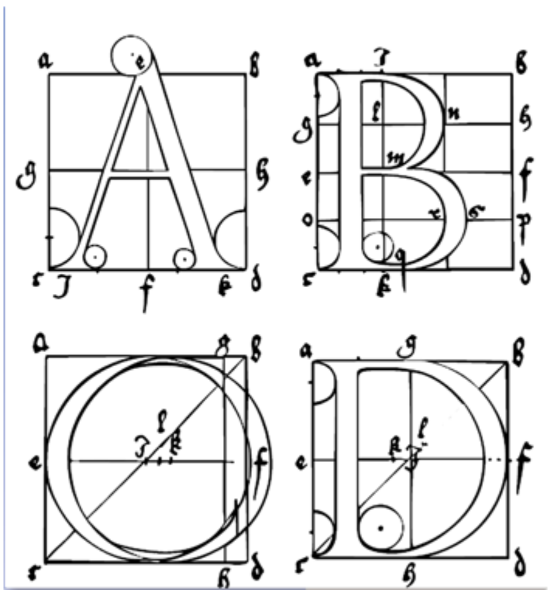
Chapter 7

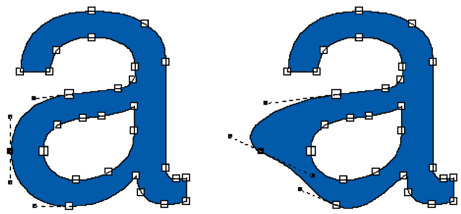
Polynomial and Rational Functions



The graphs of linear, quadratic, exponential and power functions all have a characteristic shape. But the graphs of polynomials have a huge variety of different shapes.

Ever since Gutenberg's invention of movable type in 1455, artists and printers have been interested in the design of pleasing and practical fonts. In 1525, Albrecht Durer published *On the Just Shaping of Letters*, which set forth a system of rules for the geometric construction of Roman capitals. The letters shown above are examples of Durer's font. Until the twentieth century, a ruler and compass were the only practical design tools, so straight lines and circular arcs were the only geometric objects that could be accurately reproduced.

With the advent of computers, complex curves and surfaces, such as the smooth contours of modern cars, can be defined precisely. In the 1960s the French automobile engineer Pierre Bézier developed a new design tool based on polynomials. **Bézier curves** are widely used today in all fields of design, from technical plans and blueprints to the most creative artistic projects. Many computer drawing programs and printer languages use quadratic and cubic Bézier curves.



Investigation 48 Bézier Curves. A Bézier curve is actually a sequence of short curves pieced together. Each piece has two endpoints and (for nonlinear curves) at least one control point. The control points do not lie on the curve itself, but they determine its shape. Two polynomials define the curve, one for the x-coordinate and one for the y-coordinate.

A Linear Bézier Curves

The linear Bézier curve for two endpoints, (x_1, y_1) and (x_2, y_2), is the straight line segment joining those two points. The curve is defined by the two functions

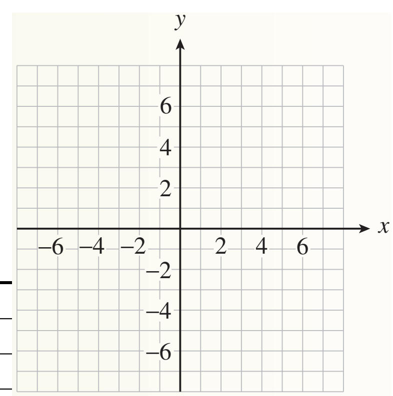
$$x = f(t) = x_1 \cdot (1 - t) + x_2 \cdot t$$
$$y = g(t) = y_1 \cdot (1 - t) + y_2 \cdot t$$

for $0 \le t \le 1$.

1 Find the functions f and g defining the linear Bézier curve joining the two points $(-4, 7)$ and $(2, 0)$. Simplify the formulas defining each function.

2 Fill in the table of values and plot the curve.

t	0	0.25	0.5	0.75
x				
y				



B Quadratic Bézier Curves: Drawing a Simple Numeral 7

The quadratic Bézier curve is defined by two endpoints, (x_1, y_1) and (x_3, y_3), and a control point, (x_2, y_2).

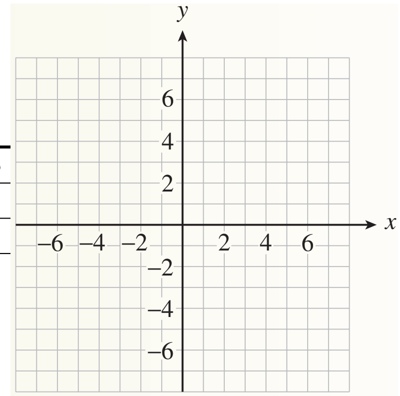
$$x = f(t) = x_1 \cdot (1 - t)^2 + 2x_2 \cdot t(1 - t) + x_3 \cdot t^2$$
$$y = g(t) = y_1 \cdot (1 - t)^2 + 2y_2 \cdot t(1 - t) + y_3 \cdot t^2$$

for $0 \le t \le 1$.

1 Find the functions f and g for the quadratic Bézier curve defined by the endpoints $(-4, 7)$ and $(2, 0)$, and the control point $(0, 5)$. Simplify the formulas defining each function.

2 Fill in the table of values and plot the curve.

t	0	0.25	0.5	0.75
x				
y				

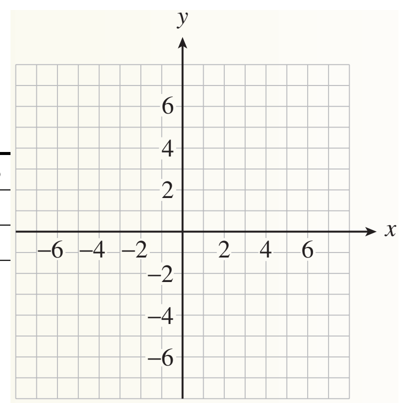


3 Draw a line segment from $(-4, 7)$ to $(4, 7)$ on the grid above to complete the numeral 7.

4 We can adjust the curvature of the diagonal stroke of the 7 by moving the control point. Find the functions f and g for the quadratic Bézier curve defined by the endpoints $(4, 7)$ and $(0, -7)$, and the control point $(0, -3)$. Simplify the formulas defining each function.

5 Fill in the table of values and plot the curve.

t	0	0.25	0.5	0.75
x				
y				



6 Draw a line segment from $(-4, 7)$ to $(4, 7)$ on the grid above to complete the numeral 7.

7 On your graphs in steps B.3, p. 697 and B.6, p. 697, plot the three points that defined the curved section of the numeral 7, then connect them in order with line segments. How does the position of the control point change the curve?

C Cubic Bézier Curves: Drawing a Letter y

A cubic Bézier curve is defined by two endpoints, (x_1, y_1) and (x_4, y_4), and two control points, (x_2, y_2) and (x_3, y_3).

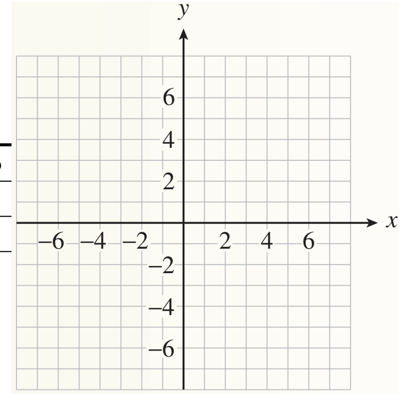
$$x = f(t) = x_1 \cdot (1-t)^2 + 3x_2 \cdot t(1-t)^2 + 3x_3 \cdot t^2(1-t) + x_4 \cdot t^3$$
$$y = g(t) = y_1 \cdot (1-t)^2 + 3y_2 \cdot t(1-t)^2 + 3y_3 \cdot t^2(1-t) + y_4 \cdot t^3$$

for $0 \le t \le 1$.

1 Find the functions f and g for the cubic Bézier curve defined by the endpoints $(4, 7)$ and $(-4, -5)$, and the control points $(3, 3)$ and $(0, -8)$. Simplify the formulas defining each function.

2 Fill in the table of values and plot the curve.

t	0	0.25	0.5	0.75
x				
y				



3 Connect the four given points in order using three line segments. How does the position of the control points affect the curve? Finish the letter y by including the linear Bézier curve you drew for step A.2, p. 696.

7.1 Polynomial Functions

We have already encountered some examples of polynomial functions. Linear functions,

$$f(x) = ax + b$$

and quadratic functions

$$f(x) = ax^2 + bx + c$$

are special cases of polynomial functions. In general, we make the following definition.

Polynomial Function.

A **polynomial function** has the form

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_2x^2 + a_1x + a_0$$

where a_0, a_1, a_2, ..., a_n are constants and $a_n \neq 0$. The coefficient a_n of the highest power term is called the **lead coefficient**.

Some examples of polynomials are

$$f(x) = 6x^3 - 4x^2 + x - 2 \qquad g(x) = 9x^5 - 2$$
$$p(x) = x^4 + x^2 + 1 \qquad q(x) = 2x^{10} - x^7 + 3x^6 + 5x^3 + 3x$$

Each of the polynomials above is written in **descending powers**, which means that the highest-degree term comes first, and the degrees of the terms decrease from largest to smallest. Sometimes it is useful to write a polynomial in **ascending powers**, so that the degrees of the terms increase. For example, the polynomial $f(x)$ above would be written as

$$f(x) = -2 + x - 4x^2 + 6x^3$$

in ascending powers.

7.1.1 Products of Polynomials

When we multiply two or more polynomials together, we get another polynomial of higher degree. (See Algebra Skills Refresher A.7 for the definition of degree.)

Example 7.1.1 Compute the products.

a $(x+2)(5x^3-3x^2+4)$

b $(x-3)(x+2)(x-4)$

Solution.

a. $(x+2)(5x^3-3x^2+4)$ — Apply the distributive law.

$= x(5x^3-3x^2+4)+2(5x^3-3x^2+4)$ — Apply the distributive law again.

$= 5x^4-3x^3+4x+10x^3-6x^2+8$ — Combine like terms.

$= 5x^4+7x^3-6x^2+4x+8$

b. $(x-3)(x+2)(x-4)$ — Multiply two of the factors first.

$= (x-3)(x^2-2x-8)$ — Apply the distributive law.

$= x(x^2-2x-8)-3(x^2-2x-8)$ — Apply the distributive law again.

$= x^3-2x^2-8x-3x^2+6x+24$ — Combine like terms.

$= x^3-5x^2-2x+24$

□

Checkpoint 7.1.2 Multiply $(y+2)(y^2-2y+3)$.

Answer. y^3-y+6

In Example 7.1.1, p. 699a, we multiplied a polynomial of degree 1 by a polynomial of degree 3, and the product was a polynomial of degree 4. In Example 7.1.1, p. 699b, the product of three first degree polynomials is a third-degree polynomial.

Degree of a Product.

The degree of a product of nonzero polynomials is the sum of the degrees of the factors. That is,

If $P(x)$ has degree m and $Q(x)$ has degree n, then their product $P(x)Q(x)$ has degree $n+m$.

Example 7.1.3 Let $P(x)=5x^4-2x^3+6x^2-x+2$, and $Q(x)=3x^3-4x^2+5x+3$.

a What is the degree of their product? What is the coefficient of the lead term?

b Find the coefficient of the x^3-term of the product.

Solution.

a The degree of P is 4, and the degree of Q is 3, so the degree of their product is $4+3=7$. The only degree 7 term of the product is $(5x^4)(3x^3)=15x^7$, which has coefficient 15.

b In the product, each term of $P(x)$ is multiplied by each term of $Q(x)$. We get degree 3 terms by multiplying together terms of degree 0 and 3, or 1 and 2. For these polynomials, the possible combinations are:

$P(x)$	$Q(x)$	Product
2	$3x^3$	$6x^3$
$-2x^3$	3	$-6x^3$
$-x$	$-4x^2$	$4x^3$
$6x^2$	$5x$	$30x^3$

The sum of the third-degree terms of the product is $34x^3$, with coefficient 34.

□

Checkpoint 7.1.4 Find the coefficient of the fourth-degree term of the product of $f(x) = 2x^6 + 2x^4 - x^3 + 5x^2 + 1$ and $g(x) = x^5 - 3x^4 + 2x^3 + x^2 - 4x - 2$.

Answer. 2

7.1.2 Special Products

In the Algebra Skills Refresher Section A.8, p. 912, you can review the following special products involving quadratic expressions.

Special Products of Binomials.

$$(a+b)^2 = (a+b)(a+b) = a^2 + 2ab + b^2$$
$$(a-b)^2 = (a-b)(a-b) = a^2 - 2ab + b^2$$
$$(a+b)(a-b) = a^2 - b^2$$

There are also special products resulting in cubic polynomials. In the Homework problems, you will be asked to verify the following products.

Cube of a Binomial.

1. $(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$
2. $(x-y)^3 = x^3 - 3x^2y + 3xy^2 - y^3$

If you become familiar with these general forms, you can use them as patterns to find specific examples of such products.

Example 7.1.5 Write $(2w-3)^3$ as a polynomial.

Solution. Use product 2, p. 789 above, with x replaced by $\mathbf{2w}$ and y replaced by $\mathbf{3}$.

$$\begin{aligned}(x-y)^3 &= x^3 - 3x^2y + 3xy^2 - y^3\\ (\mathbf{2w}-\mathbf{3})^3 &= (\mathbf{2w})^3 - 3(\mathbf{2w})^2(\mathbf{3}) + 3(\mathbf{2w})(\mathbf{3})^2 - \mathbf{3}^3 && \text{Simplify.}\\ &= 8w^3 - 36w^2 + 54w - 27\end{aligned}$$

□

Of course, we can also expand the product in Example 7.1.5, p. 700 simply by polynomial multiplication and arrive at the same answer.

Checkpoint 7.1.6 Write $\left(5+x^2\right)^3$ as a polynomial.

Answer. $125 + 75x^2 + 15x^4 + x^6$

7.1.3 Factoring Cubics

Another pair of products is useful for factoring cubic polynomials. In the Homework problems, you will be asked to verify the following products:

$$(x+y)(x^2-xy+y^2) = x^3+y^3$$
$$(x-y)(x^2+xy+y^2) = x^3-y^3$$

Viewing these products from right to left, we have the following special factorizations for the sum and difference of two cubes.

Factoring the Sum or Difference of Two Cubes.

1 $x^3+y^3 = (x+y)(x^2-xy+y^2)$

2 $x^3-y^3 = (x-y)(x^2+xy+y^2)$

When we recognize a polynomial as a sum or difference of two perfect cubes, we then identify the two cubed expressions and apply the formula.

Example 7.1.7 Factor each polynomial.

a $8a^3+b^3$

b $1-27h^6$

Solution.

a This polynomial is a sum of two cubes. The cubed expressions are $2a$, because $(2a)^3 = 8a^3$, and b. Use formula 1, p. 789 as a pattern, replacing x with $\boldsymbol{2a}$, and y with $\boldsymbol{b}$.

$$\begin{aligned} x^3+y^3 &= (x+y)(x^2-xy+y^2) \\ (\boldsymbol{2a})3+\boldsymbol{b}3 &= (\boldsymbol{2a}+\boldsymbol{b})((\boldsymbol{2a})^2-(\boldsymbol{2a})\boldsymbol{b}+\boldsymbol{b}^2) \qquad \text{Simplify.} \\ &= (2a+b)(4a^2-2ab+b^2) \end{aligned}$$

b This polynomial is a difference of two cubes. The cubed expressions are 1, because $1^3 = 1$, and $3h^2$, because $(3h^2)^3 = 27h^6$. Use formula 2, p. 789 above as a pattern, replacing x by $\mathbf{1}$, and y by $\boldsymbol{3h^2}$:

$$\begin{aligned} x^3-y^3 &= (x-y)(x^2+xy+y^2) \\ \mathbf{1}^3-(\boldsymbol{3h^2})^3 &= (\mathbf{1}-\boldsymbol{3h^2})(\mathbf{1}^2+\mathbf{1}(\boldsymbol{3h^2})+(\boldsymbol{3h^2})^2) \qquad \text{Simplify.} \\ &= (1-3h^2)(1+3h^2+9h^4) \end{aligned}$$

□

Checkpoint 7.1.8 Factor $125n^3-p^3$

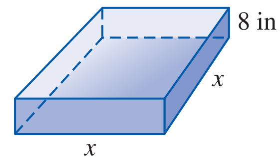
Answer. $(5n-p)(25n^2+5np+p^2)$

7.1.4 Modeling with Polynomials

Polynomials model many variable relationships, including volume and surface area.

Example 7.1.9

A closed box has a square base of length and width x inches and a height of 8 inches, as shown at right.



a Write a polynomial function $S(x)$ that gives the surface area of the box in terms of the dimensions of the base.

b What is the surface area of a box of length and width 18 inches?

Solution.

a The surface area of a box is the sum of the areas of its six faces,

$$S = 2lh + 2wh + 2lw$$

Substituting x for l and w, and 8 for h gives us

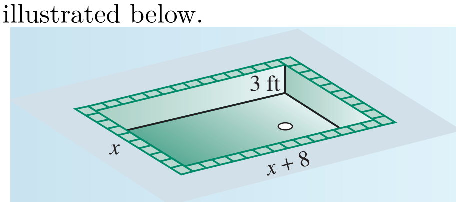
$$S(x) = 2(8)x + 2(8)x + 2x^2 = 2x^2 + 32x$$

b We evaluate the polynomial for $x = 18$ to find

$$S(18) = 2(18)^2 + 32(18) = 1224 \text{ square inches}$$

□

Checkpoint 7.1.10 An empty reflecting pool is 3 feet deep. It is 8 feet longer than it is wide, as illustrated below.



a Write a polynomial function $S(x)$ that gives the surface area of the empty pool.

b Write a polynomial function $V(x)$ for the volume of the pool.

Answer.

a $S(x) = x^2 + 20x + 48$

b $V(x) = 3x^2 + 24x$

Cubic polynomials are often used in economics to model cost functions. The cost of producing x items is an increasing function of x, but its rate of increase is usually not constant.

