, are

$$\boldsymbol{x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}}$$

This formula expresses the solutions of a quadratic equation in terms of its coefficients. (The proof of the formula is considered in the Homework problems.) The symbol $\pm$, read plus or minus, is used to combine the two equations

$$x = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \text{and} \quad x = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

into a single equation.

To solve a quadratic equation using the quadratic formula, all we have to do is substitute the coefficients a, b, and c into the formula.

Example 6.2.1 Solve $2x^2 + 1 = 4x$.

Solution. Write the equation in standard form as

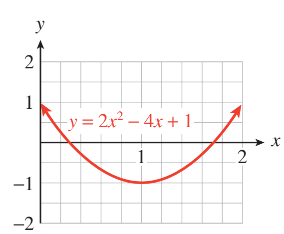
$$2x^2 - 4x + 1 = 0$$

Substitute **2** for a, **−4** for b, and **1** for c into the quadratic formula, then simplify.

$$\begin{aligned} x &= \frac{-(\mathbf{-4}) \pm \sqrt{(\mathbf{-4})^2 - 4(\mathbf{2})(\mathbf{1})}}{2(\mathbf{2})} \\ &= \frac{4 \pm \sqrt{8}}{4} \end{aligned}$$

Using a calculator, we find that the solutions are approximately 1.707 and 0.293.

We can also verify that the x-intercepts of the graph of $y = 2x^2 - 4x + 1$ are approximately 1.707 and 0.293, as shown below.



□

Checkpoint 6.2.2 Use the quadratic formula to solve $x^2 - 3x = 1$.

Answer. $x = \dfrac{3 \pm \sqrt{13}}{2}$

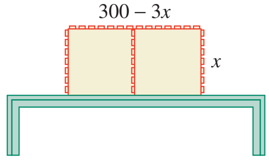
6.2.2 Applications

We have now seen four different algebraic methods for solving quadratic equations:

1. Factoring
2. Extraction of roots
3. Completing the square
4. Quadratic formula

Factoring and extraction of roots are relatively fast and simple, but they do not work on all quadratic equations. The quadratic formula will work on any quadratic equation.

Example 6.2.3 The owners of a day-care center plan to enclose a divided play area against the back wall of their building, as shown below. They have 300 feet of picket fence and would like the total area of the playground to be 6000 square feet. Can they enclose the playground with the fence they have, and if so, what should the dimensions of the playground be?



Solution. Suppose the width of the play area is x feet. Because there are three sections of fence along the width of the play area, that leaves $300 - 3x$ feet of fence for its length. The area of the play area should be 6000 square feet, so we have the equation

$$x(300 - 3x) = 6000$$

This is a quadratic equation. In standard form,

$$\begin{aligned} 3x^2 - 300x + 6000 &= 0 \quad \text{Divide each term by 3.} \\ x^2 - 100x + 2000 &= 0 \end{aligned}$$

The left side cannot be factored, so we use the quadratic formula with $a = \mathbf{1}$, $b = \mathbf{-100}$, and $c = \mathbf{2000}$.

$$\begin{aligned} x &= \frac{-(\mathbf{-100}) \pm \sqrt{(\mathbf{-100})^2 - 4(\mathbf{1})(\mathbf{2000})}}{2(\mathbf{1})} \\ &= \frac{100 \pm \sqrt{2000}}{2} \approx \frac{100 \pm 44.7}{2} \end{aligned}$$

Simplifying the last fraction, we find that $x \approx 72.35$ or $x \approx 27.65$. Both values give solutions to the problem.

- If the width of the play area is 72.35 feet, then the length is $300-3(72.35)$, or 82.95 feet.
- If the width is 27.65 feet, the length is $300 - 3(27.65)$, or 217.05 feet.

□

Checkpoint 6.2.4 In Investigation 40, p. 600, we considered the height of a baseball, given by the equation

$$h = -16t^2 + 64t + 4$$

Find two times when the ball is at a height of 20 feet. Give your answers to two decimal places.

Answer. 0.27 sec, 3.73 sec

Sometimes it is useful to solve a quadratic equation for one variable in terms of the others.

Example 6.2.5 Solve $x^2 - xy + y = 2$ for x in terms of y.

Solution. We first write the equation in standard form as a quadratic equation in the variable x.

$$x^2 - yx + (y - 2) = 0$$

Expressions in y are treated as constants with respect to x, so that $a = \mathbf{1}$, $b = \mathbf{-y}$, and $c = \mathbf{y - 2}$. Substitute these expressions into the quadratic formula.

$$\begin{aligned} x &= \frac{-(\mathbf{-y}) \pm \sqrt{(\mathbf{-y})^2 - 4(\mathbf{1})(\mathbf{y - 2})}}{2(\mathbf{1})} \\ &= \frac{y \pm \sqrt{y^2 - 4y + 8}}{2} \end{aligned}$$

□

Checkpoint 6.2.6 Solve $2x^2 + kx + k^2 = 1$ for x in terms of k.

Answer. $x = \dfrac{-k \pm \sqrt{8 - 7k^2}}{4}$

6.2.3 Introduction to complex numbers

You know that not all quadratic equations have real solutions.

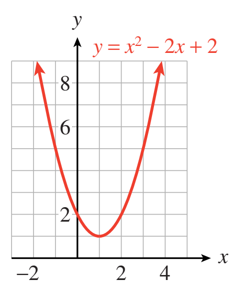
For example, the graph of

$$f(x) = x^2 - 2x + 2$$

has no x-intercepts (as shown at right), and the equation

$$x^2 - 2x + 2 = 0$$

has no real solutions.



We can still use completing the square or the quadratic formula to solve the equation.

Example 6.2.7 Solve the equation $x^2 - 2x + 2 = 0$ by using the quadratic formula.

Solution. We substitute $a = 1$, $b = -2$, and $c = 2$ into the quadratic formula to get

$$x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(2)}}{2(1)} = \frac{2 \pm \sqrt{-4}}{2}$$

Because $\sqrt{-4}$ is not a real number, the equation $x^2 - 2x + 2 = 0$ has no real solutions. □

Checkpoint 6.2.8 Solve the equation $x^2 - 6x + 13 = 0$ by using the quadratic formula.

Answer. $x = \dfrac{6 \pm \sqrt{-16}}{2}$

6.2.3.1 Imaginary Numbers

Although square roots of negative numbers such as $\sqrt{-4}$ are not real numbers, they occur often in mathematics and its applications.

Mathematicians began working with square roots of negative numbers in the sixteenth century, in their attempts to solve quadratic and cubic equations. René Descartes gave them the name imaginary numbers, which reflected the mistrust with which mathematicians regarded them at the time. Today, however, such numbers are well understood and used routinely by scientists and engineers.

We begin by defining a new number, i, whose square is -1.

Imaginary Unit.

We define the **imaginary unit** i by

$$i^2 = -1 \quad \text{or} \quad i = \sqrt{-1}$$

Caution 6.2.9 The letter i used in this way is not a variable; it is the name of a specific number and hence is a constant.

The square root of any negative number can be written as the product of a real number and i. For example,

$$\begin{aligned}\sqrt{-4} &= \sqrt{-1 \cdot 4}\\ &= \sqrt{-1}\sqrt{4} = i \cdot 2\end{aligned}$$

or $\sqrt{-4} = 2i$. Any number that is the product of i and a real number is called an **imaginary number**.

Imaginary Numbers.

For $a > 0$,

$$\sqrt{-a} = \sqrt{-1} \cdot \sqrt{a} = i\sqrt{a}$$

Examples of imaginary numbers are

$$3i, \quad \frac{7}{8}i, \quad -38i, \quad \text{and} \quad i\sqrt{5}$$

Example 6.2.10 Write each radical as an imaginary number.

a $\sqrt{-25}$

b $2\sqrt{-3}$

Solution.

a

$$\begin{aligned}\sqrt{-25} &= \sqrt{-1}\sqrt{25}\\ &= i\sqrt{25} = 5i\end{aligned}$$

b

$$\begin{aligned}2\sqrt{-3} &= 2\sqrt{-1}\sqrt{3}\\ &= 2i\sqrt{3}\end{aligned}$$

□

Checkpoint 6.2.11 Write each radical as an imaginary number.

a $\sqrt{-18}$

b $-6\sqrt{-5}$

Answer.

a $3i\sqrt{2}$

b $-6i\sqrt{5}$

Note 6.2.12 Every negative real number has two imaginary square roots, $i\sqrt{a}$ and $-i\sqrt{a}$, because

$$\left(i\sqrt{a}\right)^2 = i^2(\sqrt{a})^2 = -a$$

and

$$\left(-i\sqrt{a}\right)^2 = (-i)^2(\sqrt{a})^2 = -a$$

For example, the two square roots of -9 are $3i$ and $-3i$.

6.2.3.2 Complex Numbers

Consider the quadratic equation

$$x^2 - 2x + 5 = 0$$

Using the quadratic formula to solve the equation, we find

$$x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(5)}}{2} = \frac{2 \pm \sqrt{-16}}{2}$$

If we now replace $\sqrt{-16}$ with $4i$, we have

$$x = \frac{2 \pm 4i}{2} = 1 \pm 2i$$

The two solutions are $1+2i$ and $1-2i$. These numbers are examples of **complex numbers**.

Complex Numbers.

A **complex number** can be written in the form $a + bi$, where a and b are real numbers.

Examples of complex numbers are

$$3 - 5i, \quad 2 + \sqrt{7}i, \quad \frac{4-i}{3}, \quad 6i, \quad \text{and } -9$$

In a complex number $a + bi$, a is called the **real part**, and b is called the **imaginary part**. All real numbers are also complex numbers (with the imaginary part equal to zero). A complex number whose real part equals zero is called a **pure imaginary** number.

Example 6.2.13 Write the solutions to Example 7.3.1, p. 735, $\dfrac{2 \pm \sqrt{-4}}{2}$, as complex numbers.

Solution. Because $\sqrt{-4} = \sqrt{-1}\sqrt{4} = 2i$, we have $\dfrac{2 \pm \sqrt{-4}}{2} = \dfrac{2 \pm 2i}{2}$, or $1 \pm i$. The solutions are $1 + i$ and $1 - i$. □

Checkpoint 6.2.14 Use extraction of roots to solve $(2x + 1)^2 + 9 = 0$. Write your answers as complex numbers.

Answer. $x = \dfrac{-1}{2} \pm \dfrac{3}{2}i$

6.2.4 Arithmetic of Complex Numbers

All the properties of real numbers listed in Algebra Skills Refresher Section A.13, p. 970 are also true of complex numbers. We can carry out arithmetic operations with complex numbers.

We add and subtract complex numbers by combining their real and imaginary parts separately. For example,

$$\begin{aligned}(4+5i)+(2-3i) &= (4+2)+(5-3)i \\ &= 6+2i\end{aligned}$$

Sums and Differences of Complex Numbers.

$$(a+bi)+(c+di) = (a+c)+(b+d)i$$

$$(a+bi)-(c+di) = (a-c)+(b-d)i$$

Example 6.2.15 Subtract: $(8-6i)-(5+2i)$.

Solution. Combine the real and imaginary parts.

$$\begin{aligned}(8-6i)-(5+2i) &= (8-5)+(-6-2)i \\ &= 3+(-8)i = 3-8i\end{aligned}$$

□

Checkpoint 6.2.16 Subtract: $(-3+2i)-(-3-2i)$.

Answer. $4i$

6.2.5 Section Summary

6.2.5.1 Vocabulary

Look up the definitions of new terms in the Glossary.

- Quadratic trinomial
- Complete the square
- Quadratic formula

6.2.5.2 CONCEPTS

1 The square of the binomial is a **quadratic trinomial**,

$$(x+p)^2 = x^2+2px+p^2$$

2 **To Solve a Quadratic Equation by Completing the Square.**

1
  a Write the equation in standard form.
  b Divide both sides of the equation by the coefficient of the quadratic term, and subtract the constant term from both sides.

2 Complete the square on the left side:
  a Multiply the coefficient of the first-degree term by one-half, then square the result.

b Add the value obtained in (a) to both sides of the equation.

3 Write the left side of the equation as the square of a binomial. Simplify the right side.

4 Use extraction of roots to finish the solution.

3 **The Quadratic Formula.**

The solutions of the equation $ax^2 + bx + c = 0, \quad a \neq 0$, are

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

4 We have four methods for solving quadratic equations: extracting of roots, factoring, completing the square, and using the quadratic formula. The first two methods are faster, but they don't work on all equations. The last two methods work on any quadratic equation.

6.2.5.3 STUDY QUESTIONS

1 Name four algebraic methods for solving a quadratic equation.

2 Give an example of a quadratic trinomial that is the square of a binomial.

3 What number must be added to $x^2 - 26x$ to make it the square of a binomial?

4 After completing the square, how do we finish solving the quadratic equation?

5 What is the first step in solving the equation $2x^2 - 6x = 5$ by completing the square?

6.2.5.4 SKILLS

Practice each skill in the Homework 6.2.6, p. 626 problems listed.

1 Solve quadratic equations by completing the square: #3–24

2 Solve quadratic equations by using the quadratic formula: #27–36

3 Solve problems by writing and solving quadratic equations: #37–44

4 Solve formulas: #45–64

6.2.6 Solving Quadratic Equations (Homework 6.2)

For Problems 1-2, complete the square and write the result as the square of a binomial.

1.

(a) $x^2 + 8x$ (b) $x^2 - 7x$ (c) $x^2 + \frac{3}{2}x$ (d) $x^2 - \frac{4}{5}x$

Answer.

(a) $(x+4)^2$ (b) $\left(x-\frac{7}{2}\right)^2$ (c) $\left(x+\frac{3}{4}\right)^2$ (d) $\left(x-\frac{2}{5}\right)^2$

2. (a) x^2-14x (b) x^2+3x (c) $x^2-\frac{5}{2}x$ (d) $x^2+\frac{2}{3}x$

For Problems 3-18, solve by completing the square.

3. $x^2-2x+1=0$

Answer. 1

4. $x^2+4x+4=0$

5. $x^2+9x+20=0$

Answer. $-4, \quad -5$

6. $x^2-x-20=0$

7. $x^2=3-3x$

Answer. $\frac{3}{2}\pm\sqrt{\frac{21}{4}}=\frac{-3\pm\sqrt{21}}{2}$

8. $x^2=5-5x$

9. $2x^2+4x-3=0$

Answer. $-1\pm\sqrt{\frac{5}{2}}$

10. $3x^2+12x+2=0$

11. $3x^2+x=4$

Answer. $\frac{-4}{3}, \ 1$

12. $4x^2+6x=3$

13. $4x^2-3=2x$

Answer. $\frac{1}{4}\pm\sqrt{\frac{13}{16}}=\frac{1\pm\sqrt{13}}{4}$

14. $2x^2-5=3x$

15. $3x^2-x-4=0$

Answer. $-1, \ \frac{4}{3}$

16. $2x^2-x-3=0$

17. $5x^2+8x=4$

Answer. $-2, \ \frac{2}{5}$

18. $9x^2-12x-5=0$

For Problems 19-24, solve by completing the square. Your answers will involve a, b, or c.

19. $x^2+2x+c=0$

Answer. $-1\pm\sqrt{1-c}$

20. $x^2-4x+c=0$

21. $x^2+bx+1=0$

Answer. $-\frac{b}{2}\pm\sqrt{\frac{b^2-4}{4}}=\frac{-b\pm\sqrt{b^2-4}}{2}$

22. $x^2+bx-4=0$

23. $ax^2+2x-4=0$

Answer. $\frac{-1\pm\sqrt{4a+1}}{a}$

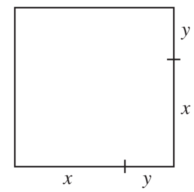
24. $ax^2-4x+9=0$

25.

(a) Write an expression for the area of the square in the figure.

(b) Express the area as a polynomial.

(c) Divide the square into four pieces whose areas are given by the terms of your answer to part (b).

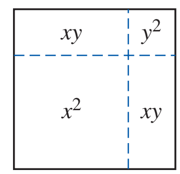


Answer.

(a) $A = (x + y)^2$

(b) $A = x^2 + 2xy + y^2$

(c) x^2, xy, xy, y^2

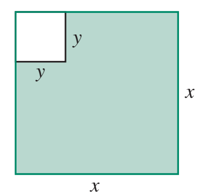


26.

(a) Write an expression for the area of the shaded region in the figure.

(b) Express the area in factored form.

(c) By making one cut in the shaded region, rearrange the pieces into a rectangle whose area is given by your answer to part (b).



For Problems 23-36, solve using the quadratic formula. Round your answers to three decimal places.

27. $x^2 - x - 1 = 0$

Answer. 1.618, − 0.618

28. $x^2 + x + 1 = 0$

29. $y^2 + 2y = 5$

Answer. 1.449, − 3.449

30. $y^2 - 4y = -4$

31. $3z^2 = 4.2z + 1.5$

Answer. 1.695, − 0.295

32. $2z^2 = 7.5z - 6.3$

33. $0 = x^2 - \frac{5}{3}x + \frac{1}{3}$

Answer. 1.434, 0.232

34. $0 = -x^2 + \frac{5}{2}x - \frac{1}{2}$

35. $-5.2z^2 + 176z + 1218 = 0$

Answer. −5.894, 39.740

36. $15z^2 - 18z - 2750 = 0$

37. A car traveling at s miles per hour on a dry road surface requires approximately d feet to stop, where d is given by the function

$$d = f(s) = \frac{s^2}{24} + \frac{s}{2}$$

(a) Make a table showing the stopping distance, d, for speeds of 10, 20, . . . , 100 miles per hour. (Use the **Table** feature of your calculator.)

(b) Graph the function for d in terms of s. Use your table values to help you choose appropriate window settings.

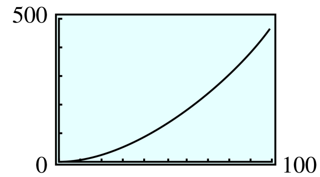
(c) Write and solve an equation to answer the question: If a car must be able to stop in 50 feet, what is the maximum safe speed it can travel?