Example 7.1.11 Pegasus Printing, Ltd. is launching a new magazine. The cost of printing x thousand copies is given by

$$C(x) = x^3 - 24x^2 + 195x + 250$$

a What are the **fixed costs**, that is, the costs incurred before any copies are printed?

b Graph the cost function in the window below and describe the graph.

Xmin $= 0$ Xmax $= 20$

Ymin = 0 Ymax = 1500

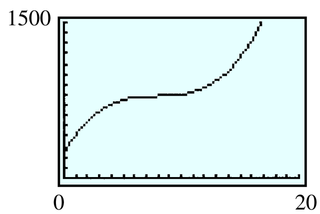
c How many copies can be printed for \$1200?

d What does the concavity of the graph tell you about the cost function?

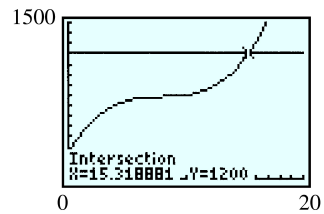
Solution.

a Fixed costs are given by $C(0) = 250$, or \$250. The fixed costs include expenses like utility bills that must be paid even if no magazines are produced.

b The graph is shown in figure (a). It is increasing from a vertical intercept of 250. The graph is concave down for $x < 8$ approximately, and concave up for $x > 8$.



(a)



(b)

c We must solve the equation

$$x^3 - 24x^2 + 195x + 250 = 1200$$

We will solve the equation graphically, as shown in figure (b). Graph $y = 1200$ along with the cost function, and use the *intersect* command to find the intersection point of the graphs, $(15.319, 1200)$. $C(x) = 1200$ when x is about 15.319, so 15,319 copies can be printed for \$1200.

d Although the cost is always increasing, it increases very slowly from about $x = 5$ to about $x = 11$. The flattening of the graph in this interval is a result of economy of scale: By buying supplies in bulk and using time efficiently, the cost per magazine can be minimized. However, if the production level is too large, costs begin to rise rapidly again.

□

In Example 7.1.11, p. 702c, we solved a cubic equation graphically. There is a cubic formula, analogous to the quadratic formula, that allows us to solve cubic equations algebraically, but it is complicated and not often used. See the Projects for Chapter 7 if you would like to know more about the cubic formula.

Cubic polynomials are also used to model smooth curves connecting given points. Such a curve is called a **cubic spline**.

Checkpoint 7.1.12 Leon is flying his plane to Au Gres, Michigan. He maintains a constant altitude until he passes over a marker just outside the neighboring town of Omer, when he begins his descent for landing. During the descent, his altitude, in feet, is given by

$$A(x) = 128x^3 - 960x^2 + 8000$$

where x is the number of miles Leon has traveled since passing over the marker in Omer.

a What is Leon's altitude when he begins his descent?

b Graph $A(x)$ in the window

$$\text{Xmin} = 0 \quad \text{Xmax} = 5$$
$$\text{Ymin} = 0 \quad \text{Ymax} = 8000$$

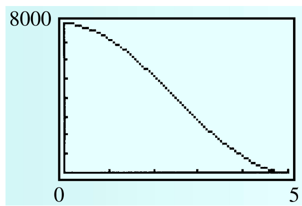
c Use the *Trace* feature to discover how far from Omer Leon will travel before landing. (In other words, how far is Au Gres from Omer?)

d Verify your answer to part (c) algebraically.

Answer.

a 8000 ft

b



c 5 mi

d $A(5) = 0$

7.1.5 Section Summary

7.1.5.1 Vocabulary

Look up the definitions of new terms in the Glossary.

- Polynomial function
- Degree
- Ascending powers
- Lead coefficient
- Descending powers

7.1.5.2 CONCEPTS

1 The degree of a product of nonzero polynomials is the sum of the degrees of the factors.

2 **Cube of a Binomial.**

1 $(x + y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$

2 $(x - y)^3 = x^3 - 3x^2y + 3xy^2 - y^3$

3 If we know the vertex of a parabola, we need only one other point to find its equation.

4 **Factoring the Sum or Difference of Two Cubes.**

1 $x^3 + y^3 = (x + y)(x^2 - xy + y^2)$

2 $x^3 - y^3 = (x - y)(x^2 + xy + y^2)$

7.1.5.3 STUDY QUESTIONS

1 If you add two polynomials of degree 3, what can you say about the degree of the sum?

2 If you multiply a polynomial of degree 3 and a polynomial of degree 4, what can you say about the degree of the product?

3 If you multiply together

$$(2x-1)(2x-2)(2x-3)\cdots(2x-8)$$

what will be the degree of the product? What will be the lead coefficient?

4 What are the two middle terms in the expansion of $(x+y)^3$?

5 Is it possible to factor the sum of two cubes? What about the sum of two squares?

7.1.5.4 SKILLS

Practice each skill in the Homework 7.1.6, p. 705 problems listed.

1 Multiply polynomials: #1–8

2 Find specific terms of polynomial products: #9–22

3 Use the formula for the cube of a binomial: #23–34

4 Factor the sum or difference of two cubes: #35–46

5 Write and analyze polynomial models: #47–64

7.1.6 Polynomial Functions (Homework 7.1)

For Problems 1-8, multiply.

1. $(3x-2)(4x^2+x-2)$
Answer. $12x^3-5x^2-8x+4$

2. $(2x+3)(3x^2-4x+2)$

3. $(x-2)(x-1)(x-3)$
Answer. $x^3-6x^2+11x-6$

4. $(z-5)(z+6)(z-1)$

5. $(2a^2-3a+1)(3a^2+2a-1)$
Answer.
$6a^4-5a^3-5a^2+5a-1$

6. $(b^2-3b+5)(2b^2-b+1)$

7. $(y-2)(y+2)(y+4)(y+1)$
Answer. $y^4+5y^3-20y-16$

8. $(z+3)(z+2)(z-1)(z+1)$

For Problems 9-12, find the first three terms of the product in ascending powers. (Do not compute the entire product!)

9. $(2-x+3x^2)(3+2x-x^2+2x^4)$
Answer. $6+x+5x^2$

10. $(1+x-2x^2)(-3+2x-4x^3)$

11. $(1-2x^2-x^4)(4+x^2-2x^4)$
Answer. $4-7x^2-8x^4$

12. $(3+2x)(5-2x^2-3x^3-x^5+2x^6)$

For Problems 13-16, find the indicated term in each product. (Do not compute the entire product!)

13. $(4+2x-x^2)(2-3x+2x^2)$; x^2

Answer. $0x^2$

14. $(1-2x+3x^2)(6-x-x^3)$; x^3

15. $(3x+x^3-7x^5)(1+4x-3x^2)$; x^3

Answer. $-8x^3$

16. $(2+3x^2+2x^4)(2-x-x^2-x^4)$; x^4

For Problems 17-18, without performing the multiplication, give the degree of each product and the leading coefficient.

17.

(a) $(x^2-4)(3x^2-6x+2)$

(b) $(x-3)(2x-5)(x^3-x+2)$

(c) $(3x^2+2x)(x^3+1)(-2x^2+8)$

Answer.

(a) 4 (b) 5 (c) 7

18.

(a) $(6x^2-1)(4x^2-9)$

(b) $(3x+4)(3x+1)(2x^3+x^2-7)$

(c) $(x^2-3)(2x^3-5x^2+2)(-x^3-5x)$

For Problems 19-22, verify the following products discussed in the text.

19. $(x+y)^3 = x^3+3x^2y+3xy^2+y^3$

Answer.
$$\begin{aligned}(x+y)^3 &= (x+y)(x+y)^2\\ &= (x+y)(x^2+2xy+y^2)\\ &= x^3+2x^2y+xy^2+x^2y+2xy^2+y^3\\ &= x^3+3x^2y+3xy^2+y^3\end{aligned}$$

20. $(x-y)^3 = x^3-3x^2y+3xy^2-y^3$

21. $(x+y)(x^2-xy+y^2) = x^3+y^3$

Answer.
$$\begin{aligned}(x+y)(x^2-xy+y^2) &= x^3-x^2y+xy^2+x^2y-xy^2+y^3\\ &= x^3+y^3\end{aligned}$$

22. $(x-y)(x^2+xy+y^2) = x^3-y^3$

23.

(a) As if you were addressing a classmate, explain how to remember the formula for expanding $(x+y)^3$. In particular, mention the exponents on each term and the numerical coefficients.

(b) Explain how to remember the formula for expanding $(x-y)^3$, assuming your listener already knows the formula for $(x+y)^3$.

Answer.

(a) The formula begins with x^3 and ends with y^3. As you proceed from term to term, the exponents on x decrease while the exponents on y increase, and on each term the sum of the exponents is 3. The coefficients of the two middle terms are both 3.

(b) The formula is the same as for $(x - y)^3$, except that the terms alternate in sign.

24.

(a) As if you were addressing a classmate, explain how to remember the formula for factoring a sum of two cubes. Pay particular attention to the placement of the variables and the signs of the terms.

(b) Explain how to remember the formula for factoring a difference of two cubes, assuming your listener already knows how to factor a sum of two cubes.

For Problems 25-28, use the formulas for the cube of a binomial to expand the products.

25. $(1 + 2z)^3$

Answer. $1 + 6z + 12z^2 + 8z^3$

26. $(1 - x^2)^3$

27. $(1 - 5\sqrt{t})^3$

Answer. $1 - 15\sqrt{t} + 75t - 125t\sqrt{t}$

26. $\left(1 - \dfrac{3}{a}\right)^3$

For Problems 29-34, write each product as a polynomial and simplify.

29. $(x - 1)(x^2 + x + 1)$

Answer. $x^3 - 1$

30. $(x + 2)(x^2 - 2x + 4)$

31. $(2x + 1)(4x^2 - 2x + 1)$

Answer. $8x^3 + 1$

32. $(3x - 1)(9x^2 + 3x + 1)$

33. $(3a - 2b)(9a^2 + 6ab + 4b^2)$

Answer. $27a^3 - 8b^3$

34. $(2a + 3b)(4a^2 - 6ab + 9b^2)$

For Problems 35-46, factor completely.

35. $x^3 + 27$

Answer. $(x+3)(x^2-3x+9)$

36. $y^3 - 1$

37. $a^3 - 8b^3$

Answer. $(a - 2b)(a^2 + 2ab + 4b^2)$

38. $27a^3 + b^3$

39. $x^3y^6 - 1$

Answer. $(xy^2 - 1)(x^2y^4 + xy^2 + 1)$

40. $8 + x^{12}y^3$

41. $27a^3 + 64b^3$

Answer. $(3a + 4b)(9a^2 - 12ab + 16b^2)$

42. $8a^3 - 125b^3$

43. $125a^3b^3 - 1$

Answer. $(5ab - 1)(25a^2b^2 + 5ab + 1)$

44. $64a^3b^3 + 1$

45. $64t^9 + w^6$

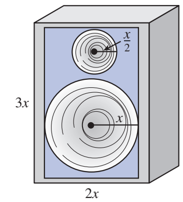
Answer. $(4t^3 + w^2)(16t^6 - 4t^3w^2 + w^4)$

46. $w^{15} - 125t^9$

47.

(a) Write a polynomial function, $A(x)$, that gives the area of the front face of the speaker frame (the region in color) in the figure.

(b) If $x = 8$ inches, find the area of the front face of the frame.



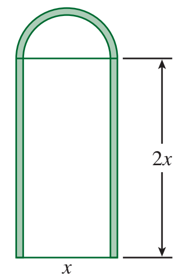
Answer.

(a) $\left(6 - \frac{5}{4}\pi\right) x^2$ (b) ≈ 132.67 square inches

48.

(a) A Norman window is shaped like a rectangle whose length is twice its width, with a semicircle at the top (see the figure). Write a polynomial, $A(x)$, that gives its area.

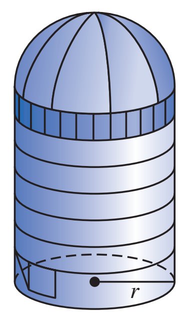
(b) If $x = 3$ feet, find the area of the front face of the frame.



49.

(a) A grain silo is built in the shape of a cylinder with a hemisphere on top (see the figure). Write an expression for the volume of the silo in terms of the radius and height of the cylindrical portion of the silo.

(b) If the total height of the silo is five times its radius, write a polynomial function $V(r)$ in one variable for its volume.



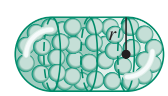
Answer.

(a) $\frac{2}{3}\pi r^3 + \pi r^2 h$

(b) $V(r) = \frac{14}{3}\pi r^3$

50.

(a) A cold medication capsule is shaped like a cylinder with a hemispherical cap on each end (see the figure). Write an expression for the volume of the capsule in terms of the radius and length of the cylindrical portion.



(b) If the radius of the capsule is one-fourth of its overall length, write a polynomial function $V(r)$ in one variable for its volume.

51. Jack invests \$500 in an account bearing interest rate r, compounded annually. This means that each year his account balance is increased by a factor of $1 + r$.

(a) Write expressions for the amount of money in Jack's account after 2 years, after 3 years, and after 4 years.

(b) Expand as polynomials the expressions you found in part (a).

(c) How much money will be in Jack's account at the end of 2 years, 3 years, and 4 years if the interest rate is 8%?

Answer.

(a) $500(1+r)^2$; $500(1+r)^3$; $500(1+r)^4$

(b) $500r^2+1000r+500$; $500r^3+1500r^2+1500r+500$; $500r^4+2000r^3+3000r^2+2000r+500$

(c) \$583.20, \$629.86, \$680.24

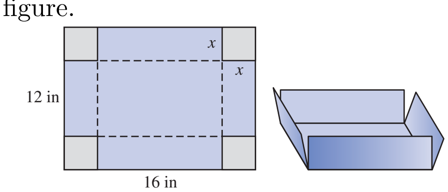
52. A small company borrows \$800 for start-up costs and agrees to repay the loan at interest rate r, compounded annually. This means that each year the debt is increased by a factor of $1+r$.

(a) Write expressions for the amount of money the company will owe if it repays the loan after 2 years, after 3 years, or after 4 years.

(b) Expand as polynomials the expressions you found in part (a).

(c) How much money will the company owe after 2 years, after 3 years, or after 4 years at an interest rate of 12%?

53. A paper company plans to make boxes without tops from sheets of cardboard 12 inches wide and 16 inches long. The company will cut out four squares of side x inches from the corners of the sheet and fold up the edges as shown in the figure.



(a) Write expressions in terms of x for the length, width, and height of the resulting box.

(b) Write a formula for the volume, V, of the box as a function of x.

(c) What is the domain of the function V? (What are the largest and smallest reasonable values for x?)

(d) Make a table of values for $V(x)$ on its domain.

(e) Graph your function V in a suitable window.

(f) Use your graph to find the value of x that will yield a box with maximum possible volume. What is the maximum possible volume?

Answer.

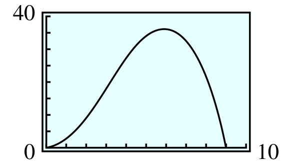
(a) Length: $16-2x$; Width: $12-2x$; Height: x

(b) $V=x(16-2x)(12-2x)$

(c) Real numbers between 0 and 6

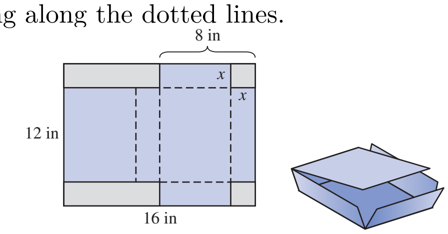
(d)

x	1	2	3	4	5
V	140	192	180	128	60

(e) 

(f) 2.26 in, 194.07 cu in

54. The paper company also plans to make boxes with tops from 12-inch by 16-inch sheets of cardboard by cutting out the shaded areas shown in the figure and folding along the dotted lines.



(a) Write expressions in terms of x for the length, width, and height of the resulting box.

(b) Write a formula for the volume, V, of the box as a function of x.

(c) What is the domain of the function V? (What are the largest and smallest reasonable values for x?)

(d) Make a table of values for $V(x)$ on its domain.

(e) Graph your function V in a suitable window.

(f) Use your graph to find the value of x that will yield a box with maximum possible volume. What is the maximum possible volume?

Use your graphing calculator to help you answer the questions in Problems 55–62. Then verify your answers algebraically.

55. A doctor who is treating a heart patient wants to prescribe medication to lower the patient's blood pressure. The body's reaction to this medication is a function of the dose administered. If the patient takes x milliliters of the medication, his blood pressure should decrease by $R = f(x)$ points, where

$$f(x) = 3x^2 - \frac{1}{3}x^3$$

(a) For what values of x is $R = 0$?

(b) Find a suitable domain for the function and explain why you chose this domain.

(c) Graph the function f on its domain.

(d) How much should the patient's blood pressure drop if he takes 2 milliliters of medication?

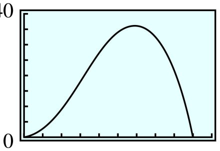
(e) What is the maximum drop in blood pressure that can be achieved with this medication?

(f) There may be risks associated with a large change in blood pressure. How many milliliters of the medication should be administered to produce half the maximum possible drop in blood pressure?

Answer.

(a) $0, 9$

(b) $0 \le x \le 9$; $R \ge 0$ for these values

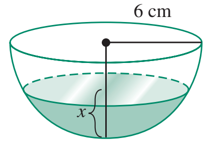
(c) 

(d) $\dfrac{28}{3}$ points

(e) 36 points

(f) 3 ml or 8.2 ml

56. A soup bowl has the shape of a hemisphere of radius 6 centimeters. The volume of the soup in the bowl, $V = f(x)$, is a function of the depth, x, of the soup.



(a) What is the domain of f? Why?

(b) The function f is given by

$$f(x) = 6\pi x^2 - \frac{\pi}{3}x^3$$

Graph the function on its domain.

(c) What is the volume of the soup if it is 3 centimeters deep?

(d) What is the maximum volume of soup that the bowl can hold?

(e) Find the depth of the soup (to within 2 decimal places of accuracy) when the bowl is filled to half its capacity.

57. The population, $P(t)$, of Cyberville has been growing according to the formula

$$P(t) = t^3 - 63t^2 + 1403t + 900$$

where t is the number of years since 1970.

(a) Graph $P(t)$ in the window

Xmin = 0	Xmax = 47
Ymin = 0	Ymax = 20,000

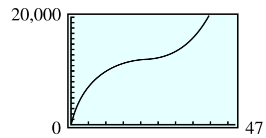
(b) What was the population in 1970? In 1985? In 2004?

(c) By how much did the population grow from 1970 to 1971? From 1985 to 1986? From 2004 to 2005?

(d) Approximately when was the population growing at the slowest rate, that is, when is the graph the least steep?

Answer.

(a)



(b) 900; $11,145$; $15,078$

(c) 1341; 171; 627

(d) Between 1990 and 1991

58. The annual profit, $P(t)$, of the Enviro Company, in thousands of dollars, is given by

$$P(t) = 2t^3 - 152t^2 + 3400t + 30$$

where t is the number of years since 1960, the first year that the company showed a profit.

(a) Graph $P(t)$ in the window

$$\text{Xmin} = 0 \qquad \text{Xmax} = 94$$
$$\text{Ymin} = 0 \qquad \text{Ymax} = 50,000$$

(b) What was the profit in 1960? In 1980? In 2000?

(c) How did the profit change from 1960 to 1961? From 1980 to 1981? From 2000 to 2001?

(d) During which years did the profit decrease from one year to the next?

59. The total annual cost of educating postgraduate research students at an Australian university, in thousands of dollars, is given by the function

$$C(x) = 0.0173x^3 - 0.647x^2 + 9.587x + 195.366$$

where x is the number of students, in hundreds. (Source: Creedy, Johnson, and Valenzuela, 2002)

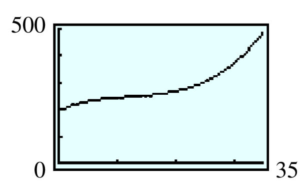
(a) Graph the function in a suitable window for up to 3500 students.

(b) Describe the concavity of the graph. For what value of x is the cost growing at the slowest rate?

(c) Approximately how many students can be educated for $\$350,000$?

Answer.

(a)



(b) The graph is concave down until about $x = 12.5$ and is concave up afterwards. The cost is growing at the slowest rate at the inflection point at about $x = 12.5$, or 1250 students.

(c) About 2890

60. It has been proposed that certain cubic functions model the response of wheat and barley to nitrogen fertilizer. These functions exhibit a "plateau" that fits observations better than the standard quadratic model. (See Problem 8.2.5.36, p. 848 of Section 8.2, p. 832.) In trials in Denmark, the yield per acre was a function of the amount of nitrogen applied. A typical response function is

$$Y(x) = 54.45 + 0.305x - 0.001655x^2 + 2.935 \times 10^{-6}x^3$$

where x is the amount of fertilizer, in kilograms per acre.(Source: Beattie, Mortensen, and Knudsen, 2005)

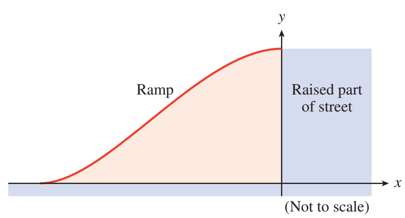
(a) Graph the function on the domain $[0, 400]$.

(b) Describe the concavity of the graph. In reality, the yield does not increase after reaching its plateau. Give a suitable domain for the model in this application.

(c) Estimate the maximum yield attainable and the optimum application of fertilizer.

61. During an earthquake, Nordhoff Street split in two, and one section shifted up several centimeters. Engineers created a ramp from the lower section to the upper section. In the coordinate system shown in the figure below, the ramp is part of the graph of

$$y = f(x) = -0.00004x^3 - 0.006x^2 + 20$$



(a) By how much did the upper section of the street shift during the earthquake?

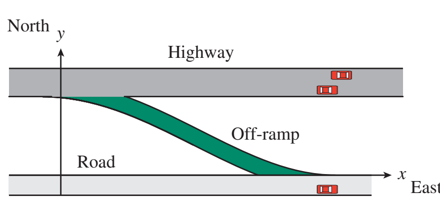
(b) What is the horizontal distance from the bottom of the ramp to the raised part of the street?

Answer.

(a) 20 cm (b) 100 cm

62. The off-ramp from a highway connects to a parallel one-way road. The accompanying figure shows the highway, the off-ramp, and the road. The road lies on the x-axis, and the off-ramp begins at a point on the y-axis. The offramp is part of the graph of the polynomial

$$y = f(x) = 0.00006x^3 - 0.009x^2 + 30$$



(a) How far east of the exit does the off-ramp meet the one-way road?

(b) How far apart are the highway and the road?

63. The number of minutes of daylight per day in Chicago is approximated by the polynomial

$$H(t) = 0.000\,000\,525t^4 - 0.0213t^2 + 864$$

where t is the number of days since the summer solstice. The approximation is valid for $-74 < t < 74$. (A negative value of t corresponds to a number of days before the summer solstice.)

(a) Use a table of values with increments of 10 days to estimate the range of the function on its domain.

(b) Graph the polynomial on its domain.

(c) How many minutes of daylight are there on the summer solstice?

(d) How much daylight is there two weeks before the solstice?

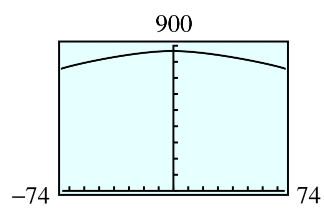
(e) When are the days more than 14 hours long?

(f) When are the days less than 13 hours long?

Answer.

(a) $763.10 < H(t) < 864$

(b)



(c) 864 min

(d) 859.8 min

(e) Within 34 days of the summer solstice

(f) More than 66 days from the summer solstice

64. The water level (in feet) at a harbor is approximated by the polynomial

$$W(t) = 0.00733t^4 - 0.332t^2 + 9.1$$

where t is the number of hours since the high tide. The approximation is valid for $-4 \le t \le 4$. (A negative value of t corresponds to a number of hours before the high tide.)

(a) Use a table of values to estimate the range of the function on its domain.

(b) Graph the polynomial on its domain.

(c) What is the water level at high tide?

(d) What is the water level 3 hours before high tide?

(e) When is the water level below 8 feet?

(f) When is the water level above 7 feet?

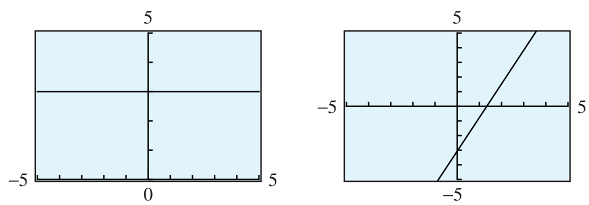
7.2 Graphing Polynomial Functions

In Section 7.1, p. 698, we considered applications of polynomial functions. Although most applications use only a portion of the graph of a particular polynomial, we can learn a lot about these functions by taking a more global view of their behavior.

7.2.1 Classifying Polynomials by Degree

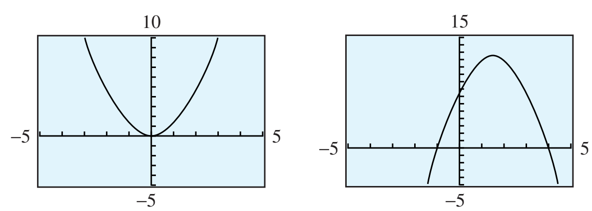
The graph of a polynomial function depends first of all on its degree. We have already studied the graphs of polynomials of degrees 0, 1, and 2.

- A polynomial of degree 0 is a constant, and its graph is a horizontal line. An example of such a polynomial function is $f(x) = 3$.
- A polynomial of degree 1 is a linear function, and its graph is a straight line. The function $f(x) = 2x - 3$ is an example of a polynomial of degree 1.



(a) $f(x) = 3$ **(b)** $f(x) = 2x - 3$

- Quadratic functions, such as $f(x) = -2x^2 + 6x + 8$, are polynomials of degree 2. The graph of every quadratic function is a parabola, with the same basic shape as the standard parabola, $y = x^2$. It has one turning point, where the graph changes from increasing to decreasing or vice versa. The turning point of a parabola is the same as its vertex.



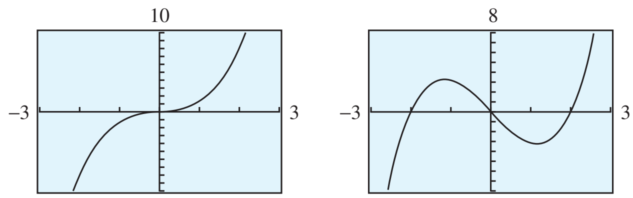
(a) $f(x) = x^2$ **(b)** $f(x) = -2x^2 + 6x + 8$

7.2.2 Cubic Polynomials

Do the graphs of all **cubic**, or third-degree, polynomials have a basic shape in common? We can graph a few examples and find out. Unlike the basic parabola, the graph of $y = x^3$ is always increasing. At the origin, however, it changes from concave down to concave up. A point where the graph changes concavity is called an **inflection point**.

Example 7.2.1 Graph the cubic polynomial $P(x) = x^3 - 4x$ and compare its graph with that of the basic cubic, $y = x^3$.

Solution. The graph of the basic cubic is shown in figure (a) below. To help us understand the graph of the polynomial $P(x) = x^3 - 4x$, we will evaluate the function to make a table of values. We can do this by hand or use the Table feature on the graphing calculator.



(a) $y = x^3$ **(b)** $P(x) = x^3 - 4x$

x	−3	−2	−1	0	1	2	3
$P(x)$	−15	0	3	0	−3	0	15

The graph of $P(x) = x^3 - 4x$ is shown in figure (b). It is not exactly the same shape as the basic cubic -- it has two turning points -- but it is similar, especially at the edges of the graphs. □

Despite the differences in the central portions of the two graphs, they exhibit similar *long-term* behavior.

- For very large and very small values of x, both graphs look like the power function $y = x^3$.
- The y-values increase from $-\infty$ toward zero in the third quadrant, and they increase from zero toward $+\infty$ in the first quadrant. Or we might say that the graphs start at the lower left and extend to the upper right.

All cubic polynomials display this behavior when their lead coefficients (the coefficient of the x^3 term) are positive.

- Both of the graphs in Example 7.2.1, p. 715 are smooth curves without any breaks or holes. This smoothness is a feature of the graphs of all polynomial functions.
- The domain of any polynomial function is the entire set of real numbers.

Checkpoint 7.2.2

a Complete the table of values below for $C(x) = -x^3 - 2x^2 + 4x + 4$.

x	−4	−3	−2	−1	0	1	2	3	4
y									

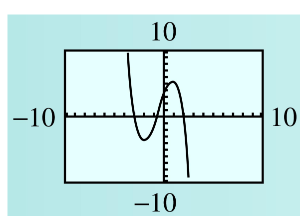
b Graph $y = C(x)$ in the standard window. Compare the graph to the graphs in Example 7.2.1, p. 715: What similarities do you notice? What differences?

Answer.

a

x	−4	−3	−2	−1	0	1	2	3	4
y	20	1	−4	−1	4	5	−4	−29	−76

b



Both graphs have three x-intercepts, but the function in Example 7.2.1, p. 715 has long-term behavior like $y = x^3$, and this function has long-term behavior like $y = -x^3$.

7.2.3 Quartic Polynomials

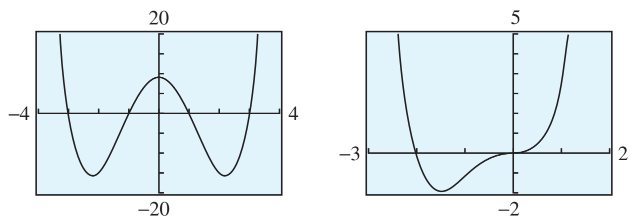
Now let's compare the long-term behavior of two **quartic**, or fourth-degree, polynomials.

Example 7.2.3 Graph the polynomials $f(x) = x^4 - 10x^2 + 9$ and $g(x) = x^4 + 2x^3$, and compare.

Solution. For each function we make a table of values.

x	-4	-3	-2	-1	0	1	2	3	4
$f(x)$	105	0	-15	0	9	0	-15	0	105
$G(x)$	128	27	0	-1	0	3	32	135	384

The graphs are shown below. All the essential features of the graphs are shown in these viewing windows. The graphs continue forever in the directions indicated, without any additional twists or turns. You can see that the graph of f has three turning points, and the graph of g has one turning point.



(a) $f(x) = x^4 - 10x^2 + 9$ **(b)** $g(x) = x^4 + 2x^3$

As in Example 7.2.1, p. 715, both graphs have similar long-term behavior. The y-values decrease from $-\infty$ toward zero as x increases from $-\infty$, and the y-values increase toward $+\infty$ as x increases to $+\infty$. This long-term behavior is similar to that of the power function $y = x^4$. Its graph also starts at the upper left and extends to the upper right. ☐

Checkpoint 7.2.4

a Complete the following table of values for $Q(x) = -x^4 - x^3 - 6x^2 + 2$.

x	-4	-3	-2	-1	0	1	2	3	4
y									

b Graph $y = Q(x)$ in the window

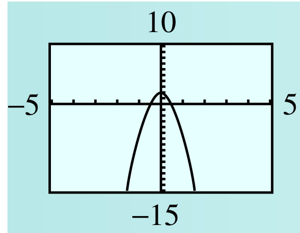
$$\begin{array}{ll} \text{Xmin} = -5 & \text{Xmax} = 5 \\ \text{Ymin} = -15 & \text{Ymax} = 10 \end{array}$$

Compare the graph to the graphs in Example 7.2.3, p. 717: What similarities do you notice? What differences?

Answer.

a

x	-4	-3	-2	-1	0	1	2	3	4
y	-286	-106	-30	-4	2	-6	-48	-160	-414

b 

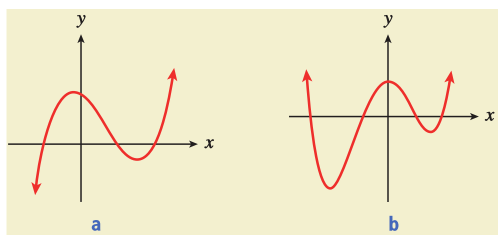
The graphs all have long-term behavior like a fourth degree power function, $y = ax^4$. The long-term behavior of the graphs in Example 7.2.3, p. 717 is the same as that of $y = x^4$, but the graph here has long-term behavior like $y = -x^4$.

In Examples 7.2.1, p. 715 and 7.2.3, p. 717, we have seen polynomials of degree 3 and degree 4, whose graphs are illustrated in the next box. In the Homework Problems, you will consider more graphs to help you verify the following observations.

Long-Term Behavior of Polynomial Functions.

a A polynomial of odd degree (with positive lead coefficient) has negative y-values for large negative x and positive y-values for large positive x.

b A polynomial of even degree (with positive lead coefficient) has positive y-values for both large positive and large negative x.



Note 7.2.5 Another way to describe the long-term behavior of a polynomial graph is to note that for large values of $|x|$, the shape is similar to a power function of the same degree. It is the presence of the lower-degree terms in the polynomial that are responsible for any extra wiggles or turning points in the graph.

7.2.4 x-Intercepts and the Factor Theorem

In Chapter 6, p. 599, we saw that the x-intercepts of a quadratic polynomial, $f(x) = ax^2 + bx + c$, occur at values of x for which $f(x) = 0$, that is, at the real-valued solutions of the equation $ax^2 + bx + c = 0$. The same holds true for polynomials of higher degree.

Solutions of the equation $P(x) = 0$ are called **zeros** of the polynomial P. In Example 7.2.1, p. 715, we graphed the cubic polynomial $P(x) = x^3 - 4x$. Its x-intercepts are the solutions of the equation $x^3 - 4x = 0$, which we can solve by factoring the polynomial $P(x)$.

$$x^3 - 4x = 0$$
$$x(x-2)(x+2) = 0$$

The zeros of P are 0, 2, and -2. Each zero of P corresponds to a factor of $P(x)$. This result suggests the following theorem, which holds for any polynomial P.

Factor Theorem.

Let $P(x)$ be a polynomial with real number coefficients. Then $(x-a)$ is a factor of $P(x)$ if and only if $P(a)=0$.

The factor theorem follows from the division algorithm for polynomials. We will consider both of these results in more detail in the Homework problems.

Because a polynomial function of degree n can have at most n linear factors of the form $(x-a)$, it follows that P can have at most n distinct zeros.

Another way of saying this is that if $P(x)$ is a polynomial of nth degree, the equation $P(x)=0$ can have at most n distinct solutions, some of which may be complex numbers. (We consider complex numbers in Section 7.3, p. 735.)

Because only real-valued solutions appear on the graph as x-intercepts, we have the following corollary to the factor theorem.

x-Intercepts of Polynomials.

A polynomial of degree n can have at most n x-intercepts.

If some of the zeros of P are complex numbers, they will not appear on the graph, so a polynomial of degree n *may have fewer* than n x-intercepts.