Answer.

(a)

s	10	20	30	40	50	60	70	80	90	100
d	9	27	53	87	129	180	239	307	383	467

(b)



(c) $\frac{s^2}{24} + \frac{s}{2} = 50$; 29.16 mph

38. A car traveling at s miles per hour on a wet road surface requires approximately d feet to stop, where d is given by the function

$$d = f(s) = \frac{s^2}{12} + \frac{s}{2}$$

(a) Make a table showing the stopping distance, d, for speeds of 10, 20, ... , 100 miles per hour. (Use the **Table** feature of your calculator.)

(b) Graph the function for d in terms of s. Use your table values to help you choose appropriate window settings.

(c) Insurance investigators at the scene of an accident find skid marks 100 feet long leading up to the point of impact. Write and solve an equation to discover how fast the car was traveling when it put on the brakes. Verify your answer on your graph.

39. A skydiver jumps out of an airplane at 11,000 feet. While she is in free-fall, her altitude in feet t seconds after jumping is given by the function

$$h = f(t) = -16t^2 - 16t + 11{,}000$$

(a) Make a table of values showing the skydiver's altitude at 5-second intervals after she jumps from the airplane. (Use the **Table** feature of your calculator.)

(b) Graph the function. Use your table of values to choose appropriate window settings.

(c) If the skydiver must open her parachute at an altitude of 1000 feet, how long can she free-fall? Write and solve an equation to find the answer.

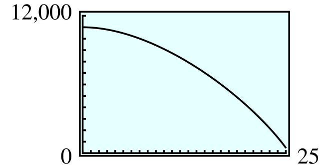
(d) If the skydiver drops a marker just before she opens her parachute, how long will it take the marker to hit the ground? (*Hint*: The marker continues to fall according to the equation given above.)

(e) Find points on your graph that correspond to your answers to parts (c) and (d).

Answer.

(a)

t	0	5	10	15	20	25
h	11,000	10,520	9240	7160	4280	600

(b) 

(c) $-16t^2 - 16t + 11,000 = 1000$; 24.5 sec

(d) 1.2 sec

40. A high diver jumps from the 10-meter springboard. His height in meters above the water t seconds after leaving the board is given by the function

$$h = f(t) = -4.9t^2 + 8t + 10$$

(a) Make a table of values showing the diver's altitude at 0.25-second intervals after he jumps from the airplane. (Use the **Table** feature of your calculator.)

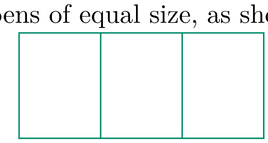
(b) Graph the function. Use your table of values to choose appropriate window settings.

(c) How long is it before the diver passes the board on the way down?

(d) How long is it before the diver hits the water?

(e) Find points on your graph that correspond to your answers to parts (c) and (d).

41. A dog trainer has 100 meters of chain link fence. She wants to enclose 250 square meters in three pens of equal size, as shown in the figure.



(a) Let l and w represent the length and width, respectively, of the entire area. Write an equation about the amount of chain link fence.

(b) Solve your equation for l in terms w.

(c) Write and solve an equation in w for the total area enclosed.

(d) Find the dimensions of each pen.

Answer.

(a) $2l + 4w = 100$

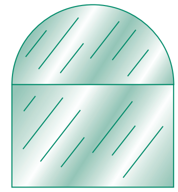
(b) $l = 50 - 2w$

(c) $w(50 - 2w) = 250$; $w = 6.91, \; 18.09$

(d) 12.06 m by 6.91 m, or 4.61 m by 18.09 m

42.

An architect is planning to include a rectangular window topped by a semicircle in his plans for a new house, as shown in the figure. In order to admit enough light, the window should have an area of 120 square feet. The architect wants the rectangular portion of the window to be 2 feet wider than it is tall.

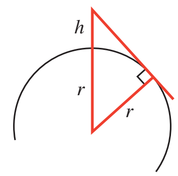


(a) Let x stand for the horizontal width of the window. Write expressions for the height of the rectangular portion and for the radius of the semicircular portion.

(b) Write an expression for the total area of the window.

(c) Write and solve an equation to find the width and overall height of the window.

43. When you look down from a height, say a tall building or a mountain peak, your line of sight is tangent to the Earth at the horizon, as shown in the figure.



(a) Suppose you are standing on top of the Petronas Tower in Kuala Lumpur, 1483 feet high. How far can you see on a clear day? (You will need to use the Pythagorean theorem and the fact that the radius of the Earth is 3960 miles. Do not forget to convert the height of the Petronas Tower to miles.)

(b) How tall a building should you stand on in order to see 100 miles?

Answer.

(a) 47.2 mi (b) 1.26 mi

44.

(a) If the radius of the Earth is 6370 kilometers, how far can you see from an airplane at an altitude of 10,000 meters? (*Hint*: See Problem 43.)

(b) b. How high would the airplane have to be in order for you to see a distance of 10 kilometers?

For Problems 45-52, use the quadratic formula to solve each equation for the indicated variable.

45. $A = 2w^2 + 4lw$, for w

Answer.
$$w = \frac{-4l \pm \sqrt{16l^2 + 8A}}{4} = \frac{-2l \pm \sqrt{4l^2 + 2A}}{2}$$

46. $A = \pi r^2 + \pi rs$, for r

47. $h = 4t - 16t^2$, for t

Answer.
$$t = \frac{4 \pm \sqrt{16 + 64h}}{32} = \frac{1 \pm \sqrt{1 + 4h}}{8}$$

48. $P = IE - RI^2$, for I

49. $s = vt - \dfrac{1}{2}at^2, \quad \text{for } t$

Answer.
$t = \dfrac{v \pm \sqrt{v^2 - 2as}}{a}$

50. $S = \dfrac{n^2 + n}{2}, \quad \text{for } n$

51. $3x^2 + xy + y^2 = 2, \quad \text{for } y$

Answer.
$y = \dfrac{-x \pm \sqrt{8 - 11x^2}}{2}$

52. $y^2 - 3xy + x^2 = 3, \quad \text{for } x$

For Problems 53-60, solve for y in terms of x. Use whichever method of solution seems easiest.

53. $x^2y - y^2 = 0$

Answer. $0, \ x^2$

54. $x^2y^2 - y = 0$

55. $(2y + 3x)^2 = 9$

Answer.
$\dfrac{-3x \pm 3}{2}$

56. $(3y - 2x)^2 = 4$

57. $4x^2 - 9y^2 = 36$

Answer.
$\dfrac{\pm\sqrt{4x^2 - 36}}{3} = \dfrac{\pm 2\sqrt{x^2 - 9}}{3}$

58. $9x^2 + 4y^2 = 36$

59. $4x^2 - 25y^2 = 0$

Answer. $\dfrac{\pm 2x}{5}$

60. $(2x - 5y)^2 = 0$

For Problems 61-66, solve the formula for the indicated variable.

61. $V = \pi(r - 3)^2h, \quad \text{for } r$

Answer. $3 \pm \sqrt{\dfrac{V}{\pi h}}$

62. $A = P(1 + r)^2, \quad \text{for } P$

63. $E = \dfrac{1}{2}mv^2 + mgh, \quad \text{for } v$

Answer. $\pm\sqrt{\dfrac{2(E - mgh)}{m}}$

64. $h = \dfrac{1}{2}gt^2 + dl, \quad \text{for } t$

65. $V = 2(s^2 + t^2)w, \quad \text{for } t$

Answer. $\pm\sqrt{\dfrac{V}{2w} - s^2}$

66. $V = \pi(r^2 + R^2)h, \quad \text{for } R$

67. What is the sum of the two solutions of the quadratic equation $ax^2+bx+c = 0$?

Hint. The two solutions are given by the quadratic formula.

Answer. $\dfrac{-b}{2a}$

68. What is the product of the two solutions of the quadratic equation $ax^2 + bx + c = 0$?

Hint. Do *not* try to multiply the two solutions given by the quadratic formula! Think about the factored form of the equation

In Problems 69 and 70, we prove the quadratic formula.

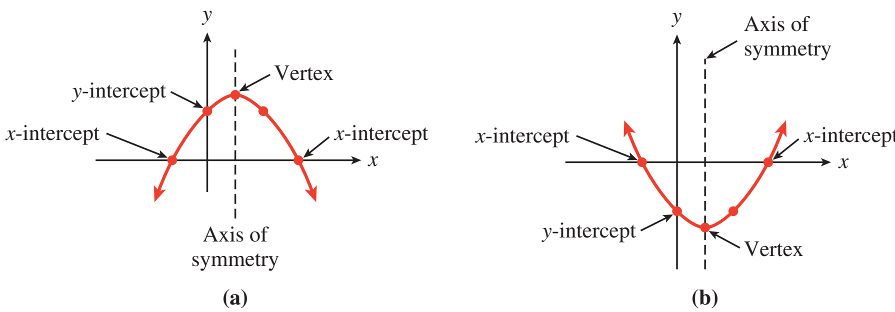
69. Complete the square to find the solutions of the equation $x^2+bx+c = 0$. (Your answers will be expressions in b and c.)

Answer. $\dfrac{-b \pm \sqrt{b^2 - 4c}}{2}$

70. Complete the square to find the solutions of the equation $ax^2+bx+c = 0$. (Your answers will be expressions in a, b, and c.)

6.3 Graphing Parabolas

The graph of a quadratic function $f(x) = ax^2 + bx + c$ is called a **parabola**. Some parabolas are shown below.



All these parabolas share certain features.

- The graph has either a highest point (if the parabola opens downward, as in figure (a) or a lowest point (if the parabola opens upward, as in figure (b). This high or low point is called the **vertex** of the graph.
- The parabola is symmetric about a vertical line, called the **axis of symmetry**, that runs through the vertex.
- The y-intercept is the point where the parabola intersects the y-axis. The graph of a quadratic function always has exactly one **y-intercept**.
- However, the graph may cross the x-axis at one point, at two points, or not at all. Points where the parabola intersects the x-axis are called the **x-intercepts**. If there are two x-intercepts, they are equidistant from the axis of symmetry.
- The values of the constants a, b, and c determine the location and orientation of the parabola. We will begin by considering each of these constants separately.

6.3.1 The Graph of $y = ax^2$

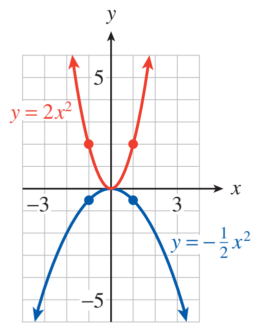
In Chapter 2, p. 149, we saw that the graph of $y = af(x)$ is a transformation of the graph of $y = f(x)$. The scale factor, a, stretches or compresses the graph vertically, and if a is negative, the graph is reflected about the x-axis.

Example 6.3.1 Sketch a graph of each quadratic function by hand.

a $y = 2x^2$ b $y = -\dfrac{1}{2}x^2$

Solution. Both functions are of the form $y = ax^2$. The graph of $y = 2x^2$ opens upward because $a = 2 > 0$, and the graph of $y = -\dfrac{1}{2}x^2$ opens downward because $a = -\frac{1}{2} < 0$.

To make a reasonable sketch by hand, it is enough to plot a few *guidepoints*; the points with x-coordinates 1 and -1 are easy to compute.



x	$y = 2x^2$	$y = -\frac{1}{2}x^2$
-1	2	$-\frac{1}{2}$
0	0	0
1	2	$-\frac{1}{2}$

We sketch parabolas through each set of guidepoints, as shown at left.

□

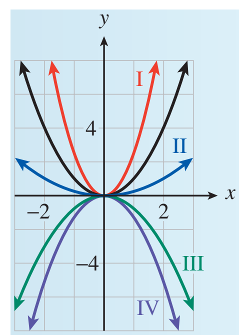
Checkpoint 6.3.2 Match each parabola in the figure at right with its equation. The basic parabola is shown in black.

a $y = -\frac{3}{4}x^2$

b $y = \frac{1}{4}x^2$

c $y = \frac{5}{2}x^2$

d $y = -\frac{5}{4}x^2$



Answer.

a III b II c I d IV

6.3.2 The Graph of $y = x^2 + c$

Next, we consider the effect of the constant term, c, on the graph. Adding a constant c to the formula for $y = f(x)$ causes a vertical translation of the graph.

Example 6.3.3 Sketch graphs for the following quadratic functions.

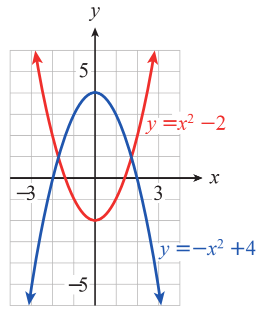
a $y = x^2 - 2$ b $y = -x^2 + 4$

Solution.

a The graph of $y = x^2 - 2$ is shifted downward by two units, compared to the basic parabola. The vertex is the point $(0, -2)$ and the x-intercepts are the solutions of the equation

$$0 = x^2 - 2$$

or $\sqrt{2}$ and $-\sqrt{2}$. The graph is shown below.



b The graph of $y = -x^2 + 4$ opens downward and is shifted 4 units up, compared to the basic parabola. Its vertex is the point $(0, 4)$. Its x-intercepts are the solutions of the equation

$$0 = -x^2 + 4$$

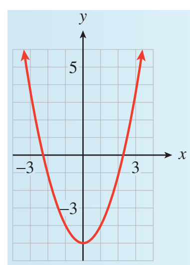
or 2 and -2. You can verify both graphs with your graphing calculator.

□

Checkpoint 6.3.4

a Find an equation for the parabola shown below.

b Give the x- and y-intercepts of the graph.



Answer.

a $y = x^2 - 5$

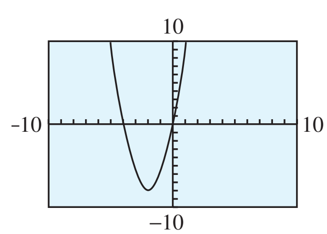
b $(-\sqrt{5}, 0), (\sqrt{5}, 0), (0, -5)$

6.3.3 The Graph of $y = ax^2 + bx$

How does the linear term, bx, affect the graph? Let us begin by considering an example. Graph the function

$$y = 2x^2 + 8x$$

on your calculator. The graph is shown at right.



Note that $a = 2$ and that $2 > 0$, so the parabola opens upward. We can find the x-intercepts of the graph by setting y equal to zero:

$$\begin{aligned} 0 &= 2x^2 + 8x \\ &= 2x(x + 4) \end{aligned}$$

The solutions of this equation are 0 and -4, so the x-intercepts are the points $(0, 0)$ and $(-4, 0)$.

Recall that the parabola is symmetric about a vertical line through its vertex. (We will prove that this is true in the Homework problems.) The two

x-intercepts are equidistant from this line of symmetry, so the x-coordinate of the vertex lies exactly halfway between the x-intercepts. We can average their values to find

$$x = \frac{1}{2}[0 + (-4)] = -2$$

To find the y-coordinate of the vertex, substitute $x = -2$ into the equation for the parabola:

$$\begin{aligned} y &= 2(\mathbf{-2})^2 + 8(\mathbf{-2}) \\ &= 8 - 16 = -8 \end{aligned}$$

Thus, the vertex is the point $(-2, -8)$.

Checkpoint 6.3.5

a Find the x-intercepts and the vertex of the parabola $y = 6x - x^2$.

b Verify your answers by graphing the function in the window

$$\begin{array}{ll} \text{Xmin} = -9.4 & \text{Xmax} = 9.4 \\ \text{Ymin} = -10 & \text{Ymax} = 10 \end{array}$$

Answer. x-intercepts: $(0, 0)$ and $(6, 0)$; vertex: $(3, 9)$

6.3.4 Finding the Vertex

We can use the same method to find a formula for the vertex of any parabola of the form

$$y = ax^2 + bx$$

We proceed as we did in the previous example.

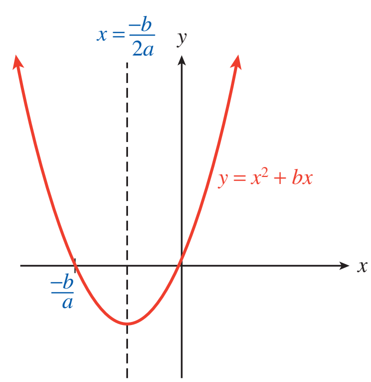
First, find the x-intercepts of the graph by setting y equal to zero and solving for x.

$$\begin{aligned} 0 &= ax^2 + bx \quad \text{Factor.} \\ &= x(ax + b) \end{aligned}$$

Thus,

$$\begin{array}{lll} x = 0 & \text{or} & ax + b = 0 \\ x = 0 & \text{or} & x = \dfrac{-b}{a} \end{array}$$

The x-intercepts are the points $(0, 0)$ and $(\frac{-b}{a}, 0)$.



Next, we find the x-coordinate of the vertex by taking the average of the two x-intercepts found above:

$$x = \frac{1}{2}\left[0 + \left(\frac{-b}{a}\right)\right] = \frac{-b}{2a}$$

This gives us a formula for the x-coordinate of the vertex.

Vertex of a Parabola.

For the graph of $y = ax^2 + bx$, the x-coordinate of the vertex is

$$x_v = \frac{-b}{2a}$$

Also, the axis of symmetry is the vertical line $x = \dfrac{-b}{2a}$ as shown in the figure above. Finally, we find the y-coordinate of the vertex by substituting its x-coordinate into the equation for the parabola.

Example 6.3.6

a Find the vertex of the graph of $f(x) = -1.8x^2 - 16.2x$.

b Find the x-intercepts of the graph.

Solution.

a The x-coordinate of the vertex is

$$x_v = \frac{-b}{2a} = \frac{-(-16.2)}{2(-1.8)} = -4.5$$

To find the y-coordinate of the vertex, evaluate $f(x)$ at $x = \mathbf{-4.5}$.