Example 7.2.6 Find the real-valued zeros of each of the following polynomials, and list the x-intercepts of its graph.

a $f(x)=x^3+6x^2+9x$ b $g(x)=x^4-3x^2-4$

Solution.

a Factor the polynomial to obtain

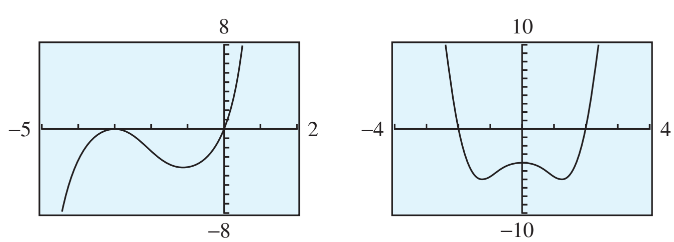
$$\begin{aligned} f(x) &= x(x^2+6x+9) \\ &= x(x+3)(x+3) \end{aligned}$$

By the factor theorem, the zeros of f are 0, -3, and -3. (We say that f has a zero *of multiplicity two* at -3.) Because all of these are real numbers, all will appear as x-intercepts on the graph. Thus, the x-intercepts occur at $(0,0)$ and at $(-3,0)$.

b Factor the polynomial to obtain

$$\begin{aligned} g(x) &= (x^2-4)(x^2+1) \\ &= (x-2)(x+2)(x^2+1) \end{aligned}$$

Because x^2+1 cannot be factored in real numbers, the graph has only two x-intercepts, at $(-2,0)$ and $(2,0)$. The graphs of both polynomials are shown below.



(a) $f(x)=x^3+6x^2+9x$ **(b)** $g(x)=x^4-3x^2-4$

□

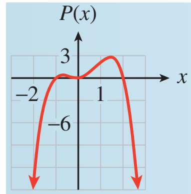
Checkpoint 7.2.7

a Find the real-valued zeros of $P(x) = -x^4 + x^3 + 2x^2$ by factoring.

b Sketch a rough graph of $y = P(x)$ by hand.

Answer.

a $-1, 0, 2$

b 

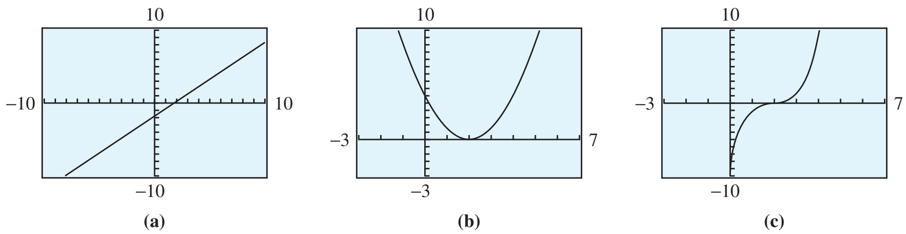
7.2.5 Zeros of Multiplicity Two or Three

The appearance of the graph near an x-intercept is determined by the multiplicity of the zero there.

- Both real zeros of the polynomial $g(x) = x^4 - 3x^2 - 4$ in Example 7.2.6, p. 719b are of multiplicity one, and the graph *crosses* the x-axis at each intercept.

- The polynomial $f(x) = x^3 + 6x^2 + 9x$ in Example 7.2.6, p. 719a has a zero of multiplicity two at $x = -3$. The graph of f just *touches* the x-axis and then reverses direction without crossing the axis.

To understand what happens in general, compare the graphs of the three polynomials below.



a In figure (a), $L(x) = x - 2$ has a zero *of multiplicity one* at $x = 2$, and its graph crosses the x-axis there.

b In figure (b), $Q(x) = (x - 2)^2$ has a zero *of multiplicity two* at $x = 2$, and its graph touches the x-axis there but changes direction without crossing.

c In figure (c), $C(x) = (x - 2)^3$ has a zero *of multiplicity three* at $x = 2$. In this case, the graph makes an S-shaped curve at the intercept, like the graph of $y = x^3$.

Near its x-intercepts, the graph of a polynomial takes one of the characteristic shapes illustrated above.

Note 7.2.8 Although we will not consider zeros of multiplicity greater than three, they correspond to similar behavior in the graph:

- At a zero of odd multiplicity, the graph has an inflection point at the intercept; its graph makes an S-shaped curve.

- At a zero of even multiplicity, the graph has a turning point; it changes direction without crossing the x-axis.

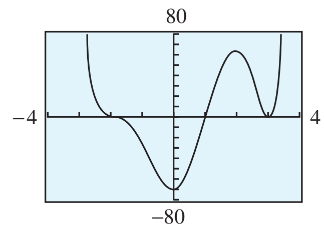
Example 7.2.9 Graph the polynomial

$$f(x) = (x + 2)^3(x - 1)(x - 3)^2$$

Solution. The polynomial has degree six, an even number, so its graph starts at the upper left and extends to the upper right. Its y-intercept is

$$f(0) = (2)^3(-1)(-3)^2 = -72$$

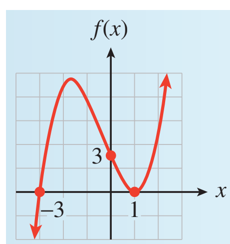
f has a zero of multiplicity three at $x = -2$, a zero of multiplicity one at $x = 1$, and a zero of multiplicity two at $x = 3$. The graph has an S-shaped curve at $x = -2$, crosses the x-axis at $x = 1$, touches the x-axis at $x = 3$, and then changes direction, as shown below.



□

Checkpoint 7.2.10 Sketch a rough graph of $f(x) = (x + 3)(x - 1)^2$ by hand. Label the x- and y-intercepts.

Answer.



7.2.6 Section Summary

7.2.6.1 Vocabulary

Look up the definitions of new terms in the Glossary.

- Degree
- Multiplicity
- Inflection point
- Quartic
- Turning point
- Cubic
- Corollary
- Zero

7.2.6.2 CONCEPTS

1 The graphs of all polynomials are smooth curves without breaks or holes.

2 The graph of a polynomial of degree n (with positive leading coefficient) has the same long-term behavior as the power function of the same degree.

3 Factor Theorem.

Let $P(x)$ be a polynomial with real number coefficients. Then $(x - a)$ is a factor of $P(x)$ if and only if $P(a) = 0$.

4 A polynomial of degree n can have at most n x-intercepts.

5 At a zero of multiplicity two, the graph of a polynomial has a turning point. At a zero of multiplicity three, the graph of a polynomial has an inflection point.

7.2.6.3 STUDY QUESTIONS

1 Describe the graphs of polynomials of degrees 0, 1, and 2.

2 What does the degree of a polynomial tell you about its long-term behavior?

3 What is a zero of a polynomial?

4 How are zeros related to the factors of a polynomial?

5 What do the zeros tell you about the graph of a polynomial?

6 Explain the difference between a turning point and an inflection point.

7.2.6.4 SKILLS

Practice each skill in the Homework 7.2.7, p. 722 problems listed.

1 Identify x-intercepts, turning points, and inflection points: #1–8, 11–18

2 Use a graph to factor a polynomial: #21–28

3 Sketch the graph of a polynomial: #29–46

4 Find a possible formula for a polynomial whose graph is shown: #47–52

5 Graph translations of polynomials: #53–56

7.2.7 Graphing Polynomial Functions (Homework 7.2)

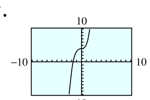
In Problems 1–8.

(a) Describe the long-term behavior of each graph. How does this behavior compare to that of the basic cubic? How does the sign of the lead coefficient affect the graph?

(b) What is the maximum number of x-intercepts? What is the maximum number of turning points? What is the maximum number of inflection points

1. $y = x^3 + 4$

Answer.



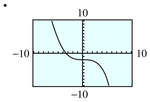
(a) The end behavior is the same as for the basic cubic because the lead coefficient is positive.

(b) There is one x-intercept, no turning points, one inflection point.

2. $y = x^3 - 8$

3. $y = -2 - 0.05x^3$

Answer.



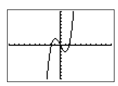
(a) The end behavior is the opposite to the basic cubic (the graph starts in the upper left and extends to the lower right) because the lead coefficient is negative.

(b) There is one x-intercept, no turning points, one inflection point.

4. $y = 5 - 0.02x^3$

5. $y = x^3 - 3x$

Answer.



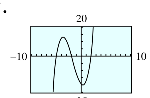
(a) The end behavior is the same as for the basic cubic because the lead coefficient is positive.

(b) There are three x-intercepts, two turning points, one inflection point.

6. $y = 9x - x^3$

7. $y = x^3 + 5x^2 - 4x - 20$

Answer.



(a) The end behavior is the same as for the basic cubic because the lead coefficient is positive.

(b) There are three x-intercepts, two turning points, one inflection point.

8. $y = -x^3 - 2x^2 + 5x + 6$

For Problems 9–10, use a calculator to graph each cubic polynomial.Which graphs are the same?

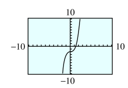
9.

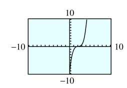
(a) $y = x^3 - 2$

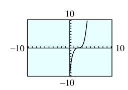
(b) $y = (x - 2)^3$

(c) $y = x^3 - 6x^2 + 12x - 8$

Answer.

(a) 

(b) 

(c) 

(b) and (c) are the same.

10.

(a) $y = x^3 + 3$

(b) $y = (x + 3)^3$

(c) $y = x^3 + 9x^2 + 27x + 27$

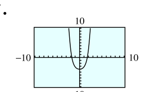
In Problems 11–18.

(a) Describe the long-term behavior of each graph. How does this behavior compare to that of the basic quartic? How does the sign of the lead coefficient affect the graph?

(b) What is the maximum number of x-intercepts? What is the maximum number of turning points? What is the maximum number of inflection points?''

11. $y = 0.5x^4 - 4$

Answer.



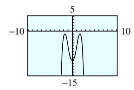
(a) The end behavior is the same as for the basic quartic because the lead coefficient is positive.

(b) There are two x-intercepts, one turning point, no inflection point.

12. $y = 0.3x^4 + 1$

13. $y = -x^4 + 6x^2 - 10$

Answer.



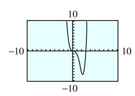
(a) The end behavior is the opposite of the basic quartic (the graph starts in the lower left and ends in the lower right) because the lead coefficient is negative.

(b) There are no x-intercepts, three turning points, two inflection points.

14. $y = x^4 - 8x^2 - 8$

15. $y = x^4 - 3x^3$

Answer.



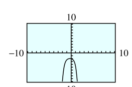
(a) The end behavior is the same as for the basic quartic because the lead coefficient is positive.

(b) There are two x-intercepts, one turning point, two inflection points.

16. $y = -x^4 - 4x^3$

17. $y = -x^4 - x^3 - 2$

Answer.



(a) The end behavior is the opposite of the basic quartic (the graph starts in the lower left and ends in the lower right) because the lead coefficient is negative.

(b) There are no x-intercepts, one turning point, two inflection points.

18. $y = x^4 + 2x^3 + 4x^2 + 10$

19. From your answers to Problems 1–8, what you can conclude about the graphs of cubic polynomials? Consider the long-term behavior, x-intercepts, turning points, and inflection points.

Answer. The graph of a cubic polynomial with a positive lead coefficient will have the same end behavior as the basic cubic, and a cubic with a negative lead coefficient will have the opposite end behavior. Each graph of a cubic polynomial has one, two, or three x-intercepts, it has two, one or no turning point, and it has exactly one inflection point.

20. From your answers to Problems 11–18, what you can conclude about the graphs of quartic polynomials? Consider the long-term behavior, x-intercepts, turning points, and inflection points.

For Problems 21–26,

a Use your calculator to graph each polynomial and locate the x-intercepts. Set Xmin $= -4.7$, Xmax $= 4.7$, and adjust **Ymin** and **Ymax** to get a good graph.

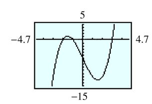
b Write the polynomial in factored form.

c Expand the factored form of the polynomial (that is, multiply the factors together). Do you get the original polynomial?

21. $P(x) = x^3 - 7x - 6$

Answer.

(a)



$(-2, 0), (-1, 0), (3, 0)$

(b) $P(x) =$ $(x+2)(x+1)(x-3)$

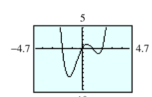
(c) Yes

22. $Q(x) = x^3 + 3x^2 - x - 3$

23. $R(x) = x^4 - x^3 - 4x^2 + 4x$

Answer.

(a)



$(-2, 0), (0, 0), (1, 0), (2, 0)$

(b) $R(x) =$ $(x+2)(x)(x-1)(x-2)$

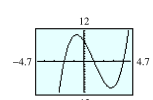
(c) Yes

24. $S(x) = x^4 + 3x^3 - x^2 - 3x$

25. $p(x) = x^3 - 3x^2 - 6x + 8$

Answer.

(a)



$(-2, 0), (1, 0), (4, 0)$

(b) $p(x) =$ $(x+2)(x-1)(x-4)$

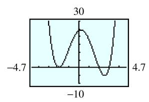
(c) Yes

26. $q(x) = x^3 + 6x^2 - x - 30$

27. $r(x) = x^4 - x^3 - 10x^2 + 4x + 24$

Answer.

(a)



$(-2, 0), (2, 0), (3, 0)$

(b) $r(x) = (x+2)^2(x-2)(x-3)$

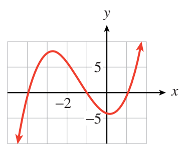
(c) Yes

28. $s(x) = x^4 - x^3 - 12x^2 - 4x + 16$

For Problems 29–36, sketch a rough graph of each polynomial function by hand, paying attention to the shape of the graph near each x-intercept.

29. $q(x) = (x+4)(x+1)(x-1)$

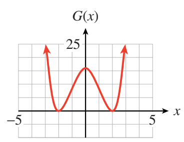
Answer.



30. $p(x) = x(x+2)(x+4)$

31. $G(x) = (x-2)^2(x+2)^2$

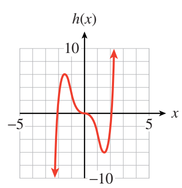
Answer.



32. $F(x) = (x-1)^2(x-3)^2$

33. $h(x) = x^3(x+2)(x-2)$

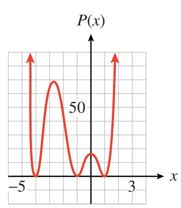
Answer.



34. $H(x) = -(x+1)^3(x-2)^2$

35. $P(x) = (x+4)^2(x+1)^2(x-1)^2$

Answer.



36. $Q(x) = x^2(x-5)(x-1)^2(x+2)$

For Problems 37–46,

a Find the zeros of each polynomial by factoring.

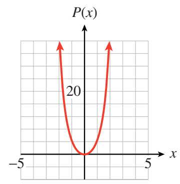
b Sketch a rough graph by hand.

37. $P(x) = x^4 + 4x^2$

Answer.

(a) 0 (multiplicity 2)

(b)



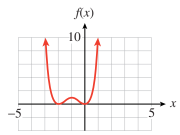
38. $P(x) = x^3 + 3x$

39. $f(x) = -x^4 + 4x^3 + 4x^2$

Answer.

(a) 0 (multiplicity 2), 2 (multiplicity 2)

(b)



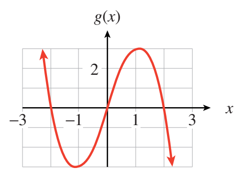
40. $g(x) = x^4 + 4x^3 + 3x^2$

41. $g(x) = 4x - x^3$

Answer.

(a) $0, \pm 2$

(b)



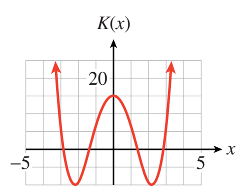
42. $f(x) = 8x - x^4$

43. $K(x) = x^4 - 10x^2 + 16$

Answer.

(a) $\pm\sqrt{2}, \pm\sqrt{8}$

(b)



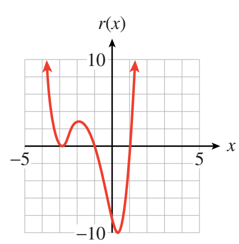
44. $m(x) = x^4 - 15x^2 + 36$

45. $r(x) = (x^2 - 1)(x + 3)^2$

Answer.

(a) $\pm 1, -3$ (multiplicity 2)

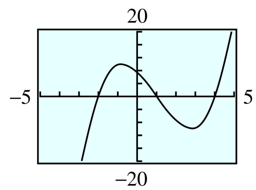
(b)



46. $s(x) = (x^2 - 9)(x - 1)^2$

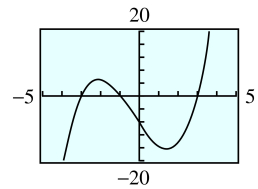
For Problems 47–52, find a possible equation of lowest possible degree for the polynomial whose graph is shown.

47.

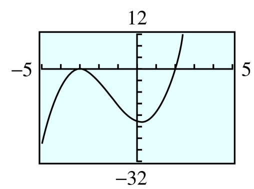


Answer.
$P(x) = (x + 2)(x - 1)(x - 4)$

48.

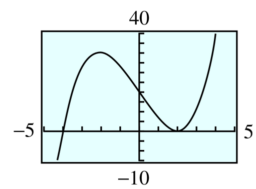


49.

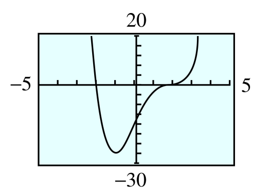


Answer.
$P(x) = (x + 3)^2(x - 2)$

50.

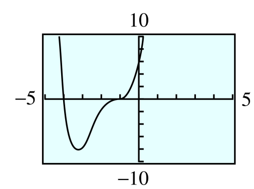


51.



Answer.
$P(x) = (x - 2)^3(x + 2)$

52.



For Problems 53–56, write the formula for each function in parts (a) through (d) and graph with a calculator. Describe how the graph differs from the graph of $y = f(x)$.

53. $f(x) = x^3 - 4x$

(a) $y = f(x) + 3$

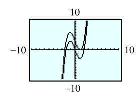
(b) $y = f(x) - 5$

(c) $y = f(x - 2)$

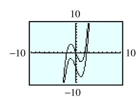
(d) $y = f(x + 3)$

Answer.

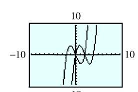
(a) $y = x^3 - 4x + 3$; The graph of $y = f(x)$ shifted 3 units up.



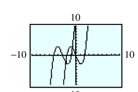
(b) $y = x^3 - 4x - 5$; The graph of $y = f(x)$ shifted 5 units down.