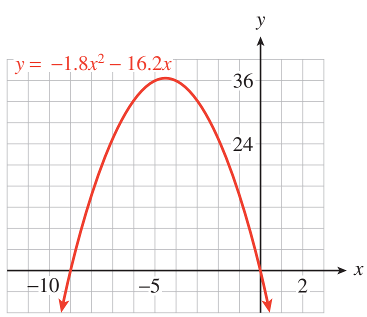
$$y_v = -1.8(\mathbf{-4.5})^2 - 16.2(\mathbf{-4.5}) = 36.45$$

The vertex is $(-4.5, 36.45)$.

b To find the x-intercepts of the graph, set $f(x) = 0$ and solve.

$$\begin{aligned} -1.8x^2 - 16.2x &= 0 && \text{Factor.} \\ -x(1.8x + 16.2) &= 0 && \text{Set each factor equal to zero.} \\ -x = 0 \qquad 1.8x + 16.2 &= 0 && \text{Solve each equation.} \\ x = 0 \qquad\qquad x &= -9 \end{aligned}$$

The x-intercepts of the graph are $(0, 0)$ and $(-9, 0)$. The graph is shown below.



□

6.3.5 The Graph of $y = ax^2 + bx + c$

Now we will see that the vertex formula holds for any parabola. Consider the function

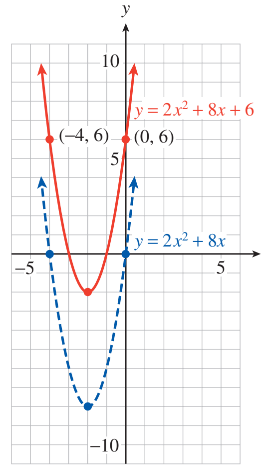
$$y = 2x^2 + 8x + 6$$

Adding 6 to $2x^2 + 8x$ shifts each point on the graph 6 units upward, as shown at right. The x-coordinate of the vertex will not be affected by an upward shift. Thus, the formula

$$x_v = \frac{-b}{2a}$$

for the x-coordinate of the vertex still holds. We have

$$x_v = \frac{-8}{2(2)} = -2$$



We find the y-coordinate of the vertex by substituting $x_v = \mathbf{-2}$ into the equation for the parabola.

$$\begin{aligned} y_v &= 2(\mathbf{-2})^2 + 8(\mathbf{-2}) + 6 \\ &= 8 - 16 + 6 = -2 \end{aligned}$$

So the vertex is the point $(-2, -2)$. (Notice that this point is shifted 6 units upward from the vertex of $y = 2x^2 + 8x$.)

We find the x-intercepts of the graph by setting y equal to zero.

$$\begin{aligned} 0 &= 2x^2 + 8x + 6 && \text{Factor the right side.} \\ &= 2(x + 1)(x + 3) && \text{Set each factor equal to zero.} \\ x + 1 = 0 &\quad \text{or} \quad x + 3 = 0 \\ x = -1 &\qquad\quad x = -3 \end{aligned}$$

The x-intercepts are the points $(-1, 0)$ and $(-3, 0)$.

The y-intercept of the graph is found by setting x equal to zero:

$$y = 2(\mathbf{0})2 + 8(\mathbf{0}) + 6 = 6$$

You can see that the y-intercept, 6, is just the constant term of the quadratic equation. The completed graph is shown above.

Example 6.3.7 Find the vertex of the graph of $f(x) = -2x^2 + x + 1$.

Solution. For this function, $a = -2$, $b = 1$, and $c = 1$. The x-coordinate of the vertex is given by

$$x_v = \frac{-b}{2a} = \frac{-1}{2(-2)} = \frac{1}{4}$$

To find the y-coordinate of the vertex, we substitute $x = \mathbf{\frac{1}{4}}$ into the equation. We can do this by hand to find

$$\begin{aligned} y_v &= -2\left(\mathbf{\frac{1}{4}}\right)^2 + \mathbf{\frac{1}{4}} + 1 \\ &= -2\left(\frac{1}{16}\right) + \frac{4}{16} + \frac{16}{16} = \frac{18}{16} = \frac{9}{8} \end{aligned}$$

So the coordinates of the vertex are $\left(\frac{1}{4}, \frac{9}{8}\right)$. Alternatively, we can use the calculator to evaluate $-2x^2 + x + 1$ for $x = 0.25$. The calculator returns the

y-value 1.125. Thus, the vertex is the point $(0.25, 1.125)$, which is the decimal equivalent of $\left(\frac{1}{4}, \frac{9}{8}\right)$. □

Checkpoint 6.3.8 Find the vertex of the graph of $f(x) = 3x^2 - 6x + 4$. Decide whether the vertex is a maximum point or a minimum point of the graph.

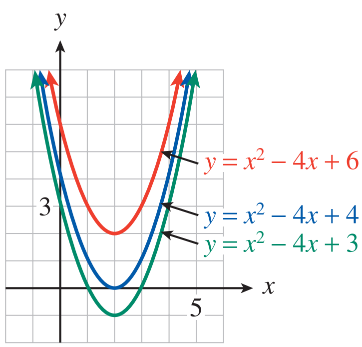
Answer. $(1, 1)$, minimum

6.3.6 Number of x-Intercepts

The graph of the quadratic function

$$f(x) = ax^2 + bx + c$$

may have two, one, or no x-intercepts, according to the number of distinct real-valued solutions of the equation $ax^2 + bx + c = 0$. Consider the three functions graphed below.



- The graph of
$$f(x) = x^2 - 4x + 3$$
has two x-intercepts, because the equation
$$x^2 - 4x + 3 = 0$$
has two real-valued solutions, $x = 1$ and $x = 3$.

- The graph of
$$g(x) = x^2 - 4x + 4$$
has only one x-intercept, because the equation
$$x^2 - 4x + 4 = 0$$
has only one (repeated) real-valued solution, $x = 2$.

- The graph of
$$h(x) = x^2 - 4x + 6$$
has no x-intercepts, because the equation
$$x^2 - 4x + 6 = 0$$
has no real-valued solutions.

A closer look at the quadratic formula reveals useful information about the solutions of quadratic equations. For the three functions above, we have the following:

$y = x^2 - 4x + 3$ two x-intercepts	$y = x^2 - 4x + 4$ one x-intercept	$y = x^2 - 4x + 6$ no x-intercepts
$x = \frac{4 \pm \sqrt{(-4)^2 - 4(1)(3)}}{2}$	$x = \frac{4 \pm \sqrt{(-4)^2 - 4(1)(4)}}{2}$	$x = \frac{4 \pm \sqrt{(-4)^2 - 4(1)(6)}}{2}$
$= \frac{4 \pm \sqrt{4}}{2}$	$= \frac{4 \pm \sqrt{0}}{2}$	$= \frac{4 \pm \sqrt{-12}}{2}$
(two solutions)	(one repeated solution)	(no solutions)

The expression $b^2 - 4ac$, which appears under the radical in the quadratic formula, is called the **discriminant**, D, of the equation. The value of the discriminant determines the nature of the solutions of the equation. In particular, if the discriminant is negative, the solutions of the quadratic equation are **complex numbers**. (We will study complex numbers in Section 7.3, p. 735.)

The Discriminant.

The **discriminant** of a quadratic equation is $D = b^2 - 4ac$.

1 If $D > 0$, there are two unequal real solutions.

2 If $D = 0$, there is one real solution of multiplicity two.

3 If $D < 0$, there are two complex solutions.

Note 6.3.9 We can also use the discriminant to decide whether a quadratic equation can be solved by factoring. First, clear the equation of fractions. If the discriminant is a perfect square, that is, the square of an integer, the solutions are rational numbers. This in turn means that the equation can be solved by factoring.

If the discriminant is not a perfect square, the solutions will be irrational. Irrational solutions always occur in conjugate pairs,

$$\frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \text{and} \quad \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

The only difference between the two solutions is the sign between the terms. For example, if we know that one solution of a particular quadratic equation is $3 + \sqrt{2}$, then the other solution must be $3 - \sqrt{2}$.

Example 6.3.10 Use the discriminant to determine the nature of the solutions of each equation. Can the equation be solved by factoring?

a $x^2 - x - 3 = 0$ b $2x^2 + x + 1 = 0$ c $x^2 - 2x - 3 = 0$

Solution.

a $D = b^2 - 4ac = (-1)^2 - 4(1)(-3) = 13 > 0$.

The equation has two real, unequal solutions. Because 13 is not a perfect square, the solutions will be irrational numbers, so the equation cannot be solved by factoring.

b $D = b^2 - 4ac = 1^2 - 4(2)(1) = -7 < 0$.

The equation has two complex solutions, which cannot be found by factoring.

c $D = b^2 - 4ac = (-2)^2 - 4(1)(-3) = 16 > 0$.

The equation has two real, unequal solutions. Because $16 = 4^2$, the solutions are rational numbers and can be found by factoring.

(You can verify the conclusions above by solving each equation.) □

Checkpoint 6.3.11 Use the discriminant to discover how many x-intercepts the graph of each function has.

a $y = x^2 + 5x + 7$

b $y = -\dfrac{1}{2}x^2 + 4x - 8$

Answer.

a None

b One

In Checkpoint 6.3.11, p. 641, you should check that the single x-intercept is also the vertex of the parabola.

6.3.7 Sketching a Parabola

Once we have located the vertex of the parabola, the x-intercepts, and the y-intercept, we can sketch a reasonably accurate graph. Recall that the graph should be symmetric about a vertical line through the vertex. We summarize the procedure as follows.

To Graph the Quadratic Function $f(x) = ax^2 + bx + c$:.

1 Determine whether the parabola opens upward (if $a > 0$) or downward (if $a < 0$).

2 Locate the vertex of the parabola.

  a The x-coordinate of the vertex is $x_v = \dfrac{-b}{2a}$.

  b Find the y-coordinate of the vertex by substituting x_v into the equation of the parabola.

3 Locate the x-intercepts (if any) by setting $y = 0$ and solving for x.

4 Locate the y-intercept by evaluating y for $x = 0$.

5 Locate the point symmetric to the y-intercept across the axis of symmetry.

Example 6.3.12 Sketch a graph of the equation $f(x) = x^2 + 3x + 1$, showing the significant points.

Solution. We follow the steps outlined above.

1 Because $a = 1 > 0$, we know that the parabola opens upward.

2 We compute the coordinates of the vertex:

$$x_v = \frac{-b}{2a} = \frac{-3}{2(1)} = -1.5$$
$$y_v = (-1.5)2 + 3(-1.5) + 1 = -1.25$$

The vertex is the point $(-1.5, -1.25)$.

3 We set y equal to zero to find the x-intercepts.

$$0 = x^2 + 3x + 1 \qquad \text{Use the quadratic formula.}$$
$$x = \frac{-3 \pm \sqrt{3^2 - 4(1)(1)}}{2(1)}$$
$$= \frac{-3 \pm \sqrt{5}}{2}$$

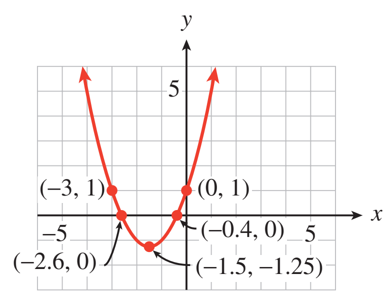
Rounding to the nearest tenth, we find that the x-intercepts are approximately $(-2.6, 0)$ and $(-0.4, 0)$.

4 We substitute $x = 0$ to find the y-intercept, $(0, 1)$.

5 The axis of symmetry is the vertical line $x = -1.5$, so the y-intercept lies 1.5 units to the right of the axis of symmetry.

There must be another point on the parabola with the same y-coordinate as the y-intercept but 1.5 units to the left of the axis of symmetry. The coordinates of this point are $(-3, 1)$.

Finally, plot the x-intercepts, the vertex, and the y-intercept and its symmetric point, and draw a parabola through them. The finished graph is shown below.



□

Checkpoint 6.3.13

a Find the intercepts and the vertex of the graph of $f(x) = x^2 - 5x + 4$.

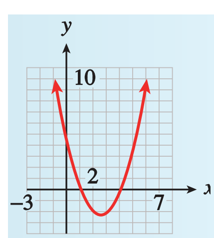
b Sketch the graph by hand.

c Use your calculator to verify your graph.

Answer.

a $(0, 4)$; $(1, 0)$, $(4, 0)$; vertex $\left(\frac{5}{2}, \frac{-9}{4}\right)$

b



6.3.8 Section Summary

6.3.8.1 Vocabulary

Look up the definitions of new terms in the Glossary.

- Vertex
- Conjugate pair
- Axis of symmetry

6.3.8.2 CONCEPTS

1 The graph of a quadratic function $f(x) = ax^2 + bx + c$ is called a **parabola**. The values of the constants a, b, and c determine the location and orientation of the parabola.

2 For the graph of $f(x) = ax^2 + bx + c$, the x-coordinate of the **vertex** is $x_v = \frac{-b}{2a}$. To find the y-coordinate of the vertex, we substitute x_v into the formula for the parabola.

3 The graph of the quadratic function $f(x) = ax^2 + bx + c$ may have two, one, or no x-intercepts, according to the number of distinct real-valued solutions of the equation $ax^2 + bx + c = 0$.

4 **The Discriminant.**

The **discriminant** of a quadratic equation is $D = b^2 - 4ac$.

1 If $D > 0$, there are two unequal real solutions.

2 If $D = 0$, there is one real solution of multiplicity two.

3 If $D < 0$, there are two complex solutions.

5 **To Graph the Quadratic Function** $f(x) = ax^2 + bx + c$**:.**

1 Determine whether the parabola opens upward (if $a > 0$) or downward (if $a < 0$).

2 Locate the vertex of the parabola.

a The x-coordinate of the vertex is $x_v = \dfrac{-b}{2a}$.

b Find the y-coordinate of the vertex by substituting x_v into the equation of the parabola.

3 Locate the x-intercepts (if any) by setting $y = 0$ and solving for x.

4 Locate the y-intercept by evaluating y for $x = 0$.

5 Locate the point symmetric to the y-intercept across the axis of symmetry.

6.3.8.3 STUDY QUESTIONS

1 Sketch a parabola that opens downward. Show the location of the x-intercepts, the y-intercept, the vertex, and the axis of symmetry.

2 Describe how the value of a in $y = ax^2$ alters the graph of the basic parabola.

3 Describe how the value of c in $y = x^2 + c$ alters the graph of the basic parabola.

4 Suppose you know that the x-intercepts of a parabola are $(-8, 0)$ and $(2, 0)$. What is the equation of the parabola's axis of symmetry?

5 State a formula for the x-coordinate of the vertex of a parabola. How can you find the y-coordinate of the vertex?

6 Suppose that a given parabola has only one x-intercept. What can you say about the vertex of the parabola?

7 Explain why a quadratic equation has one (repeated) solution if its discriminant is zero, and none if the discriminant is negative.

6.3.8.4 SKILLS

Practice each skill in the Homework 6.3.9, p. 644 problems listed.

1 Graph transformations of the basic parabola: #1 and 2, 7, and 8

2 Locate the x-intercepts of a parabola: #3–6

3 Locate the vertex of a parabola: #3–6, 13, and 14

4 Sketch the graph of a quadratic function: #15–24, 41, and 42

5 Use the discriminant to describe the solutions of a quadratic equation: #25–40

6.3.9 Graphing Parabolas (Homework 6.3)

For Problems 1–2, describe what each graph will look like compared to the basic parabola. Then sketch a graph by hand and label the coordinates of three points on the graph.

1.

(a) $y = 2x^2$

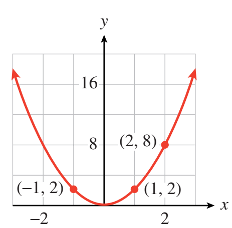
(b) $y = 2 + x^2$

(c) $y = (x + 2)^2$

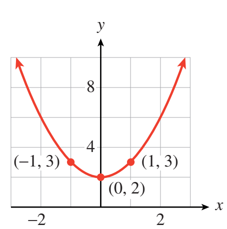
(d) $y = x^2 - 2$

Answer.

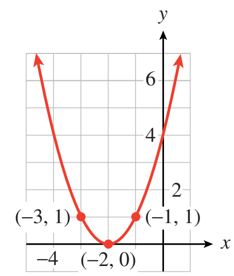
(a) The parabola opens up, twice as steep as the standard parabola.



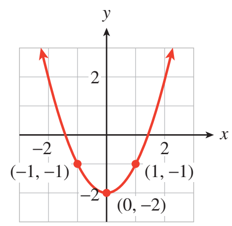
(b) The parabola is the standard parabola shifted 2 units up.



(c) The parabola is the standard parabola shifted 2 units left.



(d) The parabola is the standard parabola shifted 2 units down.



2.

(a) $y = -4x^2$ (c) $y = 4 - x^2$

(b) $y = (x - 4)^2$ (d) $y = x^2 - 4$

For problems 3–6, find the vertex and the x-intercepts (if there are any) of the graph. Then sketch the graph by hand.

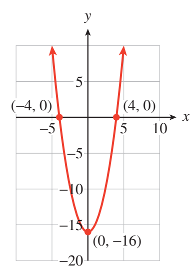
3.

a $y = x^2 - 16$ c $y = 16x - x^2$

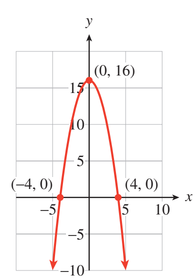
b $y = 16 - x^2$ d $y = x^2 - 16x$

Answer.