(c) $y = (x - 2)^3 - 4(x - 2)$; The graph of $y = f(x)$ shifted 2 units right.



(d) $y = (x + 3)^3 - 4(x + 3)$; The graph of $y = f(x)$ shifted 3 units left.



54. $f(x) = x^3 - x^2 + x - 1$

(a) $y = f(x) + 4$

(b) $y = f(x) - 4$

(c) $y = f(x - 3)$

(d) $y = f(x + 5)$

55. $f(x) = x^4 - 4x^2$

(a) $y = f(x) + 6$

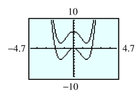
(b) $y = f(x) - 2$

(c) $y = f(x - 1)$

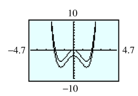
(d) $y = f(x + 2)$

Answer.

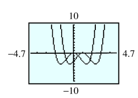
(a) $y = x^4 - 4x^2 + 6$; The graph of $y = f(x)$ shifted 6 units up.



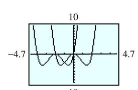
(b) $y = x^4 - 4x^2 - 2$; The graph of $y = f(x)$ shifted 2 units down.



(c) $y = (x - 1)^4 - 4(x - 1)^2$; The graph of $y = f(x)$ shifted 1 unit right.



(d) $y = (x + 2)^4 - 4(x + 2)^2$; The graph of $y = f(x)$ shifted 2 units left.



56. $f(x) = x^4 + 3x^3$

(a) $y = f(x) + 5$

(b) $y = f(x) - 3$

(c) $y = f(x - 2)$

(d) $y = f(x + 1)$

Division Algorithm for Polynomials.

If $f(x)$ and $g(x)$ are nonconstant polynomials with real coefficients, then there exist unique polynomials $q(x)$ and $r(x)$ such that

$$f(x) = g(x)q(x) + r(x),$$

where $\deg r(x) < \deg g(x)$.

In Problems 57–60, use polynomial division to divide $f(x)$ by $g(x)$, and hence find the quotient, $q(x)$, and remainder, $r(x)$. (See Algebra Skills Refresher Section A.7, p. 903 to review polynomial division.)

57. $f(x) = 2x^3 - 2x^2 - 19x - 11,\ g(x) = x - 3$

Answer. $q(x) = 2x^2 + 4x - 7$; $r(x) = -32$

58. $f(x) = 3x^3 + 12x^2 - 13x - 32,\ g(x) = x + 4$

59. $f(x) = x^5 + 2x^4 - 7x^3 - 12x^2 + 5,\ g(x) = x^2 + 2x - 1$

Answer. $q(x) = x^3 - 6x$; $r(x) = -6x + 5$

60. $f(x) = x^5 - 4x^4 + 11x^3 - 12x^2 + 5x + 2,\ g(x) = x^2 - x + 3$

61. The remainder theorem states: If $P(x)$ is a polynomial and a is any real number, there is a unique polynomial $Q(x)$ such that

$$P(x) = (x - a)Q(x) + P(a)$$

Follow the steps below to prove the remainder theorem.

(a) State the division algorithm applied to the polynomials $P(x)$ and $x - a$.

(b) What must be the degree of $r(x)$ in this case?

(c) Evaluate your expression from part (a) at $x = a$. What does this tell you about the remainder, $r(x)$?

Answer.

(a) If $P(x)$ is a nonconstant polynomial with real coefficients and a is any real number, then there exist unique polynomials $q(x)$ and $r(x)$ such that

$$P(x) = (x - a)q(x) + r(x)$$

where deg $r(x) <$ deg $(x - a)$.

(b) Zero

(c) $P(a) = (a - a)q(a) + r(a) = r(a)$. Because deg $r(x) = 0$, $r(x)$ is a constant. That constant value is $P(a)$, so $P(x) = (x - a)q(x) + P(a)$.

62. Verify the remainder theorem for the following:

(a) $P(x) = x^3 - 4x^2 + 2x - 1, a = 2$

(b) $P(x) = 3x^2 + x - 5,\ a = -3$

63. Use the remainder theorem to prove the factor theorem, stated earlier in this section. You will need to justify two statements:

(a) If $P(a) = 0$, show that $x - a$ is a factor of $P(x)$.

(b) If $x - a$ is a factor of $P(x)$, show that $P(a) = 0$.

Answer.

(a) From the remainder theorem, $\begin{aligned} P(x) &= (x - a)Q(x) + P(a) \\ &= (x - a)Q(x) + 0 \\ &= (x - a)Q(x) \end{aligned}$

(b) By definition of a factor, if $x - a$ is a factor of $P(x)$, then $P(x) = (x - a)q(x)$, so $P(x) = (x - a)q(x) + 0$. The uniqueness guaranteed in the remainder theorem tells us that $P(a) = 0$.

64. Verify the factor theorem for the following:

(a) $P(x) = x^4 - 4x^3 - 11x^2 + 3x + 2,\ a = -2$

(b) $P(x) = x^3 + 2x^2 - 31x - 20,\ a = 5$

For Problems 65–68,

(a) Verify that the given value is a zero of the polynomial.

(b) Find the other zeros. (*Hint*: Use polynomial division to write $P(x) = (x - a)Q(x)$, then factor $Q(x)$.)

65. $P(x) = x^3 - 2x^2 + 1;\ a = 1$

Answer.

(a) $P(1) = 0$ (b) $\dfrac{1 \pm \sqrt{5}}{2}$

66. $P(x) = x^3 + 2x^2 - 1;\ a = -1$

67. $P(x) = x^4 - 3x^3 - 10x^2 + 24x;\ a = -3$

Answer.

(a) $P(-3) = 0$ (b) $0, 2, 4$

68. $P(x) = x^4 + 5x^3 - x^2 - 5x;\ a = -5$

In Problems 69–70, we use polynomials to approximate other functions.

69.

(a) Graph the functions $f(x) = e^x$ and

$$p(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3$$

in the standard window. For what values of x does it appear that $p(x)$ would be a good approximation for $f(x)$?

(b) Change the window settings to

$$\text{Xmin} = -4.7 \qquad \text{Xmax} = 4.7$$
$$\text{Ymin} = 0 \qquad \text{Ymax} = 20$$

and fill in the table of values below. (You can use the **value** feature on your calculator.)

x	-1	-0.5	0	0.5	1	1.5	2
$f(x)$							
$p(x)$							

(c) The **error** in the approximation is the difference $f(x) - p(x)$. We can reduce the error by using a polynomial of higher degree. The nth degree polynomial for approximating e^x is

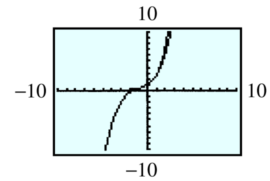
$$P_n(x) = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots + \frac{1}{n!}x^n$$

where $n! = n(n-1)(n-2)\cdots 3\cdot 2\cdot 1$. Graph $f(x)$ and $P_5(x)$ in the same window as in part (b). What is the error in approximating $f(2)$ by $P_5(2)$?

(d) Graph $f(x) - P_5(x)$ in the same window as in part (b). What does the graph tell you about the error in approximating $f(x)$ by $P_5(x)$?

Answer.

(a) About $-1 < x < 2$

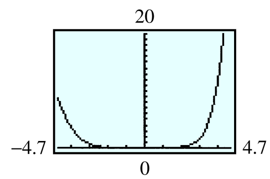


(b)

x	-1	-0.5	0	0.5	1	1.5	2
$f(x)$	0.368	0.607	1	1.649	2.718	4.482	7.389
$p(x)$	0.333	0.604	1	1.646	2.667	4.188	6.333

(c) 0.122

(d)



The error is relatively small for values of x between -3 and 2.5.

70. In Projects for Chapter 2: Periodic Functions, p. 279, we investigated periodic functions. The **sine function**, $f(x) = \sin(x)$, is a useful periodic function.

(a) Graph the functions

$$f(x) = \sin(x) \quad \text{and} \quad p(x) = x - \frac{1}{6}x^3$$

in the standard window. (Check that your calculator is set in **Radian** mode.) For what values of x does it appear that $p(x)$ would be a good approximation for $f(x)$?

(b) Change the window settings to

$$\text{Xmin} = -4.7 \qquad \text{Xmax} = 4.7$$
$$\text{Ymin} = -2 \qquad \text{Ymax} = 2$$

and fill in the table of values below. (You can use the **value** feature on your calculator.)

x	-1	-0.5	0	0.5	1	1.5	2
$f(x)$							
$p(x)$							

(c) Two more polynomials for approximating $f(x) = \sin(x)$ are

$$P_5(x) = 1 - \frac{1}{3!}x^3 + \frac{1}{5!}x^5$$
$$P_7(x) = 1 - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - + \frac{1}{7!}x^7$$

(See Problem 69, p. 733 for the definition of $n!$.) Graph $f(x)$ and $P_5(x)$ in the same window as in part (b). What is the error in approximating $f(2)$ by $P_5(2)$?

(d) Graph $f(x) - P_5(x)$ in the same window as in part (b). What does the graph tell you about the error in approximating $f(x)$ by $P_5(x)$?

7.3 Complex Numbers

7.3.1 Introduction

You know that not all quadratic equations have real solutions.

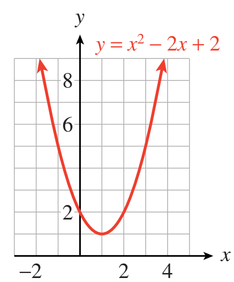
For example, the graph of

$$f(x) = x^2 - 2x + 2$$

has no x-intercepts (as shown at right), and the equation

$$x^2 - 2x + 2 = 0$$

has no real solutions.



We can still use completing the square or the quadratic formula to solve the equation.

Example 7.3.1 Solve the equation $x^2 - 2x + 2 = 0$ by using the quadratic formula.

Solution. We substitute $a = 1$, $b = -2$, and $c = 2$ into the quadratic formula to get

$$x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(2)}}{2(1)} = \frac{2 \pm \sqrt{-4}}{2}$$

Because $\sqrt{-4}$ is not a real number, the equation $x^2 - 2x + 2 = 0$ has no real solutions. □

Checkpoint 7.3.2 Solve the equation $x^2 - 6x + 13 = 0$ by using the quadratic formula.

Answer. $x = \dfrac{6 \pm \sqrt{-16}}{2}$

7.3.2 Imaginary Numbers

Although square roots of negative numbers such as $\sqrt{-4}$ are not real numbers, they occur often in mathematics and its applications.

Mathematicians began working with square roots of negative numbers in the sixteenth century, in their attempts to solve quadratic and cubic equations. René Descartes gave them the name imaginary numbers, which reflected the mistrust with which mathematicians regarded them at the time. Today, however, such numbers are well understood and used routinely by scientists and engineers.

We begin by defining a new number, i, whose square is -1.

Imaginary Unit.

We define the **imaginary unit** i by

$$i^2 = -1 \quad \text{or} \quad i = \sqrt{-1}$$

Caution 7.3.3 The letter i used in this way is not a variable; it is the name of a specific number and hence is a constant.

The square root of any negative number can be written as the product of a real number and i. For example,

$$\begin{aligned}\sqrt{-4} &= \sqrt{-1\cdot 4}\\ &= \sqrt{-1}\sqrt{4} = i\cdot 2\end{aligned}$$

or $\sqrt{-4} = 2i$. Any number that is the product of i and a real number is called an **imaginary number**.

Imaginary Numbers.

For $a > 0$,

$$\sqrt{-a} = \sqrt{-1}\cdot\sqrt{a} = i\sqrt{a}$$

Examples of imaginary numbers are

$$3i,\quad \frac{7}{8}i,\quad -38i,\quad \text{and}\quad i\sqrt{5}$$

Example 7.3.4 Write each radical as an imaginary number.

a $\sqrt{-25}$

b $2\sqrt{-3}$

Solution.

a

$$\begin{aligned}\sqrt{-25} &= \sqrt{-1}\sqrt{25}\\ &= i\sqrt{25} = 5i\end{aligned}$$

b

$$\begin{aligned}2\sqrt{-3} &= 2\sqrt{-1}\sqrt{3}\\ &= 2i\sqrt{3}\end{aligned}$$

□

Checkpoint 7.3.5 Write each radical as an imaginary number.

a $\sqrt{-18}$

b $-6\sqrt{-5}$

Answer.

a $3i\sqrt{2}$

b $-6i\sqrt{5}$

Note 7.3.6 Every negative real number has two imaginary square roots, $i\sqrt{a}$ and $-i\sqrt{a}$, because

$$\left(i\sqrt{a}\right)^2 = i^2(\sqrt{a})^2 = -a$$

and

$$\left(-i\sqrt{a}\right)^2 = (-i)^2(\sqrt{a})^2 = -a$$

For example, the two square roots of -9 are $3i$ and $-3i$.

7.3.3 Complex Numbers

Consider the quadratic equation

$$x^2 - 2x + 5 = 0$$

Using the quadratic formula to solve the equation, we find

$$x = \frac{-(-2)\pm\sqrt{(-2)^2-4(1)(5)}}{2} = \frac{2\pm\sqrt{-16}}{2}$$

If we now replace $\sqrt{-16}$ with $4i$, we have

$$x = \frac{2 \pm 4i}{2} = 1 \pm 2i$$

The two solutions are $1+2i$ and $1-2i$. These numbers are examples of **complex numbers**.

Complex Numbers.

A **complex number** can be written in the form $a + bi$, where a and b are real numbers.

Examples of complex numbers are

$$3 - 5i, \quad 2 + \sqrt{7}i, \quad \frac{4-i}{3}, \quad 6i, \quad \text{and } -9$$

In a complex number $a + bi$, a is called the **real part**, and b is called the **imaginary part**. All real numbers are also complex numbers (with the imaginary part equal to zero). A complex number whose real part equals zero is called a **pure imaginary** number.

Example 7.3.7 Write the solutions to Example 7.3.1, p. 735, $\dfrac{2 \pm \sqrt{-4}}{2}$, as complex numbers.

Solution. Because $\sqrt{-4} = \sqrt{-1}\sqrt{4} = 2i$, we have $\dfrac{2 \pm \sqrt{-4}}{2} = \dfrac{2 \pm 2i}{2}$, or $1 \pm i$. The solutions are $1 + i$ and $1 - i$. □

Checkpoint 7.3.8 Use extraction of roots to solve $(2x + 1)^2 + 9 = 0$. Write your answers as complex numbers.

Answer. $x = \dfrac{-1}{2} \pm \dfrac{3}{2}i$

7.3.4 Arithmetic of Complex Numbers

All the properties of real numbers listed in Algebra Skills Refresher Section A.13, p. 970 are also true of complex numbers. We can carry out arithmetic operations with complex numbers.

We add and subtract complex numbers by combining their real and imaginary parts separately. For example,

$$\begin{aligned}(4 + 5i) + (2 - 3i) &= (4 + 2) + (5 - 3)i \\ &= 6 + 2i\end{aligned}$$

Sums and Differences of Complex Numbers.

$$(a + bi) + (c + di) = (a + c) + (b + d)i$$

$$(a + bi) - (c + di) = (a - c) + (b - d)i$$

Example 7.3.9 Subtract: $(8 - 6i) - (5 + 2i)$.

Solution. Combine the real and imaginary parts.

$$\begin{aligned}(8 - 6i) - (5 + 2i) &= (8 - 5) + (-6 - 2)i \\ &= 3 + (-8)i = 3 - 8i\end{aligned}$$

□

Checkpoint 7.3.10 Subtract: $(-3+2i)-(-3-2i)$.

Answer. $4i$

7.3.5 Products of Complex Numbers

To find the product of two imaginary numbers, we use the fact that $i^2=-1$. For example,

$$\begin{aligned}(3i)\cdot(4i) &= 3\cdot 4i^2\\ &= 12(-1)=-12\end{aligned}$$

To find the product of two complex numbers, we use the FOIL method, as if the numbers were binomials. For example,

$$(2+3i)(3-5i)=6-10i+9i-15i^2$$

Because $i^2=-1$, the last term, $-15i^2$, can be replaced by $-15(-1)$, or 15, to obtain

$$6-10i+9i+15$$

Finally, we combine the real parts and imaginary parts to obtain

$$(6+15)+(-10i+9i)=21-i$$

Example 7.3.11 Multiply $(7-4i)(-2-i)$.

Solution.

$$\begin{aligned}(7-4i)(-2-i) &= -14-7i+8i+4i^2 && \text{Replace } \boldsymbol{i^2} \text{ by } \mathbf{-1.}\\ &= -14-7i+8i-4 && \text{Combine real parts and imaginary}\\ &= -18+i && \text{parts.}\end{aligned}$$

□

Checkpoint 7.3.12 Multiply $(-3+2i)(-3-2i)$.

Answer. 13

You can verify that in general the following rule holds.

Product of Complex Numbers.

$$(a+bi)(c+di)=(ac-bd)+(ad+bc)i$$

Caution 7.3.13 One property of real numbers that is not true of complex numbers is $\sqrt{ab}=\sqrt{a}\cdot\sqrt{b}$. This identity fails when a and b are both negative. For example, if $a=b=-2$, we have

$$\sqrt{ab}=\sqrt{(-2)(-2)}=\sqrt{4}=2$$

but

$$\sqrt{a}\cdot\sqrt{b}=\sqrt{-2}\cdot\sqrt{-2}=\sqrt{-1\cdot 2}\cdot\sqrt{-1\cdot 2}=i\sqrt{2}\cdot i\sqrt{2}=i^2(i\sqrt{2})^2=-2$$

so in this case

$$\sqrt{ab}\neq\sqrt{a}\cdot\sqrt{b}$$

We can avoid possible errors by writing square roots of negative numbers as imaginary numbers.

7.3.6 Quotients of Complex Numbers

To find the quotient of two complex numbers, we use the technique of rationalizing the denominator. (See Algebra Skills Refresher Section A.10, p. 933.)

For example, consider the quotient

$$\frac{3+4i}{2i}$$

Because i is really a radical (remember that $i=\sqrt{-1}$), we multiply the numerator and denominator of the quotient by $\boldsymbol{i}$ to obtain

$$\frac{(3+4i)\cdot \boldsymbol{i}}{2i\cdot \boldsymbol{i}} = \frac{3i+4i^2}{2i^2} \qquad \text{Apply the distributive law to the numerator.}$$
$$= \frac{3i-4}{-2} \qquad \text{Recall that } \boldsymbol{i^2=-1}.$$

To write the quotient in the form $a+bi$, we divide -2 into each term of the numerator to get

$$\frac{3i}{-2}-\frac{4}{-2} = \frac{-3}{2}i+2 = 2+\frac{-3}{2}i$$

Example 7.3.14 Divide $\dfrac{10-15i}{5i}$

Solution. We multiply numerator and denominator by $\boldsymbol{i}$.

$$\frac{10-15i}{5i} = \frac{(10-15i)\cdot \boldsymbol{i}}{5i\cdot \boldsymbol{i}}$$
$$= \frac{10i-15i^2}{5i^2} \qquad \text{Replace } \boldsymbol{i^2} \text{ by } \boldsymbol{-1}.$$
$$= \frac{10i+15}{-5}$$
$$= \frac{10i}{-5}+\frac{15}{-5} \qquad \text{Divide } \boldsymbol{-5} \text{ into each term of numerator.}$$
$$= -2i-3$$

The quotient is $-3-2i$. □

Checkpoint 7.3.15 Divide $\dfrac{8+9i}{3i}$

Answer. $3-\dfrac{8}{3}i$

If $z=a+bi$ is any nonzero complex number, then the number $\overline{z}=a-bi$ is called the **complex conjugate** of z. The product of a nonzero complex number and its conjugate is always a positive real number.

$$z\overline{z} = (a+bi)(a-bi) = a^2-b^2i^2 = a^2-b^2(-1) = a^2+b^2$$

We use this fact to find the quotient of complex numbers. If the divisor has both a real and an imaginary part, we multiply numerator and denominator by the conjugate of the denominator.

Example 7.3.16 Divide $\dfrac{2+3i}{4-2i}$

Solution. We multiply numerator and denominator by $\boldsymbol{4+2i}$, the conjugate of the denominator.

$$\frac{2+3i}{4-2i} = \frac{(2+3i)(\boldsymbol{4+2i})}{(4-2i)(\boldsymbol{4+2i})} \qquad \text{Expand numerator and denominator.}$$

$$\begin{aligned}
&= \frac{8+4i+12i+6i^2}{16+8i-8i-4i^2} && \text{Replace } \boldsymbol{i^2} \text{ by } \boldsymbol{-1.}\\
&= \frac{8+16i-6}{16-(-4)} && \text{Combine like terms.}\\
&= \frac{2+16i}{20} && \text{Divide 20 into each term of numerator.}\\
&= \frac{2}{20}+\frac{16i}{20}\\
&= \frac{1}{10}+\frac{4}{5}i
\end{aligned}$$

□

Checkpoint 7.3.17 Write the quotient $\dfrac{4-2i}{1+i}$ in the form $a+bi$.

Answer. $1-3i$

7.3.7 Zeros of Polynomials

Because we can add, subtract, and multiply any two complex numbers, we can use a complex number as an input for a polynomial function. Thus, we can extend the domain of any polynomial to include all complex numbers.

Example 7.3.18 Evaluate the polynomial $f(x)=x^2-2x+2$ for $x=1+i$, then simplify.

Solution. We substitute $x=\boldsymbol{1+i}$ to find

$$\begin{aligned}
f(\boldsymbol{1+i}) &= (\boldsymbol{1+i})^2-2(\boldsymbol{1+i})+2\\
&= 1^2+2i+i^2-2-2i+2\\
&= 1+2i+(-1)-2-2i+2\\
&= 0
\end{aligned}$$

Thus, $f(1+i)=0$, so $1+i$ is a solution of $x^2-2x+2=0$. □

Checkpoint 7.3.19 If $f(x)=x^2-6x+13$, evaluate $f(3+2i)$.

Answer. 0

In Chapter 6, p. 599, we learned that irrational solutions of quadratic equations occur in conjugate pairs,

$$x=\frac{-b}{2a}+\frac{\sqrt{b^2-4ac}}{2a} \quad \text{and} \quad x=\frac{-b}{2a}-\frac{\sqrt{b^2-4ac}}{2a}$$

If the discriminant $D=b^2-4ac$ is negative, the two solutions are complex conjugates,

$$z=\frac{-b}{2a}+\frac{i\sqrt{|D|}}{2a} \quad \text{and} \quad \overline{z}=\frac{-b}{2a}-\frac{i\sqrt{|D|}}{2a}$$

Thus, if we know that z is a complex solution of a quadratic equation, we know that $\overline{z}$ is the other solution. The quadratic equation with solutions z and $\overline{z}$ is

$$\begin{aligned}
(x-z)(x-\overline{z}) &= 0\\
x^2-(z+\overline{z})+z\overline{z} &= 0
\end{aligned}$$

Example 7.3.20

a Let $z=7-5i$. Compute $z\overline{z}$.

b Find a quadratic equation with one solution being $z=7-5i$.

Solution.

a The conjugate of $z = 7 - 5i$ is $\overline{z} = 7 + 5i$, so

$$\begin{aligned} z\overline{z} &= (7 - 5i)(7 + 5i) \\ &= 49 - 25i^2 \\ &= 49 + 25 \\ &= 74 \end{aligned}$$

b The other solution of the equation is $\overline{z} = 7 + 5i$, and the equation is $(x - z)(x - \overline{z}) = 0$. We expand the product to find

$$\begin{aligned} (x - z)(x - z) &= x^2 - (z + z)x + z\overline{z} \\ &= x^2 - (7 - 5i + 7 + 5i)x + 74 \\ &= x^2 - 14x + 74 \end{aligned}$$

The equation is $x^2 - 14x + 74 = 0$.

□

Checkpoint 7.3.21

a Let $z = -3 + 4i$. Compute $z\overline{z}$.

b Find a quadratic equation with one solution being $z = -3 + 4i$.

Answer.

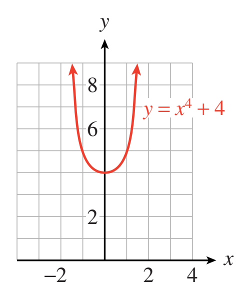
a 25 b $x^2 + 6x + 25 = 0$

One of the most important results in mathematics is the **fundamental theorem of algebra**, which says that if we allow complex numbers as inputs, then every polynomial $p(x)$ of degree $n \geq 1$ has exactly n complex number zeros.

Fundamental Theorem of Algebra.

Let $p(x)$ be a polynomial of degree $n \geq 1$. Then $p(x)$ has exactly n complex zeros.

As a result, the factor theorem tells that every polynomial of degree n can be factored as the product of n linear terms. For example, although the graph of $y = x^4 + 4$ shown at right has no x-intercepts, the fundamental theorem tells us that there are four complex solutions to $x^4 + 4 = 0$, and that $x^4 + 4$ can be factored.



You can check that the four solutions to $x^4 + 4 = 0$ are $1 + i$, $-1 + i$, $-1 - i$, and $1 - i$. For example, if $x = 1 + i$, then

$$x^2 = (1 + i)^2 = 1 + 2i + i^2 = 2i$$

and

$$x^4 = \left(x^2\right)^2 = (2i)^2 = -4,$$

so $x^4 + 4 = (-4) + 4 = 0$.

Because each zero corresponds to a factor of the polynomial, the factored form of x^4+4 is

$$x^4+4 = \left[x-(1+i)\right]\left[x-(-1+i)\right]\left[x-(-1-i)\right]\left[x-(1-i)\right]$$

Note 7.3.22 The four solutions to $x^4+4=0$ form two complex conjugate pairs, namely $1\pm i$ and $-1\pm i$. In fact, for every polynomial with real coefficients, the nonreal zeros always occur in complex conjugate pairs.

Example 7.3.23 Find a fourth-degree polynomial with real coefficients, two of whose zeros are $3i$ and $2+i$.

Solution. The other two zeros are $-3i$ and $2-i$. The factored form of the polynomial is

$$(x-3i)(x+3i)[x-(2+i)][x-(2-i)]$$

We multiply together the factors to find the polynomial. The product of $(x-3i)(x+3i)$ is x^2+9, and

$$\begin{aligned}[x-(2+i)][x-(2-i)] &= x^2-(2+i+2-i)x+(2+i)(2-i)\\ &= x^2-4x+5\end{aligned}$$

Finally, we multiply these two partial products to find the polynomial we seek,

$$(x^2+9)(x^2-4x+5) = x^4-4x^3+14x^2-36x+45$$

□

Checkpoint 7.3.24

a Find the zeros of the polynomial $f(x)=x^4+15x^2-16$.

b Write the polynomial in factored form.

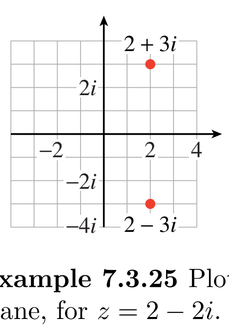
Answer.

a $\pm 1,\ \ \pm 4i$

b $(x-1)(x+1)(x-4i)(x+4i)$

7.3.8 Graphing Complex Numbers

Real numbers can be plotted on a number line, but to graph a complex number we use a plane, called the complex plane. In the **complex plane**, the real numbers lie on the horizontal or **real axis**, and pure imaginary numbers lie on the vertical or **imaginary axis**.

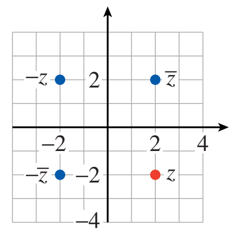


To plot a complex number a + bi, we move a units from the origin in the horizontal direction and b units in the vertical direction. The numbers $2+3i$ and $2-3i$ are plotted at left.

Example 7.3.25 Plot the numbers z, $\overline{z}$, $-z$, and $-\overline{z}$ as points on the complex plane, for $z=2-2i$.

Solution.

- To plot $z = 2 - 2i$, we move from the origin 2 units to the right and 2 units down.
- To plot $\overline{z} = 2 + 2i$, we move from the origin 2 units to the right and 2 units up.
- To plot $-z = -2 + 2i$, we move from the origin 2 units to the left and 2 units up.
- To plot $-\overline{z} = -2 - 2i$, we move from the origin 2 units to the left and 2 units down.



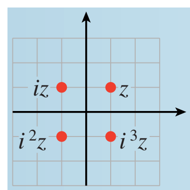
All four points are plotted at right.

□

Checkpoint 7.3.26 Plot the following numbers as points on the complex plane.

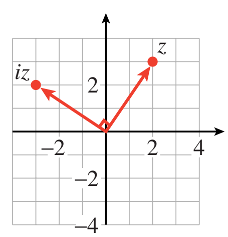
- $z = 1 + i$
- $i^2z = i^2 + i^3$
- $iz = i + i^2$
- $i^3z = i^3 + i^4$

Answer.



Note 7.3.27

If we draw an arrow from the origin to the point $a+bi$ in the complex plane, we can see that multiplication by i corresponds to rotating a point around the origin by 90° in the counterclockwise direction. For example, the figure at right shows the graphs of $z = 2 + 3i$ and $iz = 2i - 3$.



7.3.9 Section Summary

7.3.9.1 Vocabulary

Look up the definitions of new terms in the Glossary.

- Imaginary unit
- Imaginary axis
- Imaginary part
- Imaginary number
- Complex conjugate
- Complex plane
- Complex number
- Real part
- Real axis

7.3.9.2 CONCEPTS

1 The square root of a negative number is an imaginary number.

2 A complex number is the sum of a real number and an imaginary number

3 We can perform the four arithmetic operations on complex numbers

4 The product of a nonzero complex number and its conjugate is always a positive real number.

> 5 **Fundamental Theorem of Algebra.**
>
> Let $p(x)$ be a polynomial of degree $n \geq 1$. Then $p(x)$ has exactly n complex zeros.

6 The nonreal zeros of a polynomial with real coefficients always occur in conjugate pairs.

7 We can graph complex numbers in the complex plane.

8 Multiplying a complex number by i rotates its graph by 90° around the origin.

7.3.9.3 STUDY QUESTIONS

1 What are imaginary numbers, and why were they invented?

2 Simplify the following powers of i:

$$i^2, i^3, i^4, i^5, i^6, i^7, i^8$$

What do you notice?

3 Explain how the complex conjugate is used in dividing complex numbers.

4 If one solution of a quadratic equation is $3 + i\sqrt{2}$, what is the other solution?

5 If $P(x)$ is a polynomial of degree 7, how many zeros does $P(x)$ have? How many x-intercepts could its graph have? How many complex zeros could $P(x)$ have?

7.3.9.4 SKILLS

Practice each skill in the Homework 7.3.10, p. 744 problems listed.

1 Write and simplify complex numbers: #1–10

2 Perform arithmetic operations on complex numbers: #11–36

3 Evaluate polynomials at complex numbers, expand polynomials: #37–48

4 Find a polynomial with given zeros: #53–56, 59–62

5 Graph complex numbers: #63–70

7.3.10 Complex Numbers (Homework 7.3)

For Problems 1–6, write the complex number in the form $a + bi$, where a and b are real numbers.

1. $\sqrt{-25} - 4$

Answer.
$-4 + 5i$

2. $\sqrt{-9} + 3$

3. $\dfrac{-8 + \sqrt{-4}}{2}$

Answer.
$-4 + i$

4. $\dfrac{6 - \sqrt{-36}}{2}$

5. $\dfrac{-5 - \sqrt{-2}}{6}$

Answer.
$\dfrac{-5}{6} - \dfrac{\sqrt{2}}{6}i$

6. $\dfrac{7 + \sqrt{-3}}{4}$

For Problems 7–10, find the zeros of the quadratic polynomial. Write each in the form $a+bi$, where a and b are real numbers.

7. $x^2+6x+13$

Answer. $-3\pm 2i$

8. $x^2-2x+10$

9. $3x^2-x+1$

Answer. $\dfrac{1}{6}\pm\dfrac{\sqrt{11}}{6}i$

10. $5x^2+2x+2$

For Problems 11–14, add or subtract.

11. $(11-4i)-(-2-8i)$

Answer. $13+4i$

12. $(7i-2)+(6-4i)$

13. $(2.1+5.6i)+(-1.8i-2.9)$

Answer. $-0.8+3.8i$

14. $\left(\dfrac{1}{5}i-\dfrac{2}{5}\right)-\left(\dfrac{4}{5}-\dfrac{3}{5}i\right)$

For Problems 15–24, multiply.

15. $5i(2-4i)$

Answer. $20+10i$

16. $-7i(-1+4i)$

17. $(4-i)(-6+7i)$

Answer. $-17+34i$

18. $(2-3i)(2-3i)$

19. $(7+i\sqrt{3})^2$

Answer. $46+14i\sqrt{3}$

20. $(5-i\sqrt{2})^2$

21. $(7+i\sqrt{3})(7-i\sqrt{3})$

Answer. 52

22. $(5-i\sqrt{2})(5+i\sqrt{2})$

23. $(1-i)^3$

Answer. $-2-2i$

24. $(2+i)^3$

For Problems 25–36, divide.

25. $\dfrac{12+3i}{-3i}$

Answer. $-1+4i$

26. $\dfrac{12+4i}{8i}$

27. $\dfrac{10+15i}{2+i}$

Answer. $7+4i$

28. $\dfrac{4-6i}{1-i}$

29. $\dfrac{5i}{2-5i}$

Answer. $\dfrac{-25}{29}+\dfrac{10}{29}i$

30. $\dfrac{-2i}{7+2i}$

31. $\dfrac{\sqrt{3}}{\sqrt{3}+i}$

Answer. $\dfrac{3}{4}-\dfrac{\sqrt{3}}{4}i$

32. $\dfrac{2\sqrt{2}}{1-i\sqrt{2}}$

33. $\dfrac{1+i\sqrt{5}}{1-i\sqrt{5}}$

Answer. $\dfrac{-2}{3}+\dfrac{\sqrt{5}}{3}i$

34. $\dfrac{\sqrt{2}-i}{\sqrt{2}+i}$

35. $\dfrac{3+2i}{2-3i}$

Answer. i

36. $\dfrac{4-6i}{-3-2i}$

For Problems 37–42, evaluate the polynomial for the given values of the variable.

37. $z^2 + 9$

a $z = 3i$

b $z = -3i$

Answer.

(a) 0 (b) 0

38. $2y^2 - y - 2$

a $y = 2 - i$

b $y = -2 - i$

39. $x^2 - 2x + 2$

a $x = 1 - i$

b $x = 1 + i$

Answer.

(a) 0 (b) 0

40. $3w^2 + 5$

a $w = 2i$

b $w = -2i$

41. $q^2 + 4q + 13$

a $q = -2 + 3i$

b $q = -2 - 3i$

Answer.

(a) 0 (b) 0

42. $v^2 + 2v + 3$

a $v = 1 + i$

b $v = -1 + i$

For Problems 43–48, expand each product of polynomials.

43. $(2z + 7i)(2z - 7i)$

Answer. $4z^2 + 49$

44. $(5w + 3i)(5w - 3i)$

45. $[x + (3 + i)][x + (3 - i)]$

Answer. $x^2 + 6x + 10$

46. $[s - (1 + 2i)][s - (1 - 2i)]$

47. $[v - (4 + i)][v - (4 - i)]$

Answer. $v^2 - 8v + 17$

48. $[Z + (2 + i)][Z + (2 - i)]$

49. For what values of x will $\sqrt{x - 5}$ be real? Imaginary?

Answer. $x \geq 5$; $x < 5$

50. For what values of x will $\sqrt{x + 3}$ be real? Imaginary?

51. Simplify.

(a) i^6 (b) i^{12} (c) i^{15} (d) i^{102}

Answer.