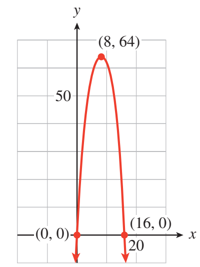
a Vertex $(0, -16)$; x-intercepts $(\pm 4, 0)$



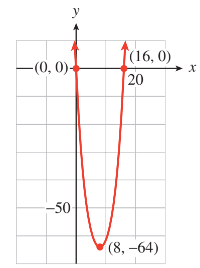
b Vertex $(0, 16)$; x-intercepts $(\pm 4, 0)$



c Vertex $(8, 64)$; x-intercepts $(0, 0)$ and $(16, 0)$



d Vertex $(8,-64)$; x-intercepts $(0,0)$ and $(16,0)$



4.

a $y=x^2-1$

b $y=1-x^2$

c $y=x^2-x$

d $y=x-x^2$

5.

a $y=3x^2+6x$

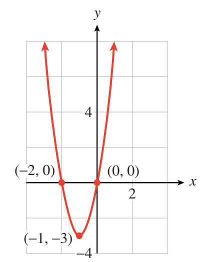
b $y=3x^2-6x$

c $y=3x^2+6$

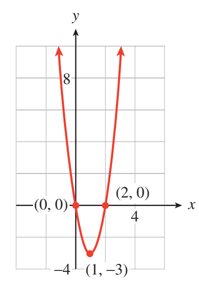
d $y=3x^2-6$

Answer.

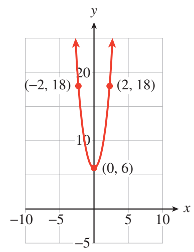
a Vertex $(1,-3)$; x-intercepts $(0,0)$ and $(-2,0)$



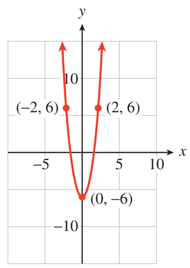
b Vertex $(1,-3)$; x-intercepts $(0,0)$ and $(2,0)$



c Vertex $(0,6)$; no x-intercepts



d Vertex $(0, -6)$; x-intercepts $(\pm\sqrt{2}, 0)$



6.

a $y = 12x - 2x^2$

b $y = 12 - 2x^2$

c $y = 12 + 2x^2$

d $y = 12x + 2x^2$

7. Match each function with its graph. In each equation, $a > 0$.

(a) $y = x^2 + a$

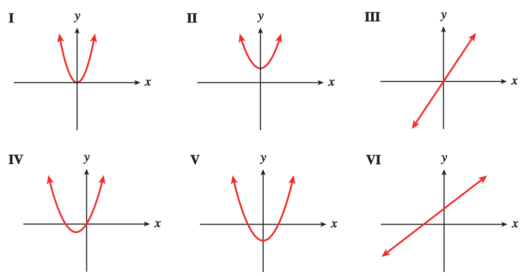
(b) $y = x^2 + ax$

(c) $y = ax^2$

(d) $y = ax$

(e) $y = x + a$

(f) $y = x^2 - a$



Answer.

(a) II (b) IV (c) I (d) III (e) VI (f) V

8. Match each function with its graph. In each equation, $b > 0$.

(a) $y = -bx$

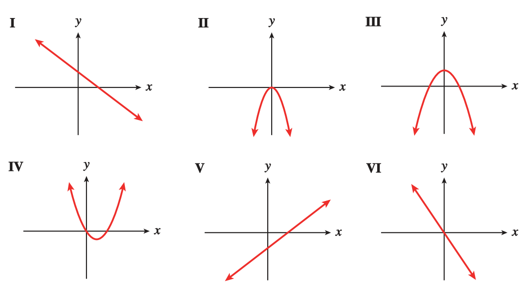
(b) $y = -bx^2$

(c) $y = b - x^2$

(d) $y = x - b$

(e) $y = b - x$

(f) $y = x^2 - bx$



9. Commercial fishermen rely on a steady supply of fish in their area. To avoid overfishing, they adjust their harvest to the size of the population.

The function

$$g(x) = 0.4x - 0.0001x^2$$

gives the annual rate of growth, in tons per year, of a fish population of biomass x tons.

(a) Find the vertex of the graph. What does it tell us about the fish population?

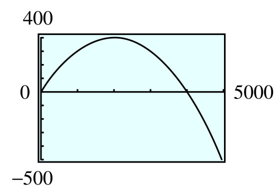
(b) Sketch the graph for $0 \le x \le 5000$.

(c) For what values of x does the fish population decrease rather than increase? Suggest a reason why the population might decrease.

Answer.

(a) $(2000, 400)$; The largest annual increase in biomass, 400 tons, occurs when the biomass is 2000 tons.

(b)



(c) $4000 < x \le 5000$; When there are too many fish, there will not be enough food to support all of them.

10. The annual increase, I, in the deer population in a national park depends on the size, x, of the population that year, according to the function

$$I = f(x) = 1.2x - 0.0002x^2$$

(a) Find the vertex of the graph. What does it tell us about the deer population?

(b) Sketch the graph for $0 \le x \le 7000$.

(c) For what values of x does the deer population decrease rather than increase? Suggest a reason why the population might decrease.

11. Many animals live in groups. A species of marmot found in Colorado lives in harems composed of a single adult male and several females with their young. The number of offspring each female can raise depends on the number of females in the harem. On average, if there are x females in the harem, each female can raise $y = 2 - 0.4x$ young marmots each year.

(a) Complete the table of values for the average number of offspring per female, and the total number of young marmots, A, produced by the entire harem in one year.

x	1	2	3	4	5
y					
A					

(b) Write a formula for A in terms of x.

(c) Graph A as a function of x.

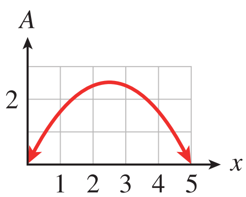
(d) What is the maximum number of young marmots a harem can produce (on average)? What is the optimal number of female marmots per harem?

Answer.

(a)

x	1	2	3	4	5
y	1.6	1.2	0.8	0.4	0
A	1.6	2.4	2.4	1.6	0

(b) $A = x(2 - 0.4x)$ or $A = 2x - 0.4x^2$

(c) 

(d) The maximum number of young marmots, on average, is 2.5; the optimal number of female marmots is 2.5.

12. Greenshield's model for traffic flow assumes that the average speed, u, of cars on a highway is a linear function of the traffic density, k, in vehicles per mile, given by

$$u = u_f\left(1 - \frac{k}{k_j}\right)$$

where u_f is the free-flow speed and k_j is the maximum density (the point when traffic jams). Then the traffic flow, q, in vehicles per hour, is given by $q = uk$.

(a) Write a formula for q as a function of k.

(b) If the free-flow speed is 70 mph and the maximum density is 240 vehicles per mile, graph q as a function of k.

(c) What value of k gives the maximum traffic flow? What is the average speed of vehicles at that density?

13. After touchdown, the distance the space shuttle travels is given by

$$d = vT + \frac{v^2}{2a}$$

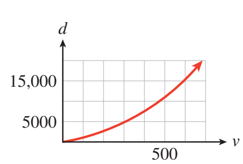
where v is the shuttle's velocity in ft/sec at touchdown, T is the pilot's reaction time before the brakes are applied, and a is the shuttle's deceleration.

(a) Graph $d = f(v)$ for $T = 0.5$ seconds and $a = 12$ ft/sec^2. Find the coordinates of the vertex and the horizontal intercepts. Explain their meaning, if any, in this context.

(b) The runway at Edwards Air Force base is 15,000 feet long. What is the maximum velocity the shuttle can have at touchdown and still stop on the runway?

Answer.

(a)



Vertex: $(-6, -1.5)$; Horizontal intercepts $(-12, 0)$ and $(0, 0)$. The point $(0, 0)$ means that no distance is required to stop a plane that is not moving.

(b) 594 ft/sec

14. When setting the pump pressure at the engine, firefighters must take into account the pressure loss due to friction inside the fire hose. For every 100 feet of hoseline, a hose of diameter 2.5 inches loses pressure according to the formula

$$L = \begin{cases} 2Q^2 + Q, & Q \geq 1 \\ 2Q^2 + \frac{1}{2}Q, & Q < 1 \end{cases}$$

where Q is the water flow in hundreds of gallos per minute. The friction loss, L, is measured in pounds per square inch (psi) (Source: www.hcc.hawaii.edu/~jkemmer)

(a) Graph $L = g(Q)$ on the domain $[0, 5]$.

(b) The firefighters have unrolled 600 feet of 2.5-inch-diameter hose, and they would like to deliver water at a rate of 200 gallons per minute, with nozzle pressure at 100 psi. They must add the friction loss to the nozzle pressure to calculate the engine pressure required. What should the engine pressure be?

For Problems 15–16, find the coordinates of the vertex. Decide whether the vertex is a maximum point or a minimum point on the graph and explain why.

15.

(a) $y = 2 + 3x - x^2$

(b) $y = \frac{1}{2}x^2 - \frac{2}{3}x + \frac{1}{3}$

(c) $y = 2.3 - 7.2x - 0.8x^2$

Answer.

(a) $\left(\frac{3}{2}, \frac{17}{4}\right)$, maximum

(b) $\left(\frac{2}{3}, \frac{1}{9}\right)$, minimum

(c) $(-4.5, 18.5)$, maximum

16.

(a) $y = 3 - 5x + x^2$

(b) $y = \frac{-3}{4}x^2 + \frac{1}{2}x - \frac{1}{4}$

(c) $y = -5.1 - 0.2x + 4.6x^2$

In Problems 17–26,

a Find the coordinates of the intercepts and the vertex.

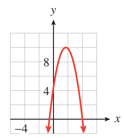
b Sketch the graph by hand.

c Use your calculator to verify your graph.

17. $y = -2x^2 + 7x + 4$

Answer.

(a) x-intercepts: $\left(\frac{-1}{2}, 0\right)$ and $(4, 0)$; y-intercept: $(0, 4)$; vertex: $\left(\frac{7}{4}, \frac{81}{8}\right)$

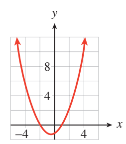
(b) 

18. $y = -3x^2 + 2x + 8$

19. $y = 0.6x^2 + 0.6x - 1.2$

Answer.

(a) x-intercepts: $(-2, 0)$ and $(1, 0)$; y-intercept: $(0, -1.2)$; vertex: $(-0.5, -1.35)$

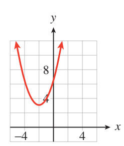
(b) 

20. $y = 0.5x^2 - 0.25x - 0.75$

21. $y = x^2 + 4x + 7$

Answer.

(a) No x-intercepts; y-intercept: $(0, 7)$; vertex: $(-2, 3)$

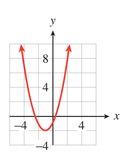
(b) 

22. $y = x^2 - 6x + 10$

23. $y = x^2 + 2x - 1$

Answer.

(a) x-intercepts: $\left(-1 \pm \sqrt{2}, 0\right)$; y-intercept: $(0, -1)$; vertex: $(-1, -2)$

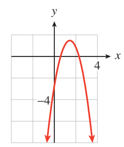
(b) 

24. $y = x^2 - 6x + 2$

25. $y = -2x^2 + 6x - 3$

Answer.

(a) x-intercepts: $\left(\dfrac{3 \pm \sqrt{3}}{2}, 0\right)$; y-intercept: $(0, -3)$; vertex: $\left(\dfrac{3}{2}, \dfrac{3}{2}\right)$

(b) 

26. $y = -2x^2 - 8x - 5$

27.

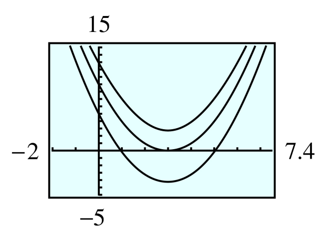
(a) Graph the three functions $f(x) = x^2 - 6x + 5$, $g(x) = x^2 - 6x + 9$, and $h(x) = x^2 - 6x + 12$ in the window

$$\text{Xmin} = -2 \qquad \text{Xmax} = 7.4$$
$$\text{Ymin} = -5 \qquad \text{Ymax} = 15$$

Use the **Trace** to locate the x-intercepts of each graph.

(b) Set $y = 0$ for each of the equations in part (a) and calculate the discriminant. What does the discriminant tell you about the solutions of the equation? How does your answer relate to the graphs in part (a)?

Answer.

(a) 

$f(x) = x^2 - 6x + 5$: x-intercepts $(1, 0)$ and $(5, 0)$; $g(x) = x^2 - 6x + 9$: x-intercept $(3, 0)$; $h(x) = x^2 - 6x + 12$: No x-intercept.

(b) $16, 0, -12$: $D = 16$ means that there are two rational x-intercepts, $D = 0$ means that there is exactly one x-intercept, $D = -12$ means that there is no x-intercept.

28.

(a) Graph the three functions $F(x) = 3 - 2x - x^2$, $G(x) = -1 - 2x - x^2$, and $H(x) = -4 - 2x - x^2$ in the window

$$\text{Xmin} = -6.4 \qquad \text{Xmax} = 3$$
$$\text{Ymin} = -10 \qquad \text{Ymax} = 5$$

Use the **Trace** to locate the x-intercepts of each graph.

(b) Set $y = 0$ for each of the equations in part (a) and calculate the discriminant. What does the discriminant tell you about the solutions of the equation? How does your answer relate to the graphs in part (a)?

For Problems 29–34, use the discriminant to determine the nature of the solutions of each equation.

29. $3x^2 + 26 = 17x$

Answer. Two complex solutions

30. $4x^2 + 23x = 19$

31. $16x^2 - 712x + 7921 = 0$

Answer. One repeated rational solution

32. $121x^2 + 1254x + 3249 = 0$

33. $65.2x = 13.2x^2 + 41.7$

Answer. Two distinct real solutions

34. $0.03x^2 = 0.05x - 0.12$

For problems 35–38, use the discriminant to decide if we can solve the equation by factoring.

35. $3x^2 - 7x + 6 = 0$

Answer. No

36. $6x^2 - 11x - 7 = 0$

37. $15x^2 - 52x - 32 = 0$

Answer. Yes

38. $17x^2 + 65x - 12 = 0$

For Problems 39–41,

a Given one zero of a quadratic equation with rational coefficients, find the other zero.

b Write a quadratic equation that has those zeros.

39. $2 + \sqrt{5}$

Answer.

(a) $2 - \sqrt{5}$

(b) $x^2 - 4x - 1 = 0$

40. $3 - 2i$

41. $4 - 3\sqrt{2}$

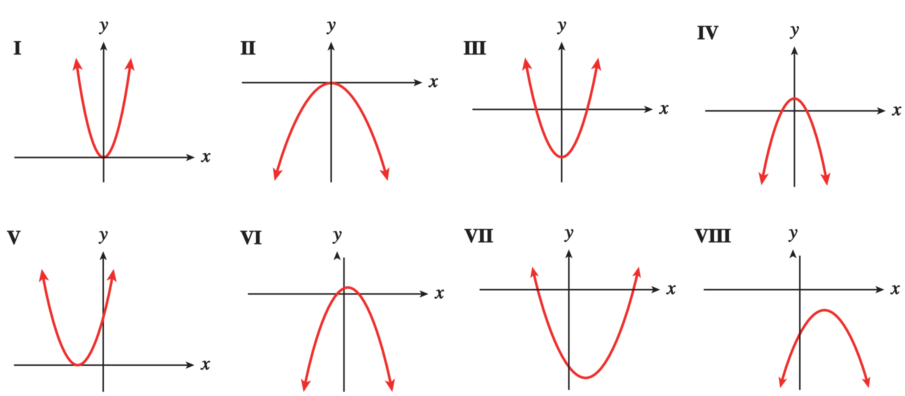
Answer.

(a) $4 + 3\sqrt{2}$

(b) $x^2 - 8x - 2 = 0$

42. If 5 is zero to a quadratic equation with rational coefficients, must -5 also be a solution?

For Problems 43 and 44, match each equation with one of the eight graphs shown.



43.

(a) $y = 1 - x^2$

(b) $y = (x+2)^2$

(c) $y = 2x^2$

(d) $y = (x-4)(x+2)$

Answer.

(a) IV (b) V (c) I (d) VII

44.

(a) $y = -2 - (x-2)^2$

(b) $y = x - x^2$

(c) $y = x^2 - 4$

(d) $y = -0.5x^2$

45.

(a) Write an equation for a parabola that has x-intercepts at $(2, 0)$ and $(-3, 0)$.

(b) Write an equation for another parabola that has the same x-intercepts.

Answer.

(a) $y = x^2 + x - 6;\ \ x = \dfrac{-1}{2}$

(b) $y = 2x^2 + 2x - 12;\ \ x = \dfrac{-1}{2}$

46.

(a) Write an equation for a parabola that opens upward and has x-intercepts at $(-1, 0)$ and $(4, 0)$. What is the equation of the parabola's axis of symmetry?

(b) Write an equation for a parabola that opens downward and has x-intercepts $(-1, 0)$ and $(4, 0)$. What is the equation of its axis of symmetry?

47.

(a) Graph the functions in the same window on your calculator:

$$f(x) = x^2 + 2x, \quad g(x) = x^2 + 4x,$$
$$h(x) = x^2 + 6x, \quad j(x) = x^2 + 8x$$

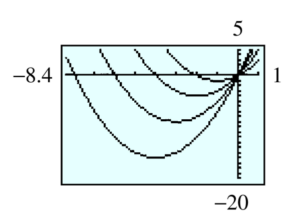
(b) Find the vertex of each graph in part (a) and plot the points.

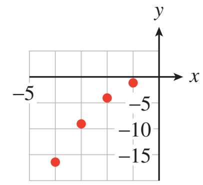
(c) Find the equation of the curve in part (b).

(d) Show that the vertex of $y = x^2 + 2kx$ lies on the curve for any value of k.

Answer.

(a)



(b) 

$(-1,-1), (-2,-4), (-3,-9), (-4,-16)$

(c) $y = -x^2$

(d) The vertex of $y = x^2 + 2kx$ is $(-k, -k^2)$

48.

(a) Graph the functions in the same window on your calculator:

$$F(x) = x - \frac{1}{2}x^2, \quad G(x) = 3x - \frac{1}{2}x^2,$$
$$H(x) = 5x - \frac{1}{2}x^2, \quad J(x) = 7x - \frac{1}{2}x^2$$

(b) Find the vertex of each graph in part (a) and plot the points.

(c) Find the equation of the curve in part (b).

(d) Show that the vertex of $y = kx - \frac{1}{2}x^2$ lies on the curve for any value of k.

49. Because of air resistance, the path of a kicked soccer ball is not actually parabolic. However, both the horizontal and vertical coordinates of points on its trajectory can be approximated by quadratic functions. For a soccer ball kicked from the ground, these functions are

$$x = f(t) = 12.8t - 1.3t^2$$
$$y = g(t) = 17.28t - 4.8t^2$$

where x and y are given in meters and t is the number of seconds since the ball was kicked.

(a) Fill in the table.

t	0	0.5	1.0	1.5	2.0	2.5	3.0	3.5
x								
y								

(b) Plot the points (x, y) from your table and connect them with a smooth curve to represent the path of the ball.

(c) Use your graph to estimate the maximum height of the ball.

(d) Estimate the horizontal distance traveled by the ball before it strikes the ground

(e) Using the formula given for y, determine how long the ball is in the air.

(f) Use your answer from part (e) and the formula for x to find the horizontal distance traveled by the ball before it strikes the ground

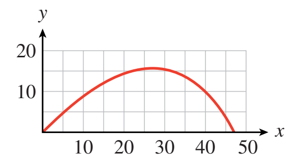
(g) Use the formula given for y to find the maximum height for the ball.

Answer.

(a)

t	0	0.5	1.0	1.5	2.0	2.5	3.0	3.5
x	0	6.075	11.5	16.275	20.4	23.875	26.7	28.875
y	0	7.44	12.48	15.12	15.36	13.2	8.64	1.68

(b)



(c) $y \approx 15.4$ m

(d) $x \approx 30$ m

(e) 3.6 sec

(f) $x \approx 29.2$ m

(g) $y \approx 15.55$ m

50. How far can you throw a baseball? The distance depends on the initial speed of the ball, v, and on the angle at which you throw it. For maximum range, you should throw the ball at 45°.

(a) If there were no air resistance, the height, x, of the ball t seconds after its release would be given in meters by the function

$$h = f(t) = \frac{vt}{\sqrt{2}} - \frac{gt^2}{2}$$

where g is the acceleration due to gravity. Find an expression for the total time the ball is in the air. (*Hint*: Set $h = 0$ and solve for t in terms of the other variables.)

(b) At time t, the ball has traveled a horizontal distance d given by

$$d = \frac{vt}{\sqrt{2}}$$

Find an expression for the range of the ball in terms of its velocity, v. (*Hint*: In part (a), you found an expression for t when $h = 0$. Use that value of t to calculate d when $h = 0$.)

(c) The fastest baseball pitch on record was 45 meters per second, or about 100 miles per hour. Use your formula from part (b) to calculate the theoretical range of such a pitch. The value of g is 9.8.

(d) The maximum distance a baseball has actually been thrown is 136 meters. Can you explain the discrepancy between this figure and your answer to part (c)?

6.4 Problem Solving

Many quadratic models arise as the product of two variables, one of which increases while the other decreases. For example, the area of a rectangle is the product of its length and its width, or $A = lw$. If we require that the rectangle have a certain perimeter, then as we increase its length, we must also decrease its width. (We analyzed this problem in Investigation 9, p. 151 of Chapter 2, p. 149.)

For Investigation 41, p. 657, recall the formula for the revenue from sales of an item:

$$\text{Revenue} = (\text{price of one item})(\text{number of items sold})$$

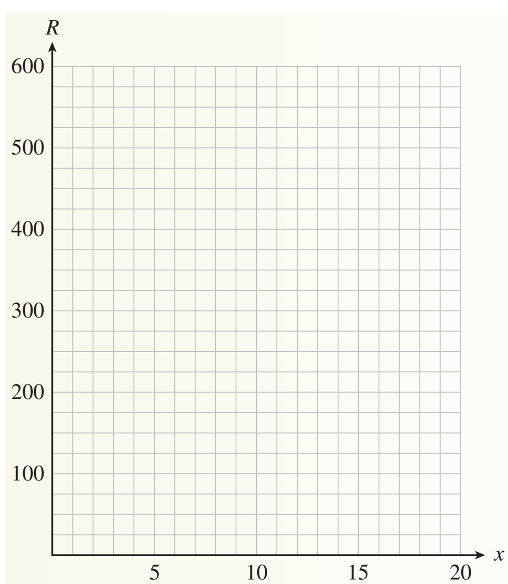
Usually, when the price of an item increases, the number of items sold decreases.

Investigation 41 Revenue from Theater Tickets. The local theater group sold tickets to its opening night performance for \$5 and drew an audience of 100 people. The next night, the group reduced the ticket price by \$0.25 and 10 more people attended; that is, 110 people bought tickets at \$4.75 apiece. In fact, for each \$0.25 reduction in ticket price, 10 additional tickets can be sold.

1 Complete the table.

Number of price reductions	Price of ticket	Number of tickets sold	Total revenue
0	5.00	100	500
1	4.75	110	522.50
2			
3			
4			
5			
6			
7			
8			
9			
10			
11			
12			

2 Use your table to make a graph. Plot total revenue on the vertical axis versus number of price reductions on the horizontal axis.



3 Let x represent the number of price reductions, as in the first column of the table. Write algebraic expressions in terms of x for each quantity.

The price of a ticket after x price reductions:

$$\text{Price} =$$

The number of tickets sold at that price:

$$\text{Number} =$$

The total revenue from ticket sales:

$$\text{Revenue} =$$

4 Enter your expressions for the price of a ticket, the number of tickets sold, and the total revenue into the calculator as Y_1, Y_2, and Y_3. Use the Table feature to verify that your algebraic expressions agree with your table from part (1).

5 Use your calculator to graph your expression for total revenue in terms of x. Use your table to choose appropriate window settings that show the high point of the graph and both xintercepts.

6 What is the maximum revenue possible from ticket sales? What price should the theater group charge for a ticket to generate that revenue? How many tickets will the group sell at that price?

6.4.1 Maximum or Minimum Values

Finding the maximum or minimum value for a variable expression is a common problem in many applications. For example, if you own a company that manufactures blue jeans, you might like to know how much to charge for your jeans in order to maximize your revenue.

As you increase the price of the jeans, your revenue may increase for a while. But if you charge too much for the jeans, consumers will not buy as many pairs, and your revenue may actually start to decrease. Is there some optimum price you should charge for a pair of jeans in order to achieve the greatest revenue?

Example 6.4.1 Late Nite Blues finds that it can sell $600 - 15x$ pairs of jeans per week if it charges x dollars per pair. (Notice that as the price increases, the number of pairs of jeans sold decreases.)

a Write an equation for the revenue as a function of the price of a pair of jeans.

b Graph the function.

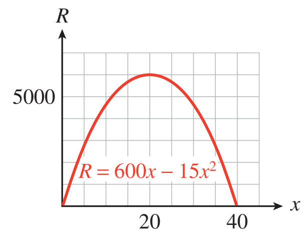
c How much should Late Nite Blues charge for a pair of jeans in order to maximize its revenue?

Solution.

a Using the formula for revenue stated above, we find

$$\begin{aligned}\text{Revenue} &= (\text{price of one item})(\text{number of items sold})\\ R &= x(600 - 15x)\\ R &= 600x - 15x^2\end{aligned}$$

b We recognize the equation as quadratic, so the graph is a parabola. You can use your calculator to verify the graph below.



c The maximum value of R occurs at the vertex of the parabola. Thus,

$$x_v = \frac{-b}{2a} = \frac{-600}{2(-15)} = 20$$
$$y_v = 600(\mathbf{20}) - 15(\mathbf{20})^2 = 6000$$

The revenue takes on its maximum value when $x = 20$, and the maximum value is $R = 6000$. This means that Late Nite Blues should charge \$20 for a pair of jeans in order to maximize revenue at \$6000 a week.

□

Note 6.4.2 If the equation relating two variables is quadratic, then the maximum or minimum value is easy to find: It is the value at the vertex. If the parabola opens downward, as in Example 6.4.1, p. 658, there is a maximum value at the vertex. If the parabola opens upward, there is a minimum value at the vertex.

Checkpoint 6.4.3 The Metro Rail service sells $1200 - 80x$ tickets each day when it charges x dollars per ticket.

a Write an equation for the revenue, R, as a function of the price of a ticket.

b What ticket price will return the maximum revenue? What is the maximum revenue?

Answer.

a $R = 1200x - 80x^2$

b \$7.50, \$4500

6.4.2 The Vertex Form for a Parabola

Consider the quadratic equation

$$y = 2(x-3)^2 - 8$$

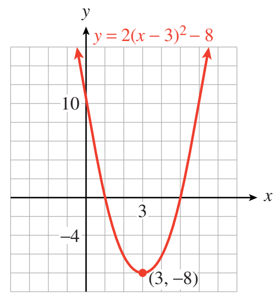
By expanding the squared expression and collecting like terms, we can rewrite the equation in standard form as

$$y = 2(x^2 - 6x + 9) - 8$$
$$y = 2x^2 - 12x + 10$$

The vertex of this parabola is

$$x_v = \frac{-(-12)}{2(2)} = 3$$
$$y_v = 2(3)^2 - 12(3) + 10 = -8$$

and its graph is shown below.



Notice that the coordinates of the vertex, $(3, -8)$, are apparent in the original equation; we don't need to do any computation to find the vertex.

$$y = 2(x - \mathbf{3})^2 \ \mathbf{-8}$$
$$\boldsymbol{x_v} \quad \boldsymbol{y_v}$$

This equation is an example of the **vertex form** for a quadratic function.

Vertex Form for a Quadratic Function.

A quadratic function $y = ax^2 + bx + c$, $a \neq 0$, can be written in the vertex form

$$y = a(x - x_v)^2 + y_v$$

where the vertex of the graph is (x_v, y_v).

Example 6.4.4 Find the vertex of the graph of $y = -3(x - 4)^2 + 6$. Is the vertex a maximum or a minimum point of the graph?

Solution. Compare the equation to the vertex form to see that the coordinates of the vertex are $(4, 6)$. For this equation, $a = -3 < 0$, so the parabola opens downward. The vertex is the maximum point of the graph. ☐

To understand why the vertex form works, substitute $x_v = \mathbf{4}$ into $y = -3(x - 4)^2 + 6$ from Example 6.4.4, p. 660 to find

$$y = -3(\mathbf{4} - 4)^2 + 6 = 6$$

which confirms that when $x = 4$, $y = 6$. Next, notice that if x is any number except 4, the expression $-3(x - 4)^2$ is negative, so $y < 6$. Therefore, 6 is the maximum value for y on the graph, so $(4, 6)$ is the high point or vertex.

You can also rewrite $y = -3(x - 4)^2 + 6$ in standard form and use the formula $x_v = \dfrac{-b}{2a}$ to confirm that the vertex is the point $(4, 6)$.

Checkpoint 6.4.5

a Find the vertex of the graph of $y = 5 - \dfrac{1}{2}(x + 2)^2$.

b Write the equation of the parabola in standard form.

Answer.

a $(-2, 5)$

b $y = -\dfrac{1}{2}x^2 - 2x + 3$

Any quadratic equation in vertex form can be written in standard form by expanding, and any quadratic equation in standard form can be put into vertex form by completing the square.

Example 6.4.6 Write the equation $y = 3x^2 - 6x - 1$ in vertex form and find the vertex of its graph.

Solution. We factor the lead coefficient, 3, from the variable terms, leaving a space to complete the square.

$$y = 3(x^2 - 2x \underline{\qquad\quad}) - 1$$

Next, we complete the square inside parentheses. Take half the coefficient of x and square the result:

$$p = \frac{1}{2}(-2) = -1, \quad \text{and} \quad p^2 = (-1)^2 = 1.$$

We must add 1 to complete the square. However, we are really adding 3(1) to the right side of the equation, so we must also subtract 3 to compensate:

$$y = 3(x^2 - 2x \mathbf{+1}) - 1 \mathbf{-3}$$

The expression inside parentheses is now a perfect square, and the vertex form is

$$y = 3(x - 1)^2 - 4$$

The vertex of the parabola is $(1, -4)$. □

Checkpoint 6.4.7 Write the equation $y = 2x^2 + 12x + 13$ in vertex form, and find the vertex of its graph.

Hint.

1 Factor 2 from the variable terms.

2 Complete the square inside parentheses.

3 Subtract $2p^2$ outside parentheses.

4 Write the vertex form.

Answer. $y = 2(x + 3)^2 - 5$; $(-3, -5)$

6.4.3 Graphing with the Vertex Form

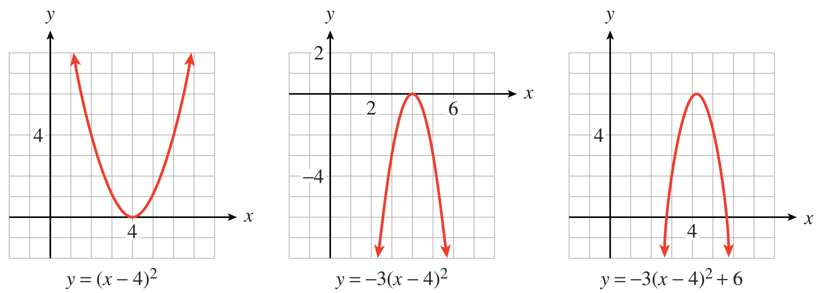
We can also use the vertex form to sketch a graph, using what we know about transformations.

Example 6.4.8 Use transformations to graph $f(x) = -3(x - 4)^2 + 6$.

Solution. We can graph $f(x)$ by applying transformations to the basic parabola, $y = x^2$. To identify the transformations, think of evaluating the function for a specific value of x. What operations would we perform on x, besides squaring?

1. Subtract 4:	$y = (x \mathbf{-4})^2$	Shift 4 units right.
2. Multiply by -3:	$y = \mathbf{-3}(x - 4)^2$	Stretch by a factor of 3, and reflect about the x-axis.
3. Add 6:	$y = -3(x - 4)^2 \mathbf{+6}$	Shift up 6 units.

We perform the same transformations on the graph of $y = x^2$, as shown in the figure.



□

Checkpoint 6.4.9

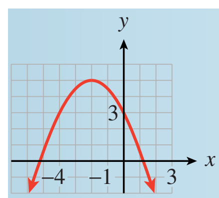
a List the transformations of $y = x^2$ needed to graph $g(x) = 5 - \frac{1}{2}(x+2)^2$.

b Use transformations to sketch the graph.

Answer.

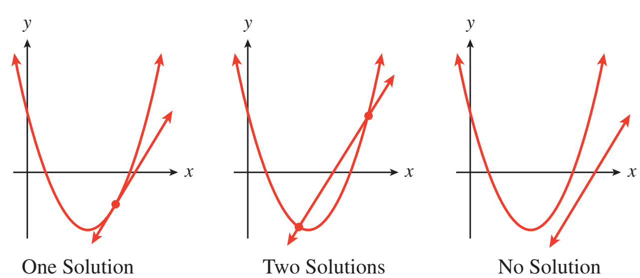
a Shift 2 units left, reflect about x-axis and compress by a factor of 2, shift 5 units up.

b



6.4.4 Systems Involving Quadratic Equations

Recall that the solution to a 2×2 system of linear equations is the intersection point of the graphs of the equations. This is also true of systems in which one or both of the equations is quadratic. Such a system may have either one solution, two solutions, or no solutions. The figure below shows the three cases for systems of one quadratic and one linear equation. (See Algebra Review Refresher Section A.5, p. 887.)



In Example 6.4.10, p. 662, we use both graphical and algebraic techniques to solve the system.

Example 6.4.10 The Pizza Connection calculates that the cost, in dollars, of producing x pizzas per day is given by

$$C = 0.15x^2 + 0.75x + 180$$

The Pizza Connection charges \$15 per pizza, so the revenue from selling x pizzas is

$$R = 15x$$

How many pizzas per day must the Pizza Connection sell in order to break even?

Solution. To break even means to make zero profit. Because

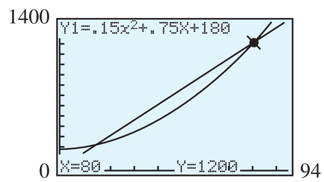
$$\text{Profit} = \text{Revenue} - \text{Cost}$$

the break-even points occur when revenue equals cost. In mathematical terms, we would like to find any values of x for which $R = C$.

If we graph the revenue and cost functions on the same axes, these values correspond to points where the two graphs intersect. Use the window settings

$$\begin{aligned} \text{Xmin} = 0 \quad \text{Xmax} = 94 \\ \text{Ymin} = 0 \quad \text{Ymax} = 1400 \end{aligned}$$

on your calculator to obtain the graph shown below. You can verify that the two intersection points are $(15, 225)$ and $(80, 1200)$.



Thus, the Pizza Connection must sell either 15 or 80 pizzas in order to break even. On the graph we see that revenue is greater than cost for x-values between 15 and 80, so the Pizza Connection will make a profit if it sells between 15 and 80 pizzas.

We can also solve algebraically for the break-even points. The intersection points of the two graphs correspond to the solutions of the system of equations

$$\begin{aligned} y &= 0.15x^2 + 0.75x + 180 \\ y &= 15x \end{aligned}$$

We equate the two expressions for y and solve for x:

$$0.15x^2 + 0.75x + 180 = 15x \quad \text{Subtract } \mathbf{15x} \text{ from both sides.}$$

$$0.15x^2 - 14.25x + 180 = 0 \quad \text{Use the quadratic formula.}$$

$$\begin{aligned} x &= \frac{14.25 \pm \sqrt{14.25^2 - 4(0.15)(180)}}{2(-0.05)} \quad \text{Simplify.} \\ &= \frac{14.25 \pm 9.75}{0.3} \end{aligned}$$

The solutions are 15 and 80, as we found from the graph. □

Checkpoint 6.4.11

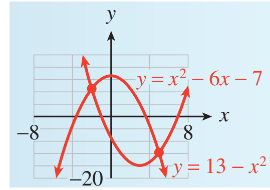
a Solve the system algebraically:

$$\begin{aligned} y &= x^2 - 6x - 7 \\ y &= 13 - x^2 \end{aligned}$$

b Graph both equations, and show the solutions on the graph.