(a) -1 (b) 1 (c) $-i$ (d) -1

52. Express with a positive exponent and simplify.

(a) i^{-1} (b) i^{-2} (c) i^{-3} (d) i^{-6}

In Problems 53–56,

a Given one solution of a quadratic equation with rational coefficients, find the other solution.

b Write a quadratic equation that has those solutions.

53. $2+\sqrt{5}$

Answer.

(a) $2-\sqrt{5}$

(b) x^2-4x-1

54. $3-\sqrt{2}$

55. $4-3i$

Answer.

(a) $4+3i$

(b) $x^2-8x+25$

54. $5+i$

Every polynomial factors into a product of a constant and linear factors of the form $(x-a)$, where a can be either real or complex. In Problems 57–58, how many linear factors are in the factored form of the given polynomial?

57.

(a) $x^4-2x^3+4x^2+8x-6$

(b) $2x^5-x^3+6x-4$

Answer.

(a) 4

(b) 5

58.

(a) x^6-6x

(b) x^3+3x^2-2x+1

For Problems 59–62, find a fourth-degree polynomial with real coefficients that has the given complex numbers as two of its zeros.

59. $1+3i,\ 2-i$

Answer.
$x^4-6x^3+23x^2-50x+50$

60. $5-4i,\ \ -i$

61. $\frac{1}{2}-\frac{\sqrt{3}}{2}i,\ 3+2i$

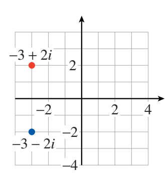
Answer.
$x^4-7x^3+20x^2-19x+13$

62. $-\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}i,\ 4-i$

For Problems 63–66, plot each number and its complex conjugate in the complex plane. What is the geometric relationship between complex conjugates?

63. $z=-3+2i$

Answer.

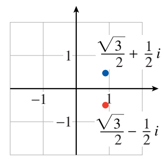


The complex conjugates are reflections of each other across the real axis.

64. $z=4-3i$

65. $z = \dfrac{\sqrt{3}}{2} - \dfrac{1}{2}i$

Answer.



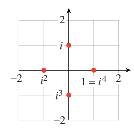
The complex conjugates are reflections of each other across the real axis.

66. $z = -\dfrac{\sqrt{2}}{2} - \dfrac{\sqrt{2}}{2}i$

For Problems 59–62, simplify and plot each complex number as a point on the complex plane.

67. $1, i, i^2, i^3$ and i^4

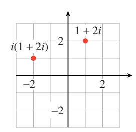
Answer.



68. $-1, -i, -i^2, -i^3$ and $-i^4$

69. $1 + 2i$ and $i(1 + 2i)$

Answer.



70. $3 - 4i$ and $i(3 - 4i)$

Problems 71–72 show that multiplication by i results in a rotation of 90°.

71. Suppose that $z = a + bi$ and that the real numbers a and b are both nonzero.

(a) What is the slope of the segment in the complex plane joining the origin to z?

(b) What is the slope of the segment in the complex plane joining the origin to zi?

(c) What is the product of the slopes of the two segments from parts (a) and (b)? What can you conclude about the angle between the two segments?

Answer.

(a) $m = \dfrac{b}{a}$

(b) $m = \dfrac{a}{-b}$

(c) -1; The angle is 90°.

72. Suppose that $z = a + bi$ and that a and b are both real numbers.

(a) If $a \neq 0$ and $b = 0$, then what is the slope of the segment in the complex plane joining the origin to z? What is the slope of the segment joining the origin to iz?

(b) If $a = 0$ and $b \neq 0$, then what is the slope of the segment in the complex plane joining the origin to z? What is the slope of the segment joining the origin to iz?

(c) What can you conclude about the angle between the two segments from parts (a) and (b)?

7.4 Graphing Rational Functions

7.4.1 Introduction

A rational function is the quotient of two polynomials. (As with rational numbers, the word *rational* refers to a ratio.)

Rational Function.

A **rational function** is one of the form

$$f(x) = \frac{P(x)}{Q(x)}$$

where $P(x)$ and $Q(x)$ are polynomials and $Q(x)$ is not the zero polynomial.

The graphs of rational functions can be quite different from the graphs of polynomials.

Example 7.4.1 Francine is planning a 60-mile training flight through the desert on her cycle-plane, a pedal-driven aircraft. If there is no wind, she can pedal at an average speed of 15 miles per hour, so she can complete the flight in 4 hours.

a If there is a headwind of x miles per hour, it will take Francine longer to fly 60 miles. Express the time it will take to complete the training flight as a function of x.

b Make a table of values for the function.

c Graph the function and explain what it tells you about the time Francine should allot for the flight.

Solution.

a If there is a headwind of x miles per hour, Francine's ground speed will be $15 - x$ miles per hour. Using the fact that time time $= \dfrac{\text{distance}}{\text{rate}}$, we find that the time needed for the flight will be

$$t = f(x) = \frac{60}{15 - x}$$

b We evaluate the function for several values of x, as shown in the table below.

x	0	3	5	7	9	10
t	4	5	6	7.5	10	12

For example, if the headwind is **5** miles per hour, then

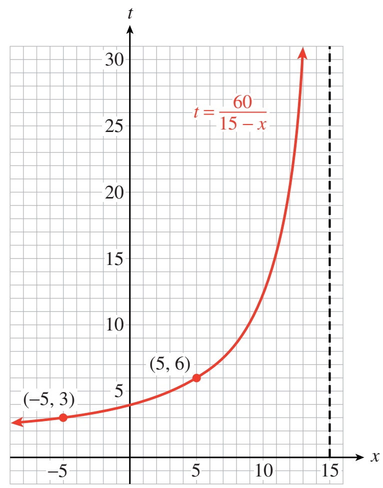
$$t = \frac{60}{15 - \mathbf{5}} = \frac{60}{10} = 6$$

Francine's effective speed is only 10 miles per hour, and it will take her 6 hours to fly the 60 miles. The table shows that as the speed of the headwind increases, the time required for the flight increases also.

c The graph of the function is shown below. You can use your calculator with the window

Xmin $= -8.5$	Xmax $= 15$
Ymin $= 0$	Ymax $= 30$

to verify the graph. In particular, the point $(0, 4)$ lies on the graph. This point tells us that if there is no wind, Francine can fly 60 miles in 4 hours, as we calculated earlier.



The graph is increasing, as indicated by the table of values. In fact, as the speed of the wind gets close to 15 miles per hour, Francine's flying time becomes extremely large. In theory, if the wind speed were exactly 15 miles per hour, Francine would never complete her flight. On the graph, the time becomes infinite at $x = 15$.

What about negative values for x? If we interpret a negative headwind as a tailwind, Francine's flying time should decrease for negative x-values. For example, if $x = -5$, there is a tailwind of 5 miles per hour, so Francine's effective speed is 20 miles per hour, and she can complete the flight in 3 hours. As the tailwind gets stronger (that is, as we move farther to the left in the x-direction), Francine's flying time continues to decrease, and the graph approaches the x-axis.

□

The vertical dashed line at $x = 15$ on the graph of $t = \dfrac{60}{15 - x}$ is a **vertical asymptote** for the graph. We first encountered asymptotes in Section 2.2, p. 170 when we studied the graph of $y = \dfrac{1}{x}$. Locating the vertical asymptotes of a rational function is an important part of determining the shape of the graph.

Checkpoint 7.4.2 Queueing theory is used to predict your waiting time in a line, or queue. For example, suppose the attendant at a toll booth can process 6 vehicles per minute. The average total time spent by a motorist negotiating the toll booth depends on the rate, r, at which vehicles arrive, according to the formula

$$T = g(r) = \frac{12 - r}{12(6 - r)}$$

a What is the average time spent at the toll booth if vehicles arrive at a rate of 3 vehicles per minute?

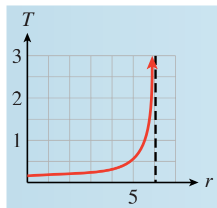
b Graph the function on the domain $[0, 6]$.

c What is the vertical asymptote of the graph? What does it tell you about the queue?

Answer.

a 0.25 min

b



c $r = 6$. The wait time becomes infinite as the arrival rate approaches 6 vehicles per minute.

Example 7.4.3 EarthCare decides to sell T-shirts to raise money. The company makes an initial investment of \$100 to pay for the design of the T-shirt and to set up the printing process. After that, the T-shirts cost \$5 each for labor and materials.

a Express the average cost per T-shirt as a function of the number of T-shirts EarthCare produces.

b Make a table of values for the function.

c Graph the function and explain what it tells you about the cost of the T-shirts.

Solution.

a If EarthCare produces x T-shirts, the total costs will be $100 + 5x$ dollars. To find the average cost per T-shirt, we divide the total cost by the number of T-shirts produced, to get

$$C = g(x) = \frac{100 + 5x}{x}$$

b We evaluate the function for several values of x, as shown in the table

x	1	2	4	5	10	20
C	105	55	40	25	15	10

If EarthCare makes only one T-shirt, its cost is \$105. But if more than one T-shirt is made, the cost of the original \$100 investment is distributed among them. For example, the average cost per T-shirt for **2** T-shirts is

$$\frac{100 + 5(\mathbf{2})}{\mathbf{2}} = 55$$

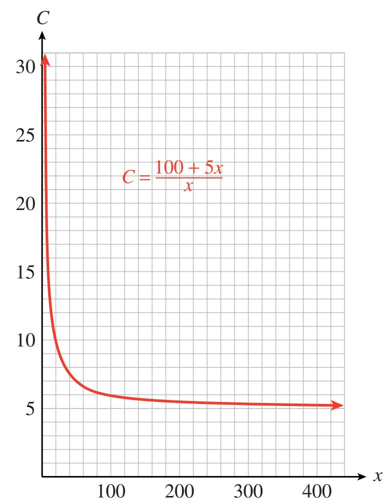
and the average cost for **5** T-shirts is

$$\frac{100 + 5(\mathbf{5})}{\mathbf{5}} = 25$$

c The graph is shown below. You can use your calculator with the window

$$\text{Xmin} = 0 \qquad \text{Xmax} = 470$$
$$\text{Ymin} = 0 \qquad \text{Ymax} = 30$$

to verify the graph. Use the *Trace* to locate on the graph several points from the table of values. For example, the point $(5, 25)$ indicates that if EarthCare makes 5 T-shirts, the cost per shirt is \$25.



The graph shows that as the number of T-shirts increases, the average cost per shirt continues to decrease, but not as rapidly as at first. Eventually the average cost levels off and approaches \$5 per T-shirt. For example, if EarthCare produces **400** T-shirts, the average cost per shirt is

$$\frac{100 + 5(\mathbf{400})}{\mathbf{400}} = 5.25$$

□

The horizontal line $C = 5$ on the graph of $C = \dfrac{100 + 5x}{x}$ is a **horizontal asymptote**. As x increases, the graph approaches the line $C = 5$ but never actually meets it. The average price per T-shirt will always be slightly more than \$5. Horizontal asymptotes are also important in sketching the graphs of rational functions.

Checkpoint 7.4.4 Delbert prepares a 20% glucose solution of by mixing 2 mL of glucose with 8 mL of water. If he adds x ml of glucose to the solution, its concentration is given by

$$C(x) = \frac{2 + x}{8 + x}$$

a How many milliliters of glucose should Delbert add to increase the concentration to 50%?

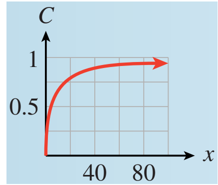
b Graph the function on the domain $[0, 100]$.

c What is the horizontal asymptote of the graph? What does it tell you about the solution?

Answer.

a 4 ml

b



c $C = 1$. As Delbert adds more glucose to the mixture, its concentration increases toward 100%.

7.4.2 Domain of a Rational Function

Most applications of rational functions have restricted domains, that is, they make sense for only a subset of the real numbers on the x-axis. Consequently, only a portion of the graph is useful for analyzing the application. However, a knowledge of the general shape and properties of the whole graph can be very helpful in understanding a rational function.

As we stated earlier, a rational function is a quotient of two polynomials. Some examples of rational functions are shown below.

$$f(x) = \frac{2}{(x-3)^2} \qquad g(x) = \frac{x}{x+1}$$

$$h(x) = \frac{2x^2}{x^2+4} \qquad k(x) = \frac{x^2-1}{x^2-9}$$

Because we cannot divide by zero, a rational function $f(x) = \dfrac{P(x)}{Q(x)}$ is undefined for any value $x = a$ where $Q(a) = 0$. These x-values are not in the domain of the function.

Example 7.4.5 Find the domains of the rational functions f, g, h, and k defined above.

Solution. The domain of f is the set of all real numbers except 3, because the denominator, $(x-3)^2$, equals 0 when $x = 3$.

The domain of g is the set of all real numbers except -1, because $x + 1$ equals zero when $x = -1$.

The denominator of the function h, $x^2 + 4$, is never equal to zero, so the domain of h is all the real numbers.

The domain of k is the set of all real numbers except 3 and -3, because $x^2 - 9$ equals 0 when $x = 3$ or $x = -3$. □

Note 7.4.6 We only need to exclude the zeros of the *denominator* from the domain of a rational function. We do not exclude the zeros of the numerator. In fact, the zeros of the numerator include the zeros of the rational function itself, because a fraction is equal to 0 when its numerator is 0 but its denominator is not 0.

Checkpoint 7.4.7

a Find the domain of $F(x) = \dfrac{x-2}{x+4}$.

b Find the zeros of $F(x)$.

Answer.

a $x \neq -4$

b $x = 2$

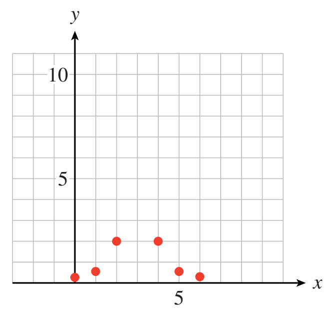
7.4.3 Vertical Asymptotes

As we saw in Section 7.2, p. 715, a polynomial function is defined for all values of x, and its graph is a smooth curve without any breaks or holes. The graph of a rational function, on the other hand, will have breaks or holes at those x-values where it is undefined.

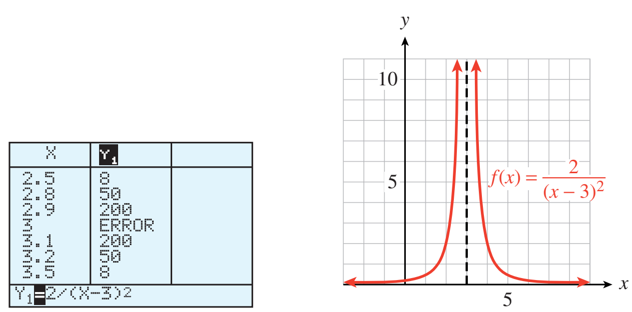
Example 7.4.8 Investigate the graph of $f(x) = \dfrac{2}{(x-3)^2}$ near $x = 3$.

Solution. This function is undefined for $x = 3$, so there is no point on the graph with x-coordinate 3. However, we can make a table of values for other values of x. Plotting the ordered pairs in the table results in the points shown below>.

x	y
0	$\frac{2}{9}$
1	$\frac{1}{2}$
2	2
3	undefined
4	2
4	$\frac{1}{2}$
6	$\frac{2}{9}$



Next, we make a table showing x-values close to 3, as shown below. As we choose x-values closer and closer to 3, $(x-3)^2$ gets closer to 0, so the fraction $\dfrac{2}{(x-3)^2}$ gets very large. This means that the graph approaches, but never touches, the vertical line $x = 3$. In other words, the graph has a vertical asymptote at $x = 3$. We indicate the vertical asymptote by a dashed line, as shown in the figure.



(a) **(b)**

□

In general, we have the following result.

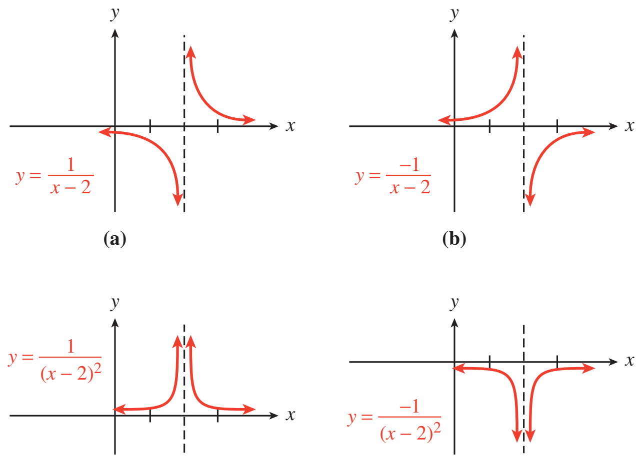
> **Vertical Asymptotes.**
>
> If $Q(a) = 0$ but $P(a) \neq 0$, then the graph of the rational function $f(x) = \dfrac{P(x)}{Q(x)}$ has a **vertical asymptote** at $x = a$.

Note 7.4.9 If $P(a)$ and $Q(a)$ are both zero, then the graph of the rational function $\dfrac{P(x)}{Q(x)}$ may have a hole at $x = a$ rather than an asymptote. (This possibility is considered in the homework exercises.)

Checkpoint 7.4.10 Find the vertical asymptotes of $G(x) = \dfrac{4x^2}{x^2 - 4}$.

Answer. $x = -2$ and $x = 2$

Near a vertical asymptote, the graph of a rational function has one of the four characteristic shapes, illustrated below. Locating the vertical asymptotes can help us make a quick sketch of a rational function.

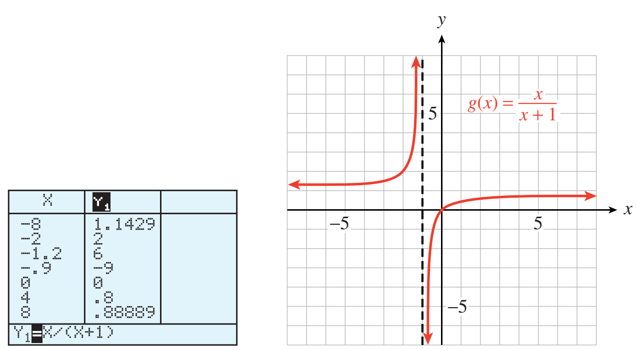


(a) (b) (c) (d)

Example 7.4.11 Locate the vertical asymptotes and sketch the graph of $g(x) = \dfrac{x}{x+1}$.