Answer.

a $(-2, 9), (5, -12)$

b 

6.4.5 Section Summary

6.4.5.1 Vocabulary

Look up the definitions of new terms in the Glossary.

- Maximum value
- Minimum value
- Vertex form

6.4.5.2 CONCEPTS

1 Quadratic models may arise as the product of two variables.

2 The maximum or minimum of a quadratic function occurs at the vertex.

3 Vertex Form for a Quadratic Function.

A quadratic function $y = ax^2 + bx + c$, $a \neq 0$, can be written in the vertex form

$$y = a(x - x_v)^2 + y_v$$

where the vertex of the graph is (x_v, y_v).

4 We can convert a quadratic equation to vertex form by completing the square.

5 We can graph a quadratic equation in vertex form using transformations.

6 A 2×2 system involving quadratic equations may have one, two, or no solutions.

6.4.5.3 STUDY QUESTIONS

1 How can you tell whether a variable given by a quadratic equation has a maximum value or a minimum value?

2 What is wrong with this statement: The maximum or minimum value given by a quadratic equation is the average of the x-intercepts?

3 Explain why -4 is the smallest function value for $f(x) = 2(x - 3)^2 - 4$.

4 In the equation $y = \dfrac{1}{3}(x + 5)^2 - 2$, what does each of the constants tell you about the graph?

5 Francine attempts to write the equation $g(x) = 2x^2 - 6x + 1$ in vertex form as follows: $g(x) = (2x^2 - 6x + \mathbf{9}) + 1 - \mathbf{9}$. What is wrong with her work?

6 Without doing any calculations, solve the system $y = x^2 + 4, y = 2 - 3x^2$. (*Hint*: Visualize the graphs.)

6.4.5.4 SKILLS

Practice each skill in the Homework 6.4.6, p. 665 problems listed.

1 Find the maximum or minimum value of a quadratic function: #1–14

2 Convert a quadratic equation from vertex form to standard form: #19–22

3 Convert a quadratic equation from standard form to vertex form: #23–28

4 Use transformations to graph a quadratic equation: #15–28

5 USolve a system involving quadratic equations: #31–50

6.4.6 Problem Solving (Homework 6.4)

1. The owner of a motel has 60 rooms to rent. She finds that if she charges \$0 per room per night, all the rooms will be rented. For every \$2 that she increases the price of a room, 3 rooms will stand vacant.

(a) Complete the table. The first two rows are filled in for you.

No. of price increases	Price of room	No. of rooms rented	Total revenue
0	20	60	1200
1	22	57	1254
2			
3			
4			
5			
6			
7			
8			
10			
12			
16			
20			

(b) Let x stand for the number of \$2 price increases the owner makes. Write algebraic expressions for the price of a room, the number of rooms that will be rented, and the total revenue earned at that price.

(c) Use your calculator to make a table of values for your algebraic expressions. Let Y_1 stand for the price of a room, Y_2 for the number of rooms rented, and Y_3 for the total revenue. Verify the values you calculated in part (a).

(d) Use your table to find a value of x that causes the total revenue to be zero.

(e) Use your graphing calculator to graph your formula for total revenue.

(f) What is the lowest price that the owner can charge for a room if she wants her revenue to exceed \$1296 per night? What is the highest price she can charge to obtain this revenue?

(g) What is the maximum revenue the owner can earn in one night? How much should she charge for a room to maximize her revenue? How many rooms will she rent at that price?

Answer.

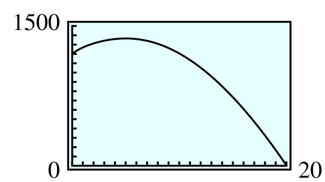
(a)

No. of price increases	Price of room	No. of rooms rented	Total revenue
0	20	60	1200
1	22	57	1254
2	24	54	1296
3	26	51	1326
4	28	48	1344
5	30	45	1350
6	32	42	1344
7	34	39	1326
8	36	36	1296
10	40	30	1200
12	44	24	1056
16	52	12	624
20	60	0	0

(b) Price of a room: $20 + 2x$; Rooms rented: $60 - 3x$; Revenue: $1200 + 60x - 6x^2$

(c) 20

(d)



(e) \$24; \$36

(f) \$1350; \$30; 45 rooms

2. The owner of a video store sells 96 blank tapes per week if he charges \$6 per tape. For every \$0.50 he increases the price, he sells 4 fewer tapes per week.

(a) Complete the table. The first two rows are filled in for you.

No. of price increases	Price of tape	No. of tapes sold	Total revenue
0	6	96	576
1	6.50	92	598
2			
3			
4			
5			
6			
7			
8			
12			
16			
20			
24			

(b) Let x stand for the number of \$0.50 price increases the owner makes. Write algebraic expressions for the price of a tape, the number of tapes sold, and the total revenue.

(c) Use your calculator to make a table of values for your algebraic expressions. Let Y_1 stand for the price of a tape, Y_2 for the number of tapes sold, and Y_3 for the total revenue. Verify the values you calculated in part (a).

(d) Use your table to find a value of x that causes the total revenue to be zero.

(e) Use your graphing calculator to graph your formula for total revenue.

(f) How much should the owner charge for a tape in order to bring in \$630 per week from tapes? (You should have two answers.)

(g) What is the maximum revenue the owner can earn from tapes in one week? How much should he charge for a tape to maximize his revenue? How many tapes will he sell at that price?

3.

(a) Give the dimensions of two different rectangles with perimeter 60 meters. Compute the areas of the two rectangles.

(b) A rectangle has a perimeter of 60 meters. If the length of the rectangle is x meters, write an expression for its width.

(c) Write an expression for the area of the rectangle.

Answer.

(a) (For example) 10 m by 20 m with area 200 sq m; or 15 m by 15 m, area 225 sq m

(b) $30 - x$

(c) $30x - x^2$

4.

(a) Give the dimensions of two different rectangles with perimeter 48 inches. Compute the areas of the two rectangles.

(b) A rectangle has a perimeter of 48 inches. If the width of the rectangle is w inches, write an expression for its length.

(c) Write an expression for the area of the rectangle.

For Problems 5–8,

(a) Find the maximum or minimum value algebraically.

(b) Obtain a good graph on your calculator and verify your answer. (Use the coordinates of the vertex and the vertical intercept to help you choose an appropriate window for the graph.)

5. Delbert launches a toy water rocket from ground level. Its distance above the ground t seconds after launch is given, in feet, by

$$d = 96t - 16t^2$$

When will the rocket reach its greatest height, and what will that height be?

Answer. 3 sec, 144 ft

6. Francine throws a wrench into the air from the bottom of a trench 12 feet deep. Its height t seconds later is given, in feet, by

$$h = -12 + 32t - 16t^2$$

When will the wrench reach its greatest height, and what will that height be?

7. The owners of a small fruit orchard decide to produce gift baskets as a sideline. The cost per basket for producing x baskets is

$$C = 0.01x^2 - 2x + 120$$

How many baskets should they produce in order to minimize the cost per basket? What will their total cost be at that production level?

Answer. 100 baskets, \$2000

8. A new electronics firm is considering marketing a line of telephones. The cost per phone for producing x telephones is

$$C = 0.001x^2 - 3x + 2270$$

How many telephones should the firm produce in order to minimize the cost per phone? What will the firm's total cost be at that production level?

9. As part of a collage for her art class, Sheila wants to enclose a rectangle with 100 inches of yarn.

(a) Let w represent the width of the rectangle, and write an expression for its length. Then write an expression that gives the area, A, of the rectangle as a function of its width, w.

(b) What is the area of the largest rectangle that Sheila can enclose with 100 inches of yarn?

Answer.

(a) Length: $50 - w$; Area: $50w - w^2$

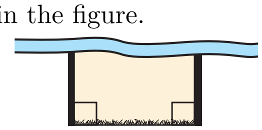
(b) 625 sq in

10. Gavin has rented space for a booth at the county fair. As part of his display, he wants to rope off a rectangular area with 80 yards of rope.

(a) Let w represent the width of the roped-off rectangle, and write an expression for its length. Then write an expression that gives the area, A, of the roped-off space as a function of its width, w.

(b) What is the largest area that Gavin can rope off? What will the dimensions of the rectangle be?

11. A farmer plans to fence a rectangular grazing area along a river with 300 yards of fence as shown in the figure.



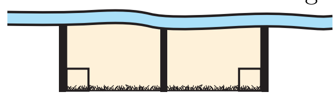
(a) Write an expression that gives the area, A, of the grazing land as a function of the width, w, of the rectangle.

(b) What is the largest area the farmer can enclose?

Answer.

(a) $300w - 2w^2$ (b) $11,250$ sq yd

12. A breeder of horses wants to fence two rectangular grazing areas along a river with 600 meters of fence as shown in the figure.



(a) Write an expression that gives the area, A, of the grazing land as a function of the width, w, of the rectangles.

(b) What is the largest area the breeder can enclose?

13. A travel agent offers a group rate of \$2400 per person for a week in London if 16 people sign up for the tour. For each additional person who signs up, the price per person is reduced by \$100.

(a) Let x represent the number of additional people who sign up. Write expressions for the total number of people signed up, the price per person, and the total revenue.

(b) How many people must sign up for the tour in order for the travel agent to maximize her revenue?

Answer.

(a) Number of people: $16 + x$; Price per person: $2400 - 100x$; Total revenue: $38,400 + 800x - 100x^2$

(b) 20

14. An entrepreneur buys an apartment building with 40 units. The previous owner charged \$240 per month for a single apartment and on the average rented 32 apartments at that price. The entrepreneur discovers that for every \$20 he raises the price, another apartment stands vacant.

(a) Let x represent the number of \$20 price increases. Write expressions for the new price, the number of rented apartments, and the total revenue.

(b) What price should the entrepreneur charge for an apartment in order to maximize his revenue?

15. During a statistical survey, a public interest group obtains two estimates for the average monthly income of young adults aged 18 to 25. The first estimate is \$860 and the second estimate is \$918. To refine its estimate, the group will take a weighted average of these two figures:

$$I = 860a + 918(1 - a) \quad \text{where} \quad 0 \le a \le 1$$

To get the best estimate, the group must choose a to minimize the function

$$V = 576a^2 + 5184(1 - a)^2$$

(The numbers that appear in this expression reflect the **variance** of the data, which measures how closely the data cluster around the mean, or average.) Find the value of a that minimizes V, and use this value to get a refined estimate for the average income.

Answer. $a = 0.9$; $I = \$865.80$

16. The rate at which an antigen precipitates during an antigen-antibody reaction depends upon the amount of antigen present. For a fixed quantity of antibody, the time required for a particular antigen to precipitate is given in minutes by the function

$$t = 2w^2 - 20w + 54$$

where w is the quantity of antigen present, in grams. For what quantity of antigen will the reaction proceed most rapidly, and how long will the precipitation take?

For Problems 17-20, use transformations to graph the parabola. What is the vertex of each graph?

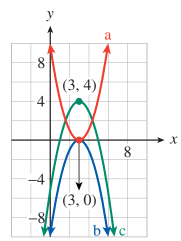
17.

(a) $y = (x-3)^2$

(b) $y = -(x-3)^2$

(c) $y = -(x-3)^2 + 4$

Answer.



18.

(a) $y = (x+1)^2$

(b) $y = 2(x+1)^2$

(c) $y = 2(x+1)^2 - 4$

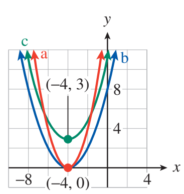
19.

(a) $y = (x+4)^2$

(b) $y = \frac{1}{2}(x+4)^2$

(c) $y = 3 + \frac{1}{2}(x+4)^2$

Answer.



20.

(a) $y = (x-2)^2$

(b) $y = -(x-2)^2$

(c) $y = -3 - (x-2)^2$

In Problems 21–24,

a Find the vertex of the parabola.

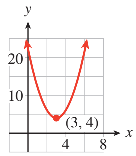
b Use transformations to sketch the graph.

c Write the equation in standard form.

21. $y = 2(x-3)^2 + 4$

Answer.

(a) $(3, 4)$

(b) 

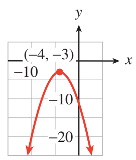
(c) $y = 2x^2 - 12x + 22$

22. $y = -3(x+1)^2 - 2$

23. $y = -\dfrac{1}{2}(x+4)^2 - 3$

Answer.

(a) $(-4, -3)$

(b) 

(c) $y = \dfrac{-1}{2}x^2 - 4x - 11$

24. $y = 4(x-2)^2 - 6$

For Problems 25–30,

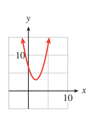
a Write each equation in the form $y = a(x-p)^2 + q$ by completing the square.

b Using horizontal and vertical translations, sketch the graph by hand.

25. $y = x^2 - 4x + 7$

Answer.

(a) $y = (x-2)^2 + 3$

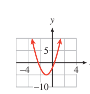
(b) 

26. $y = x^2 - 2x - 1$

27. $y = 3x^2 + 6x - 2$

Answer.

(a) $y = 3(x+1)^2 - 5$

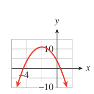
(b) 

28. $y = \dfrac{1}{2}x^2 + 2x + 5$

29. $y = -2x^2 - 8x + 3$

Answer.

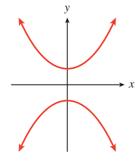
(a) $y = -2(x+2)^2 + 11$

(b) 

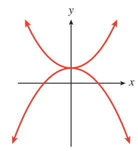
30. $y = -x^2 + 5x + 2$

31. A system of two quadratic equations may have no solution, one solution, or two solutions. Sketch a system illustrating each case. In your sketches, one of the parabolas should open up, and the other down.

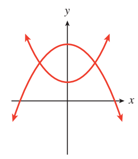
Answer. No solutions:



One solution:



Two solutions:



32. A system of two quadratic equations may have no solution, one solution, or two solutions. Sketch a system illustrating each case. In your sketches, both parabolas should open up.

For Problems 33–44, solve the system algebraically. Use your calculator to graph both equations and verify your solutions.

33. $y = x^2 - 4x + 7$
$y = 11 - x$
Answer. $(-1, 12), (4, 7)$

34. $y = x^2 + 6x + 4$
$y = 3x + 8$

35. $y = -x^2 - 2x + 7$
$y = 2x + 11$
Answer. $(-2, 7)$

36. $y = x^2 - 8x + 17$
$y + 4x = 13$

37. $y = x^2 + 8x + 8$
$3y + 2x = -36$
Answer. No solution

38. $y = -x^2 + 4x + 2$
$4y - 3x = 24$

39. $y = x^2 - 9$
$y = -2x^2 + 9x + 21$
Answer. $(-2, -5), (5, 16)$

40. $y = 4 - x^2$
$y = 3x^2 - 12x - 12$

41. $y = x^2 - 0.5x + 3.5$
$y = -x^2 + 3.5x + 1.5$
Answer. $(1, 4)$

42. $y = x^2 + 10x + 22$
$y = -0.5x^2 - 8x - 32$

43. $y = x^2 - 4x + 4$
$y = x^2 - 8x + 16$
Answer. $(3, 1)$

44. $y = 0.5x^2 + 3x + 5.5$
$y = 2x^2 + 12x + 4$

Problems 45–48 deal with wildlife management and sustainable yield.

45. In Problem 6.3.9.9, p. 647 of Section 6.3, p. 633, you graphed the annual growth rate of a population of fish,

$$R = f(x) = 0.4x - 0.0001x^2$$

where x is the current biomass of the population, in tons.

(a) Suppose that fishermen harvest 300 tons of fish each year. Sketch the graph of $H = 300$ on the same axes with your graph of y.

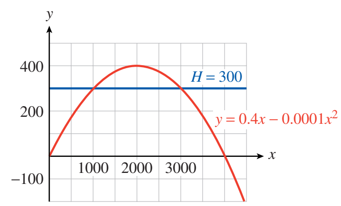
(b) If the biomass is currently 2500 tons and 300 tons are harvested, will the population be larger or smaller next year? By how much? What if the biomass is currently 3500 tons?

(c) What sizes of biomass will remain stable from year to year if 300 tons are harvested annually?

(d) If the biomass ever falls below 1000 tons, what will happen after several years of harvesting 300 tons annually?

Answer.

(a)



(b) Larger, by 75 tons. Smaller, by 125 tons.

(c) 1000 tons and 3000 tons

(d) The fish population will decrease each year until it is completely depleted.

46. In Problem 6.3.9.10, p. 648 of Section 6.3, p. 633, you graphed the annual increase, I, in the deer population in a national park,

$$I = g(x) = 1.2x - 0.0002x^2$$

where x is the current population.

(a) Suppose hunters are allowed to kill 1000 deer per year. Sketch the graph of $H = 1000$ on the same axes with a graph of y.

(b) What sizes of deer populations will remain stable from year to year if 1000 deer are hunted annually?

(c) Suppose 1600 deer are killed annually. What sizes of deer populations will remain stable?

(d) What is the largest annual harvest that still allows for a stable population? (This harvest is called the maximum sustainable yield.) What is the stable population?

(e) What eventually happens if the population falls below the stable value but hunting continues at the maximum sustainable yield?

47. The annual increase, N, in a bear population of size x is given by

$$N = F(x) = 0.0002x(2000 - x)$$

if the bears are not hunted. The number of bears killed each year by hunters is related to the bear population by the equation $K = 0.2x$. (Notice that in this model, hunting is adjusted to the size of the bear population.)

(a) Sketch the graphs of N and K on the same axes.

(b) When the bear population is 1200, which is greater, N or K? Will the population increase or decrease in the next year? By how many bears?

(c) When the bear population is 900, will the population increase or decrease in the next year? By how many bears?

(d) What sizes of bear population will remain stable after hunting?

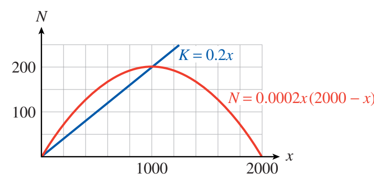
(e) What sizes of bear populations will increase despite hunting? What sizes of populations will decrease?

(f) Toward what size will the population tend over time?

(g) Suppose hunting limits are raised so that $K = 0.3x$. Toward what size will the population tend over time?

Answer.

(a)



(b) $K > N$. The population will decrease by 48 bears.

(c) The population will increase by 18 bears.

(d) 1000

(e) Populations between 0 and 1000 will increase; populations over 1000 will decrease.

(f) 1000 (unless the population is 0)

(g) 500 (unless the population is 0)

48. The annual increase in the biomass of a whale population is given in tons by

$$w = G(x) = 0.001x(1000 - x)$$

where x is the current population, also in tons.

(a) Sketch a graph of w for $0 \leq x \leq 1100$. What size biomass remains stable?

(b) Each year hunters are allowed to harvest a biomass given by $H = 0.6x$. Sketch H on the same graph with w. What is the stable biomass with hunting?

(c) What sizes of populations will increase despite hunting? What sizes will decrease?

(d) What size will the population approach over time? What biomass are hunters allowed to harvest for that size population?

(e) Find a value of k so that the graph of $H = kx$ will pass through the vertex of $w = 0.001x(1000 - x)$.

(f) For the value of k found in part (e), what size will the population approach over time? What biomass are hunters allowed to harvest for that size population?

(g) Explain why the whaling industry should prefer hunting quotas of kx rather than $0.6x$ for a long-term strategy, even though $0.6x > kx$ for any positive value of x.

For Problems 49–52,

a Find the break-even points by solving a system of equations.

b Graph the equations for Revenue and Cost in the same window and verify your solutions on the graph.

c Use the fact that

$$\text{Profit} = \text{Revenue} - \text{Cost}$$

to find the value of x for which profit is maximum.

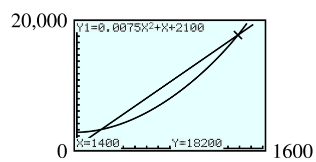
49. Writewell, Inc. makes fountain pens. It costs Writewell

$$C = 0.0075x^2 + x + 2100$$

dollars to manufacture x pens, and the company receives $R = 13x$ dollars in revenue from the sale of the pens.

Answer.

(a) $(200,\ 2600), (1400,\ 18,200)$

(b) 

(c) $x = 800$

50. It costs The Sweetshop

$$C = 0.01x^2 + 1836$$

dollars to produce x pounds of chocolate creams. The company brings in $R = 12x$ dollars revenue from the sale of the chocolates.

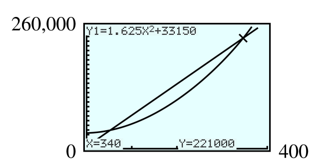
51. It costs an appliance manufacturer

$$C = 1.625x^2 + 33,150$$

dollars to produce x front-loading washing machines, which will then bring in revenues of $R = 650x$ dollars.

Answer.

(a) $(60,\ 39,000), (340,\ 221,000)$

(b) 

(c) $x = 200$

52. A company can produce x lawn mowers for a cost of

$$C = 0.125x^2 + 100,000$$

dollars. The sale of the lawn mowers will generate $R = 300x$ dollars in revenue.

Problems 53 and 54 prove that the vertical line $x = \dfrac{-b}{2a}$ is the axis of symmetry of the graph of $f(x) = ax^2 + bx + c$. A graph is **symmetric about the line**

$x = h$ if the point $(h + d, v)$ lies on the graph whenever the point $(h - d, v)$ lies on the graph.

53.

(a) Sketch a parabola $f(x) = a(x - h)^2 + k$ and the line $x = h$. We will show that the parabola is symmetric about the line $x = h$.

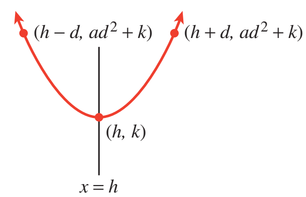
(b) Label a point on the parabola with x-coordinate $x = h+d$, where $d > 0$. What is the y-coordinate of that point?

(c) Label the point on the parabola with x-coordinate $x = h - d$. What is the y-coordinate of that point?

(d) Explain why your answers to parts (b) and (c) prove that the line $x = h$ is the axis of symmetry for the graph of $f(x) = a(x - h)^2 + k$.

Answer.

(a)



(b) See graph and (c)

(c) $ad^2 + k$

(d) The two points on the parabola that are the same horizontal distance from the line $x = h$ the axis of symmetry have the same y-coordinate, so they are symmetric about that line.

54. To find the axis of symmetry for the graph of $g(x) = ax^2 + bx + c$, we will use the results of Problem 51 and the technique of completing the square.

(a) Write the equation $y = ax^2 + bx + c$ in vertex form by completing the square. (Follow the steps in Example 6.4.6, p. 661.)

(b) Your answer to part (a) has the form $y = a(x - h)^2 + k$. What is your value of h? What is your value of k?

(c) What is the axis of symmetry for the parabola $g(x) = ax^2 + bx + c$?

6.5 Chapter Summary and Review

6.5.1 Key Concepts

1 A **quadratic** function has the form $f(x) = ax2 + bx + c$, where a, b, and c are constants and a is not equal to zero.

2 **Zero-Factor Principle.**

The product of two factors equals zero if and only if one or both of the factors equals zero. In symbols,

$$ab = 0 \quad \text{if and only if} \quad a = 0 \quad \text{or} \quad b = 0$$

3 The x-intercepts of the graph of $y = f(x)$ are the solutions of the equation $f(x) = 0$.

4 A quadratic equation written as $ax^2 + bx + c = 0$ is in **standard form**.

A quadratic equation written as $a(x - r_1)(x - r_2) = 0$ is in **factored form**.

5 **To Solve a Quadratic Equation by Factoring.**

1 Write the equation in standard form.

2 Factor the left side of the equation.

3 Apply the zero-factor principle: Set each factor equal to zero.

4 Solve each equation. There are two solutions (which may be equal).

6 Every quadratic equation has two solutions, which may be the same.

7 The value of the constant a in the factored form of a quadratic equation does not affect the solutions.

8 Each solution of a quadratic equation corresponds to a factor in the factored form.

9 An equation is called **quadratic in form** if we can use a substitution to write it as $au^2 + bu + c = 0$, where u stands for an algebraic expression.

10 The square of the binomial is a **quadratic trinomial**,

$$(x + p)^2 = x^2 + 2px + p^2$$

11 **To Solve a Quadratic Equation by Completing the Square.**

1 a Write the equation in standard form.
  b Divide both sides of the equation by the coefficient of the quadratic term, and subtract the constant term from both sides.

2 Complete the square on the left side:
  a Multiply the coefficient of the first-degree term by one-half, then square the result.
  b Add the value obtained in (a) to both sides of the equation.

3 Write the left side of the equation as the square of a binomial. Simplify the right side.

4 Use extraction of roots to finish the solution.

12 **The Quadratic Formula.**

The solutions of the equation $ax^2 + bx + c = 0, \quad a \neq 0$, are

$$\boldsymbol{x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}}$$

13 We have four methods for solving quadratic equations: extracting roots, factoring, completing the square, and using the quadratic formula. The first two methods are faster, but they do not work on all equations. The last two methods work on any quadratic equation.

14 The graph of a quadratic function $f(x) = ax^2 + bx + c$ is called a **parabola**. The values of the constants a, b, and c determine the location and orientation of the parabola.

15 For the graph of $y = ax^2 + bx + c$, the x-coordinate of the vertex is $x_v = \dfrac{-b}{2a}$.
To find the y-coordinate of the vertex, we substitute x_v into the formula for the parabola.

16 The graph of the quadratic function $y = ax^2 + bx + c$ may have two, one, or no x-intercepts, according to the number of distinct real-valued solutions of the equation $ax^2 + bx + c = 0$.

17 **The Discriminant.**

The **discriminant** of a quadratic equation is $D = b^2 - 4ac$.

1 If $D > 0$, there are two unequal real solutions.

2 If $D = 0$, there is one real solution of multiplicity two.

3 If $D < 0$, there are two complex solutions.

18 **To Graph the Quadratic Function $f(x) = ax^2 + bx + c$:.**

1 Determine whether the parabola opens upward (if $a > 0$) or downward (if $a < 0$).

2 Locate the vertex of the parabola.

  a The x-coordinate of the vertex is $x_v = \dfrac{-b}{2a}$.

  b Find the y-coordinate of the vertex by substituting x_v into the equation of the parabola.

3 Locate the x-intercepts (if any) by setting $y = 0$ and solving for x.

4 Locate the y-intercept by evaluating y for $x = 0$.

5 Locate the point symmetric to the y-intercept across the axis of symmetry.

19 Quadratic models may arise as the product of two variables.

20 The maximum or minimum of a quadratic function occurs at the vertex.

21 **Vertex Form for a Quadratic Function.**

A quadratic function $y = ax^2 + bx + c$, $a \neq 0$, can be written in the vertex form

$$y = a(x - x_v)^2 + y_v$$

where the vertex of the graph is (x_v, y_v).

22 We can convert a quadratic equation to vertex form by completing the square.

23 We can graph a quadratic equation in vertex form using transformations.

24 A 2×2 system involving quadratic equations may have one, two, or no solutions.

25 We can use a graphical technique to solve quadratic inequalities.

26 **To Solve a Quadratic Inequality Algebraically:.**

1 Write the inequality in standard form: One side is 0, and the other has the form $ax^2 + bx + c$.

2 Find the x-intercepts of the graph of $y = ax^2 + bx + c$ by setting $y = 0$ and solving for x.

3 Make a rough sketch of the graph, using the sign of a to determine whether the parabola opens upward or downward.

4 Decide which intervals on the x-axis give the correct sign for y.

27 We need three points to determine a parabola.

28 We can use the method of elimination to find the equation of a parabola through three points.

29 If we know the vertex of a parabola, we need only one other point to find its equation.

30 We can use quadratic regression to fit a parabola to a collection of data points.

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6.5.2 Chapter 6 Review Problems

For Problems 1-6, solve by factoring.

1. $x^2 + x = 4 - (x + 2)^2$

Answer. $0, \ \dfrac{-5}{2}$

2. $(n - 3)(n + 2) = 6$

3. $x(3x + 2) = (x + 2)^2$

Answer. $-1, \ 2$

4. $6y = (y + 1)^2 + 3$

5. $4x - (x+1)(x+2) = -8$

Answer. $-2,\ 3$

6. $3(x+2)^2 = 15 + 12x$

For Problems 7-8, write a quadratic equation with integer coefficients and with the given solutions.

7. $\dfrac{-3}{4}$ and 8

Answer. $4x^2 - 29x - 24 = 0$

8. $\dfrac{5}{3}$ and $\dfrac{5}{3}$

For Problems 9-10, graph the equation using the **ZDecimal** setting. Locate the x-intercepts and use them to write the quadratic expression in factored form.

9. $y = x^2 - 0.6x - 7.2$

Answer.
$y = (x-3)(x+2.4)$

10. $y = -x^2 + 0.7x + 2.6$

For Problems 11-14, use a substitution to solve.

11. $2^{2p} - 6 \cdot 2^p + 8 = 0$

Answer. $1,\ 2$

12. $3^{2r} - 6 \cdot 3^r + 5 = 0$

13. $\left(\dfrac{1}{b}\right)^2 - 3\left(\dfrac{1}{b}\right) - 4 = 0$

Answer. $-1,\ \dfrac{1}{4}$

14. $\left(\dfrac{1}{q}\right)^2 + \dfrac{1}{q} - 2 = 0$

For problems 15-18, solve by completing the square.

15. $x^2 - 4x - 6 = 0$

Answer. $2 \pm \sqrt{10}$

16. $x^2 + 3x = 3$

17. $2x^2 + 3 = 6x$

Answer. $\dfrac{3 \pm \sqrt{3}}{2}$

18. $3x^2 = 2x + 3$

For Problems 19-22, solve by using the quadratic formula.

19. $\dfrac{1}{2}x^2 + 1 = \dfrac{3}{2}x$

Answer. $1,\ 2$

20. $x^2 - 3x + 1 = 0$

21. $x^2 - 4x + 2 = 0$

Answer. $2 \pm \sqrt{2}$

22. $2x^2 + 2x = 3$

For Problems 23-26, solve the formula for the indicated variable.

23. $K = \dfrac{1}{2}mv^2$, for v

Answer. $\pm\sqrt{\dfrac{2K}{m}}$

24. $a^2 + b^2 = c^2$, for b

25. $h = 6t - 3t^2$, for t

Answer. $\dfrac{3 \pm \sqrt{9 - 3h}}{3}$

26. $D = \dfrac{n^2 - 3n}{2}$, for n

27. In a tennis tournament among n competitors, $\dfrac{n(n-1)}{2}$ matches must be played. If the organizers can schedule 36 matches, how many players should they invite?

Answer. 9

28. The formula $S = \dfrac{n(n+1)}{2}$ gives the sum of the first n positive integers. How many consecutive integers must be added to make a sum of 91?

29. Irene wants to enclose two adjacent chicken coops of equal size against the henhouse wall. She has 66 feet of chicken wire fencing and would like the total area of the two coops to be 360 square feet. What should the dimensions of the chicken coops be?

Answer. 10 ft by 18 ft or 12 ft by 15 ft

30. The base of an isosceles triangle is one inch shorter than the equal sides, and the altitude of the triangle is 2 inches shorter than the equal sides. What is the length of the equal sides?

31. A car traveling at 50 feet per second (about 34 miles per hour) can stop in 2.5 seconds after applying the brakes hard. The distance the car travels, in feet, t seconds after applying the brakes is $d = 50t - 10t^2$. How long does it take the car to travel 40 feet?

Answer. 1 sec

32. You have 300 feet of wire fence to mark off a rectangular Christmas tree lot with a center divider, using a brick wall as one side of the lot. If you would like to enclose a total area of 7500 square feet, what should be the dimensions of the lot?

33. The height, h, of an object t seconds after being thrown from ground level is given by

$$h = v_0 t - \frac{1}{2} g t^2$$

where v_0 is its starting velocity and g is a constant that depends on gravity. On the Moon, the value of g is approximately 5.6. Suppose you hit a golf ball on the Moon with an upward velocity of 100 feet per second.

(a) Write an equation for the height of the golf ball t seconds after you hit it.

(b) Graph your equation in the window

Xmin = 0	Xmax = 47
Ymin = 0	Ymax = 1000

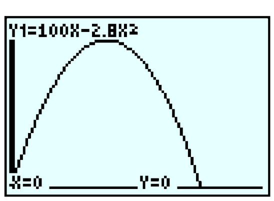
(c) Use the **Trace** to estimate the maximum height the golf ball reaches.

(d) Use your equation to calculate when the golf ball will reach a height of 880 feet.

Answer.

(a) $h = 100t - 2.8t^2$

(b)



(c) 893 ft

(d) $15\frac{5}{7}$ sec on the way up and 20 sec on the way down

34. An acrobat is catapulted into the air from a springboard at ground level. Her height, h, in meters is given by the formula

$$h = -4.9t^2 + 14.7t$$

where t is the time in seconds from launch. Use your calculator to graph the acrobat's height versus time. Use the window

$$\text{Xmin} = 0 \qquad \text{Xmax} = 4.7$$
$$\text{Ymin} = 0 \qquad \text{Ymax} = 12$$

(a) Use the **Trace** to find the coordinates of the highest point on the graph. When does the acrobat reach her maximum height, and what is that height?

(b) Use the formula to find the height of the acrobat after 2.4 seconds.

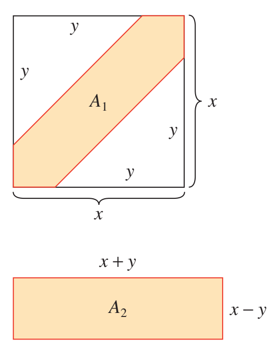
(c) Use the **Trace** to verify your answer to part (b). Find another time when the acrobat is at the same height.

(d) Use the formula to find two times when the acrobat is at a height of 6.125 meters. Verify your answers on the graph.

(e) What are the coordinates of the horizontal intercepts of your graph? What do these points have to do with the acrobat?

For problems 35-36, show that the shaded areas are equal.

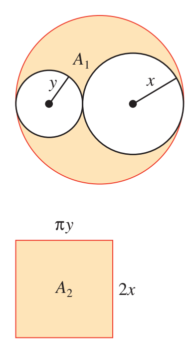
35.



Answer. A_1 is the area of a square minus the area of two triangles:

$$x^2 - 2\left(\frac{1}{2}y \cdot y\right) = x^2 - y^2$$

36.



For problems 37-46,

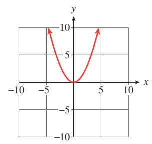
a Find the coordinates of the vertex and the intercepts.

b Sketch the graph.

37. $y = \frac{1}{2}x^2$

Answer.

(a) Vertex and intercepts are all $(0, 0)$.

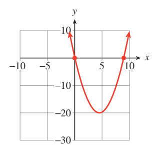
(b) 

38. $y = x^2 - 4$

39. $y = x^2 - 9x$

Answer.

(a) Vertex $(\frac{9}{2}, \frac{-81}{4})$; x-intercepts $(9, 0)$ and $(0, 0)$; y-intercept $(0, 0)$

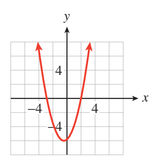
(b) 

40. $y = -2x^2 - 4x$

41. $y = x^2 + x - 6$

Answer.

(a) Vertex $(\frac{-1}{2}, \frac{-25}{4})$; x-intercepts $(-3, 0)$ and $(2, 0)$; y-intercept $(0, -6)$

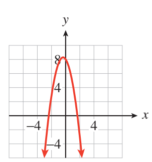
(b) 

42. $y = x^2 - 3x + 4$

43. $y = 8 - x - 2x^2$

Answer.

(a) Vertex $(\frac{-1}{4}, \frac{65}{8})$; x-intercepts $\left(\frac{-1 \pm \sqrt{65}}{4}, 0\right)$; y-intercept $(0, 8)$

(b) 

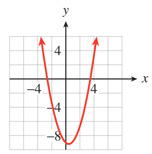
44. $y = -2x^2 + x - 4$

45. $y = x^2 - x - 9$

Answer.

(a) Vertex $(\frac{1}{2}, \frac{-37}{4})$;
x-intercepts
$\left(\frac{1 \pm \sqrt{37}}{2}, 0\right)$;
y-intercept $(0, -9)$

(b)



46. $y = -x^2 + 2x + 4$

For problems 47-48, use the discriminant to determine how many x-intercepts the graph has.

47. $y = -2x^2 + 5x - 1$

Answer. Two

48. $y = -12 - 3x + 4x^2$

For Problems 49-52, use the discriminant to determine the nature of the solution of each equation.

49. $4x^2 - 12x + 9 = 0$

Answer. One rational solution

50. $2t^2 + 6t + 5 = 0$

51. $2y^2 = 3y - 4$

Answer. No real solutions

52. $\frac{x^2}{4} = x + \frac{5}{4}$

53. The total profit Kiyoshi makes from producing and selling x floral arrangements is

$$P(x) = -0.4x^2 + 36x - 400$$

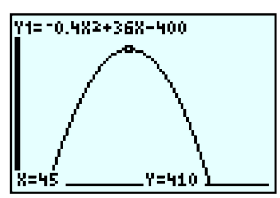
(a) How many floral arrangements should Kiyoshi produce and sell to maximize his profit? What is his maximum profit?

(b) Verify your answers on a graph.

Answer.

(a) 45; \$410

(b)



54. Lightning does about one billion dollars damage annually in the United States and kills 85 people. To study lightning, meteorologists fire small rockets at passing thunderclouds to induce lightning bolts. The rocket trails a thin copper wire that is vaporized by the lightning, leaving a plasma channel that carries the current to the grounding point. The rocket boosts the wire to a height of 250 meters, and t seconds later, its height is given in meters by

$$h(t) = -4.9t^2 + 32t + 250.$$

(a) When does the rocket reach its maximum height? What is the maximum height?

(b) Verify your answers on a graph.

55. A beekeeper has beehives distributed over 60 square miles of pastureland. When she places 4 hives per square mile, each hive produces about 32 pints of honey per year. For each additional hive per square mile, honey production drops by 4 pints per hive.

(a) Write a function for the total production of honey, in pints, in terms of the number of additional hives per square mile.

(b) How many additional hives per square mile should the beekeeper install in order to maximize honey production?

Answer.

(a) $y = 60(4 + x)(32 - 4x)$ (b) 2

56. A small company manufactures radios. When it charges \$20 for a radio, it sells 500 radios per month. For each dollar the price is increased, 10 fewer radios are sold per month.

(a) Write a function for the monthly revenue in terms of the price increase over \$20.

(b) What should the company charge for a radio in order to maximize its monthly revenue?

For Problems 57–60,

a Find all values of x for which $f(x) = 0$.

b Find all values of x for which $g(x) = 0$.

c Find all values of x for which $f(x) = g(x)$.

d Graph each pair of functions in the same window, then sketch the graph on paper. Illustrate your answers to (a)–(c) as points on the graph.

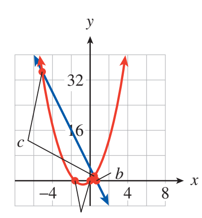
57. $f(x) = 2x^2 + 3x, \;\; g(x) = 5 - 6x$

Answer.

(a) $0, \; \dfrac{-3}{2}$

(b) $\dfrac{5}{6}$

(c) $-5, \; \dfrac{1}{2}$

(d) 

58. $f(x) = 3x^2 - 6x, \;\; g(x) = 8 + 4x$

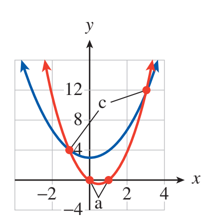
59. $f(x) = 2x^2 - 2x, \;\; g(x) = x^2 + 3$

Answer.

(a) $0, \; 1$

(b) None

(c) $-1, \; 3$

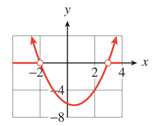
(d) 

60. $f(x) = x^2 + 4x + 6, \;\; g(x) = 4 - x^2$

For Problems 61-66, solve the inequality algebraically, and give your answers in interval notation. Verify your solutions by graphing.

61. $(x-3)(x+2)>0$

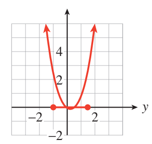
Answer. $(-\infty,-2)\cup(3,\infty)$



62. $y^2-y-12\le 0$

63. $2y^2-y\le 3$

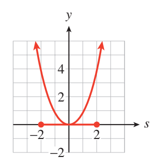
Answer. $\left[-1,\dfrac{3}{2}\right]$



64. $3z^2-5z>2$

65. $s^2\le 4$

Answer. $[-2,2]$



66. $4t^2>12$

67. The Sub Station sells $220-\dfrac{1}{4}p$ submarine sandwiches at lunchtime if it sells them at p cents each.

(a) Write a function for the Sub Station's daily revenue in terms of p.

(b) What range of prices can the Sub Station charge if it wants to keep its daily revenue from subs over $480? (Remember to convert $480 to cents.)

Answer.

(a) $R=p\left(220-\dfrac{1}{4}p\right)$

(b) Between $4.00 and $4.80

68. When it charges p dollars for an electric screwdriver, Handy Hardware will sell $30-\dfrac{1}{2}p$ screwdrivers per month.

(a) Write a function in terms of p for Handy Hardware's monthly revenue from electric screwdrivers.

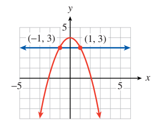
(b) How much should Handy charge per screwdriver if it wants the monthly revenue from the screwdrivers to be over $400?

For Problems 69-76, solve the system algebraically, and verify your solution with a graph.

69. $y + x^2 = 4$

$y = 3$

Answer. $(1, 3), \ (-1, 3)$



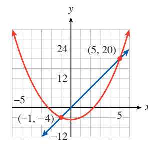
70. $y = 3 - x^2$

$5x + y = 7$

71. $y = x^2 - 5$

$y = 4x$

Answer. $(-1, -4), \ (5, 20)$



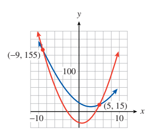
72. $y = x^2 - 2x + 1$

$y = 3 - x$

73. $y = x^2 - 6x + 20$

$y = 2x^2 - 2x - 25$

Answer. $(-9, 155), \ (5, 15)$



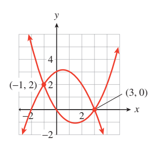
74. $y = x^2 - 5x - 28$

$y = -x^2 + 4x + 28$

75. $y = \dfrac{1}{2}x^2 - \dfrac{3}{2}x$

$y = -\dfrac{1}{2}x^2 + \dfrac{1}{2}x + 3$

Answer. $(-1, 2), \ (3, 0)$



76. $y = 2x^2 + 5x - 3$

$y = x^2 + 4x - 1$

77. Find values of a, b, and c so that the graph of the parabola $y = ax^2 + bx + c$ contains the points $(-1, -4)$, $(0, -6)$, and $(4, 6)$.

Answer. $a = 1, \ b = -1, \ c = -6$

78.

(a) Find values of a, b, and c so that the graph of the parabola $y = ax^2 + bx + c$ contains the points $(0, -2)$, $(-6, 1)$, and $(4, 6)$.

(b) Plot the data points and sketch the graph on the grid.

79. Find a parabola that fits the following data points.

x	-8	-4	2	4
y	10	18	0	-14

Answer. $p(x) = \dfrac{-1}{2}x^2 - 4x + 10$

80. Find a parabola that fits the following data points.

x	-3	0	2	4
y	-46	8	-6	-60

81. Find the equation for a parabola that has a vertex of $(15, -6)$ and passes through the point $(3, 22.8)$.

Answer. $y = 0.2(x - 15)^2 - 6$

82. Find the equation for a parabola that has a vertex of $(-3, -8)$ and passes through the point $(6, 12.25)$.

For Problems 83–86,

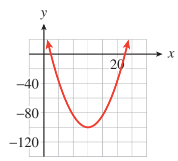
a Write the equation in vertex form.

b Use transformations to sketch the graph.

83. $f(x) = x^2 - 24x + 44$

Answer.

(a) $f(x) = (x - 12)^2 - 100$

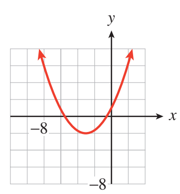
(b) 

84. $g(x) = x^2 + 30x + 300$

85. $y = \dfrac{1}{3}x^2 + 2x + 1$

Answer.

(a) $y = \dfrac{1}{3}(x + 3)^2 - 2$

(b) 

86. $y = -2x^2 + 4x + 3$

87. The height of a cannonball was observed at 0.2-second intervals after the cannon was fired, and the data were recorded in the table.

Time (seconds)	0.2	0.4	0.6	0.8	1.0	1.2	1.4	1.6	1.8	2.0
Height (meters)	10.2	19.2	27.8	35.9	43.7	51.1	58.1	64.7	71.0	76.8

(a) Find the equation of the least-squares regression line for height in terms of time.

(b) Use the linear regression equation to predict the height of the cannonball at 3 seconds and at 4 seconds after it was fired.

(c) Make a scatterplot of the data and draw the regression line on the same axes.

(d) Find the quadratic regression equation for height in terms of time.

(e) Use the quadratic regression equation to predict the height of the cannonball at 3 seconds and at 4 seconds after it was fired.

(f) Draw the quadratic regression curve on the graph from part (c).

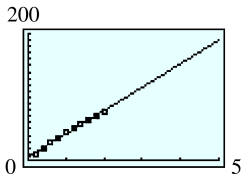
(g) Which model is more appropriate for the height of the cannonball, linear or quadratic? Why?

Answer.

(a) $h = 36.98t + 5.17$

(b) 116.1 m, 153.1 m

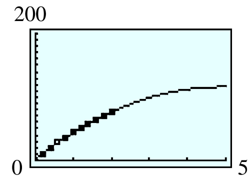
(c)



(d) $h = -4.858t^2 + 47.67t + 0.89$

(e) 100.2 m, 113.9 m

(f)



(g) Quadratic: Gravity will slow the cannonball, giving the graph a concave down shape.

88. Max took a sequence of photographs of an explosion spaced at equal time intervals. From the photographs, he was able to estimate the height and vertical velocity of some debris from the explosion, as shown in the table. (Negative velocities indicate that the debris is falling back to Earth.)

Velocity (meters/second)	67	47	27	8	−12	−31
Height (meters)	8	122	196	232	228	185

(a) Enter the data into your calculator and create a scatterplot. Fit a quadratic regression equation to the data, then graph the equation on the scatterplot.

(b) Use your regression equation to find the vertex of the parabola. What do the coordinates represent, in terms of the problem? What should the velocity of the debris be at the maximum height of the debris?

6.6 Projects for Chapter 6

Project 42 Optimum Feeding Rate. Starlings often feed in flocks, and their rate of feeding depends on the size of the flock. If the flock is too small, the birds are nervous and spend a lot of time watching for predators. If the flock is too large, the birds become overcrowded and fight each other, which interferes with feeding. Here are some data gathered at a feeding station. The data show the number of starlings in the flock and the total number of pecks per minute recorded at the station while the flock was feeding. (Source: Chapman & Reiss, 1992)

Number of starlings	Pecks per minute	Pecks per starling per minute
1	9	
2	26	
3	48	
4	80	
5	120	
6	156	
7	175	
8	152	
9	117	
10	180	
12	132	

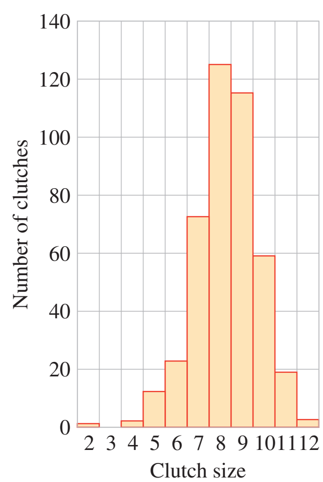
a For each flock size, calculate the number of pecks per starling per minute. For purposes of efficient feeding, what flock size appears to be optimum? How many pecks per minute would each starling make in a flock of optimal size?

b Plot the number of pecks per starling per minute against flock size. Do the data points appear to lie on (or near) a parabola?

c The quadratic regression equation for the data is $y = -0.45x^2 + 5.8x + 3.9$. Graph this parabola on the same axes with the data points.

d What are the optimum flock size and the maximum number of pecks per starling per minute predicted by the regression equation?

Project 43 Optimum clutch size.
Biologists conducted a four-year study of the nesting habits of the species **Parus major** in an area of England called Wytham Woods. The bar graph shows the clutch size (the number of eggs) in 433 nests. (Source: Perrins and Moss, 1975)



a Which clutch size was observed most frequently? Fill in the table, showing the total number of eggs produced in each clutch size.

Clutch size	2	3	4	5	6	7	8	9	10	11	12
Number of clutches	1	0	2	12	23	73	126	116	59	19	3
Number of eggs											

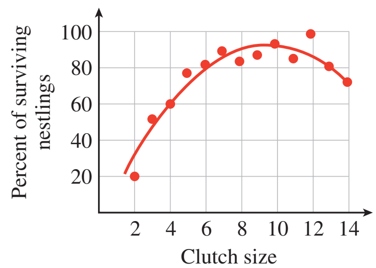
b The average weight of the nestlings declines as the size of the brood increases, and the survival of individual nestlings is linked to their weight. A hypothetical (and simplified) model of this phenomenon is described by the table below. Calculate the number of surviving nestlings for each clutch size. Which clutch size produces the largest average number of survivors?

Clutch size	1	2	3	4	5	6	7	8	9	10
Percent survival	100	90	80	70	60	50	40	30	20	10
Number of survivors										

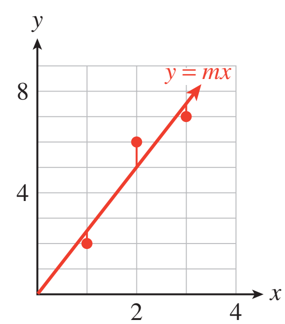
c The figure shows the number of survivors for each clutch size in Wytham Woods, along with the curve of best fit. The equation for the curve is

$$y = -0.0105x^2 + 0.2x - 0.035.$$

Find the optimal clutch size for maximizing the number of surviving nestlings. How does this optimum clutch size compare with the most frequently observed clutch size in part (a)?



Project 44 Line of best fit.
In this project, we minimize a quadratic expression to find the line of best fit. The figure shows a set of three data points and a line of best fit. For this example, the regression line passes through the origin, so its equation is $y = mx$ for some positive value of m. How shall we choose m to give the best fit for the data? We want the data points to lie as close to the line as possible. One way to achieve this is to minimize the sum of the squares of the vertical distances shown in the figure.



a The data points are $(1, 2)$, $(2, 6)$, and $(3, 7)$. Verify that the sum S we want to minimize is

$$\begin{aligned} S &= (2-m)^2 + (6-2m)^2 + (7-3m)^2 \\ &= 14m^2 - 70m + 89 \end{aligned}$$

b Graph the formula for S in the window

Xmin = 0	Xmax = 9.4
Ymin = 0	Ymax = 100

c Find the vertex of the graph of S.

d Use the value of m to write the equation of the regression line $y = mx$.

e Graph the three data points and your regression line on the same axes.

Project 45 Quadratic growth rate.

The figure shows the typical weight of two species of birds each day after hatching. (Source: Perrins, 1979)

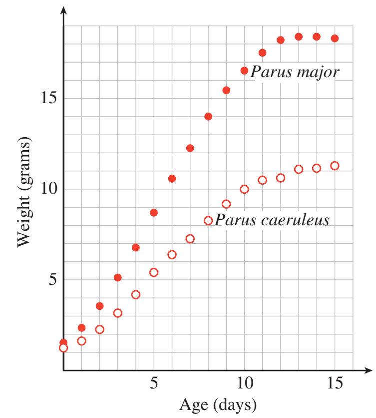


Figure 6.6.1

a Describe the rate of growth for each species over the first 15 days of life. How are the growth rates for the two species similar, and how are they different?

b Complete the tables showing the weight and the daily rate of growth for each species.

Parus major

Day	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
Weight															
Growth rate															

Parus caeruleus

Day	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
Weight															
Growth rate															

c Plot the rate of growth against weight in grams for each species. What type of curve does the growth rate graph appear to be?

d For each species, at what weight did the maximum growth rate occur? Locate the corresponding point on each original curve in Figure 6.6.1, p. 692.

Project 46 Parus major growth rate.

a Find a quadratic regression equation for the growth rate of **Parus major** in terms of its weight using the data from Project 45, p. 691.

b Make a scatterplot of the data and draw the regression curve on the same axes.

c Find the vertex of the graph of the regression equation. How does this estimate for the maximum growth rate compare with your estimate in Project 45, p. 691?

Project 47 Parus caeruleus growth rate.

a Find a quadratic regression equation for the growth rate of **Parus caeruleus** in terms of its weight using the data from Project 45, p. 691.

b Make a scatterplot of the data and draw the regression curve on the same axes.

c Find the vertex of the graph of the regression equation. How does this estimate for the maximum growth rate compare with your estimate in Project 45, p. 691?