Solution. The denominator, $x + 1$, equals zero when $x = -1$. Because the numerator does not equal zero when $x = -1$, there is a vertical asymptote at $x = -1$. The asymptote separates the graph into two pieces.

We can use the **Table** feature of a calculator to evaluate $g(x)$ for several values of x on either side of the asymptote, as shown in figure (a). We plot the points found in this way; then connect the points on either side of the asymptote to obtain the graph shown in figure (b).



(a) (b)

□

Checkpoint 7.4.12

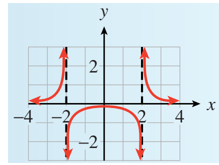
a Find the vertical asymptotes of $f(x) = \dfrac{1}{x^2 - 4}$. Locate any x-intercepts.

b Evaluate the function at $x = -3$, -1, 1, and 3. Sketch a graph of the function.

Answer.

a $x = -2$ and $x = 2$, no x-intercepts b

x	-3	-1	1	3
y	$\frac{1}{5}$	$\frac{-1}{3}$	$\frac{-1}{3}$	$\frac{1}{5}$

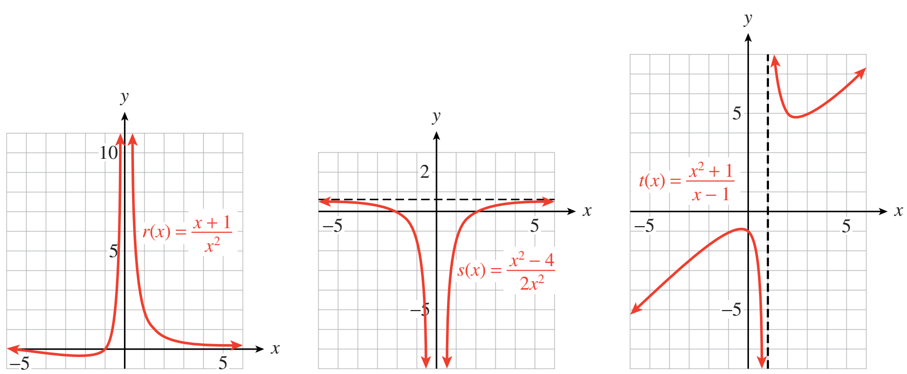


7.4.4 Horizontal Asymptotes

Look again at the graph of $g(x) = \dfrac{x}{x+1}$ in Example 7.4.11, p. 755. As $|x|$ gets large -- that is, as we move away from the origin along the x-axis in either direction -- the corresponding y-values get closer and closer to 1. The graph approaches, but never coincides with, the line $y = 1$. We say that the graph has a **horizontal asymptote** at $y = 1$.

When does a rational function $f(x) = \dfrac{P(x)}{Q(x)}$ have a horizontal asymptote? It depends on the degrees of the two polynomials $P(x)$ and $Q(x)$. The degree of the numerator of $g(x) = \dfrac{x}{x+1}$ is equal to the degree of the denominator. In other words, the highest power of x in the numerator (1, in this case) is the same as the highest power in the denominator.

Consider the three rational functions whose graphs are shown below.



- The graph of $r(x) = \dfrac{x+1}{x^2}$ in figure (a) has a horizontal asymptote at $y = 0$, the x-axis, because the degree of the denominator is larger than the degree of the numerator. Higher powers of x grow much more rapidly than smaller powers. Thus, for large values of $|x|$, the denominator is much larger in absolute value than the numerator of $r(x)$, so the function values approach 0.

- The graph of $s(x) = \dfrac{x^2-4}{2x^2}$ in figure (b) has a horizontal asymptote at $y = \dfrac{1}{2}$, because the numerator and denominator of the fraction have the same degree. For large values of $|x|$, the terms of lower degree are negligible compared to the squared terms. As x increases, $s(x)$ is approximately equal to $\dfrac{x^2}{2x^2}$, or $\dfrac{1}{2}$. Thus, the function values approach a constant value of $\dfrac{1}{2}$.

- The graph of $t(x) = \dfrac{x^2+1}{x-1}$ in figure (c) does not have a horizontal asymptote, because the degree of the numerator is larger than the degree of the denominator. As $|x|$ increases, x^2+1 grows much faster than $x-1$, so their ratio does not approach a constant value. The function values increase without bound.

We summarize our discussion as follows.

Horizontal Asymptotes.

Suppose $f(x) = \dfrac{P(x)}{Q(x)}$ is a rational function, where the degree of $P(x)$ is m and the degree of $Q(x)$ is n.

1 If $m < n$, the graph of f has a horizontal asymptote at $y = 0$.

2 If $m = n$, the graph of f has a horizontal asymptote at $y = \dfrac{a}{b}$, where a is the lead coefficient of $P(x)$ and b is the lead coefficient of $Q(x)$.

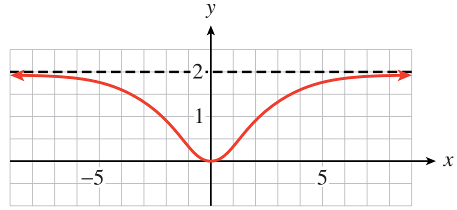
3 If $m > n$, the graph of f does not have a horizontal asymptote.

Example 7.4.13 Locate the horizontal asymptotes and sketch the graph of $h(x) = \dfrac{2x^2}{x^2+4}$.

Solution. The numerator and denominator of the fraction are both second-degree polynomials, so the graph does have a horizontal asymptote. The lead coefficients of $P(x)$ and $Q(x)$ are 2 and 1, respectively, so the horizontal asymptote is $y = \dfrac{2}{1}$, or $y = 2$.

The function h does not have a vertical asymptote because the denominator, $x^2 + 4$, is never equal to zero.

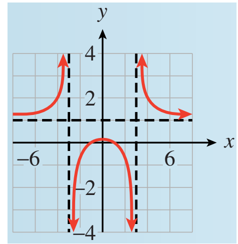
The y-intercept of the graph is the point $(0, 0)$. We can plot several points by evaluating the function at convenient x-values, and use the asymptote to help us sketch the graph, as shown below.



□

Checkpoint 7.4.14 Locate the horizontal and vertical asymptotes and sketch the graph of $k(x) = \dfrac{x^2-1}{x^2-9}$. Label the x- and y-intercepts with their coordinates.

Answer. $y = 1$; $x = -3$, $x = 3$



7.4.5 Applications

It is often useful to simplify the formula for a rational function before using it. (See Algebra Skills Refresher Section A.9, p. 918 to review operations on algebraic fractions.)

Example 7.4.15 When estimating their travel time, pilots must take into account the prevailing winds. A tailwind adds to the plane's ground speed, while a headwind decreases the ground speed. Skyhigh Airlines is setting up a shuttle service from Dallas to Phoenix, a distance of 800 miles.

a Express the time needed for a one-way trip, without wind, as a function of the speed of the plane.

b Suppose there is a prevailing wind of 30 miles per hour blowing from the west. Write expressions for the flying time from Dallas to Phoenix and from Phoenix to Dallas.

c Write an expression for the round-trip flying time, excluding stops, with a 30-mile-per-hour wind from the west, as a function of the plane's speed. Simplify your expression.

Solution.

a Recall that time $= \dfrac{\text{distance}}{\text{rate}}$. If we let r represent the speed of the plane in still air, then the time required for a one-way trip is

$$f(r) = \frac{800}{r}$$

b On the trip from Dallas to Phoenix, the plane encounters a headwind of 30 miles per hour, so its actual ground speed is $r - 30$. On the return trip, the plane enjoys a tailwind of 30 miles per hour, so its actual ground speed is $r + 30$. Therefore, the flying times are

$$\text{Dallas to Phoenix:} \qquad \frac{800}{r-30}$$

and

$$\text{Phoenix to Dallas:} \qquad \frac{800}{r+30}$$

c The round-trip flying time from Dallas to Phoenix and back is

$$F(r) = \frac{800}{r-30} + \frac{800}{r+30}$$

The LCD for these fractions is $(r - 30)(r + 30)$. Thus,

$$\frac{800}{r-30} + \frac{800}{r+30} = \frac{800\boldsymbol{(r+30)}}{(r-30)\boldsymbol{(r+30)}} + \frac{800\boldsymbol{(r-30)}}{(r+30)\boldsymbol{(r-30)}}$$

$$= \frac{(800r + 24000) + (800r - 24000)}{(r+30)(r-30)}$$
$$= \frac{1600r}{r^2 - 900}$$

(See Algebra Skills Refresher Section A.9, p. 918 to review adding fractions.)

□

Checkpoint 7.4.16 Navid took his outboard motorboat 20 miles upstream to a fishing site, returning downstream later that day. His boat travels 10 miles per hour in still water. Write an expression for the time Navid spent traveling, as a function of the speed of the current.

Answer. $\dfrac{400}{100 - x^2}$ hrs

7.4.6 Section Summary

7.4.6.1 Vocabulary

Look up the definitions of new terms in the Glossary.

- Rational function
- Vertical asymptote
- Horizontal asymptote

7.4.6.2 CONCEPTS

1 **Rational Function.**

A **rational function** is one of the form

$$f(x) = \frac{P(x)}{Q(x)}$$

where $P(x)$ and $Q(x)$ are polynomials and $Q(x)$ is not the zero polynomial.

2 A rational function $f(x) = \dfrac{P(x)}{Q(x)}$ is undefined for any value $x = 1$ where $Q(a) = 0$. These x-values are not in the domain of the function.

3 **Vertical Asymptotes.**

If $Q(a) = 0$ but $P(a) \neq 0$, then the graph of the rational function $f(x) = \dfrac{P(x)}{Q(x)}$ has a **vertical asymptote** at $x = a$.

4 **Horizontal Asymptotes.**

Suppose $f(x) = \dfrac{P(x)}{Q(x)}$ is a rational function, where the degree of $P(x)$ is m and the degree of $Q(x)$ is n.

1 If $m < n$, the graph of f has a horizontal asymptote at $y = 0$.

2 If $m = n$, the graph of f has a horizontal asymptote at

$y = \dfrac{a}{b}$, where a is the lead coefficient of $P(x)$ and b is the lead coefficient of $Q(x)$.

3 If $m > n$, the graph of f does not have a horizontal asymptote.

7.4.6.3 STUDY QUESTIONS

1 Why does the word **rational** refer to a quotient?

2 How are the graphs of rational functions different from the graphs of polynomials?

3 What do the zeros of the numerator of a rational function tell you? What about the zeros of the denominator?

4 Under what circumstances can the graph of a rational function have a horizontal asymptote?

7.4.6.4 SKILLS

Practice each skill in the Homework 7.4.7, p. 760 problems listed.

1 Find the vertical asymptotes of a rational function: #13–32

2 Find the horizontal asymptotes of a rational function: #13–32

3 Interpret the significance of horizontal and vertical asymptotes in context: #1–10

4 Sketch the graph of a rational function: #13–36, 51–54

5 Write a rational function to model a situation: #37–42

7.4.7 Graphing Rational Functions (Homework 7.4)

1. The eider duck, one of the world's fastest flying birds, can exceed an airspeed of 65 miles per hour. A flock of eider ducks is migrating south at an average airspeed of 50 miles per hour against a moderate headwind. Their next feeding grounds are 150 miles away.

(a) Express the ducks' travel time, t, as a function of the windspeed, v.

(b) Complete the table showing the travel time for various windspeeds.

v	0	5	10	15	20	25	30	35	40	45	50
t											

What happens to the travel time as the headwind increases?

(c) Use the table to choose an appropriate window and graph your function $t(v)$. Give the equations of any horizontal or vertical asymptotes. What does the vertical asymptote signify in the context of the problem?

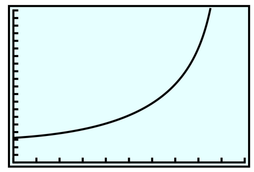
Answer.

(a) $t = \dfrac{150}{50 - v}$

(b)

v	0	5	10	15	20	25	30	35	40	45	50
t	3	3.33	3.75	4.29	5	6	7.5	10	15	30	—

The travel time increases as the headwind speed increases.

(c) 

2. The fastest fish in the sea may be the bluefin tuna, which has been clocked at 43 miles per hour in short sprints. A school of tuna is migrating a distance of 200 miles at an average speed of 36 miles per hour in still water, but they have run into a current flowing against their direction of travel.

(a) Express the tuna's travel time, t, as a function of the current speed, v.

(b) Complete the table showing the travel time for various current speeds.

v	0	4	8	12	16	20	24	28	32	36
t										

What happens to the travel time as the current increases?

(c) Use the table to choose an appropriate domain and range for your function $t(v)$. Give the equations of any horizontal or vertical asymptotes. What does the vertical asymptote signify in the context of the problem?

3. The cost, in thousands of dollars, for immunizing p percent of the residents of Emporia against a dangerous new disease is given by the function

$$C(p) = \frac{72p}{100 - p}$$

(a) What is the domain of C?

(b) Complete the table showing the cost of immunizing various percentages of the population.

p	0	15	25	40	50	75	80	90	100
C									

(c) Graph the function C. (Use Xmin $= 6$, Xmax $= 100$, and appropriate values of **Ymin** and **Ymax**.) What percentage of the population can be immunized if the city is able to spend $\$108,000$?

(d) For what values of p is the total cost more than $\$1,728,000$?

(e) The graph has a vertical asymptote. What is it? What is its significance in the context of this problem?

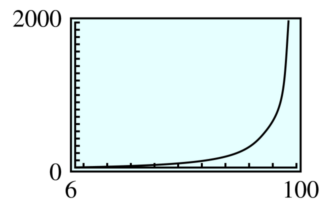
Answer.

(a) $0 \le p < 100$

(b)

p	0	15	25	40	50	75	80	90	100
C	0	12.7	24	48	72	216	288	648	—

(c) 60%



(d) $p > 96\%$

(e) $p = 100$; As the percentage immunized approaches 100, the cost grows without bound.

4. The cost, in thousands of dollars, for immunizing p percent of a precious ore from a mine is given by the equation

$$C(p) = \frac{360p}{100 - p}$$

(a) What is the domain of C?

(b) Complete the table showing the cost of extracting various percentages of the ore.

p	0	15	25	40	50	75	80	90	100
C									

(c) Graph the function C. (Use Xmin = 6, Xmax = 100, and appropriate values of **Ymin** and **Ymax**.) What percentage of the ore can be extracted if \$540,000 can be spent on the extraction?

(d) For what values of p is the total cost less than \$1,440,000?

(e) The graph has a vertical asymptote. What is it? What is its significance in the context of this problem?

5. The total cost in dollars of producing n calculators is approximately $20,000 + 8n$.

(a) Express the cost per calculator, C, as a function of the number n of calculators produced.

(b) Complete the table showing the cost per calculator for various production levels.

n	100	200	400	500	1000	2000	4000	5000	8000
C									

(c) Graph the function $C(n)$ for the cost per calculator. Use the window

$$\text{Xmin} = 0 \qquad \text{Xmax} = 9400$$
$$\text{Ymin} = 0 \qquad \text{Ymax} = 50$$

(d) How many calculators should be produced so that the cost per calculator is \$18?

(e) For what values of n is the cost less than \$12 per calculator?

(f) Find the horizontal asymptote of the graph. What does it represent in this context?

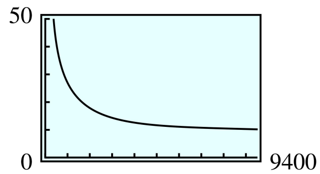
Answer.

(a) $C = 8 + \dfrac{20,000}{n}$

(b)

n	100	200	400	500	1000	2000	4000	5000	8000
C	208	108	58	48	28	18	13	12	10.5

(c)



(d) 2000

(e) $n > 5000$

(f) $C = 8$; As n increases, the average cost per calculator approaches \$8.

6. The number of loaves of Mom's Bread sold each day is approximated by the demand function

$$D(p) = \frac{100}{1 + (p - 1.10)^4}$$

where p is the price per loaf in dollars.

(a) Complete the table showing the demand for Mom's Bread at various prices per loaf. Round the values of $D(p)$ to the nearest whole number.

p	0.25	0.50	1.00	1.25	1.50	1.75	2.00	2.25	2.50	2.75	3.00
Demand											

(b) Graph the demand function $C(n)$ in the window

$$\text{Xmin} = 0 \qquad \text{Xmax} = 3.74$$
$$\text{Ymin} = 0 \qquad \text{Ymax} = 170$$

What happens to the demand for Mom's Bread as the price increases?

(c) Add a row to your table to show the daily revenue from Mom's Bread at various prices.

p	0.25	0.50	1.00	1.25	1.50	1.75	2.00	2.25	2.50	2.75	3.00
Demand											
Revenue											

(d) Using the formula for $D(p)$, write an expression $R(p)$ that approximates the total daily revenue as a function of the price, p.

(e) Graph the revenue function $R(p)$ in the same window with $D(p)$. Estimate the maximum possible revenue. Does the maximum for $D(p)$ occur at the same value of p as the maximum for $R(p)$?

(f) Find the horizontal asymptote of the graphs. What does it represent in this context?

7. A computer store sells approximately 300 of its most popular model per year. The manager would like to minimize her annual inventory cost by ordering the optimal number of computers, x, at regular intervals. If she orders x computers in each shipment, the cost of storage will be $6x$ dollars, and the cost of reordering will be $\dfrac{300}{x}(15x+10)$ dollars. The inventory cost is the sum of the storage cost and the reordering cost.

(a) Use the distributive law to simplify the expression for the reordering cost. Then express the inventory cost, C, as a function of x.

(b) Complete the table of values for the inventory cost for various reorder sizes.

x	10	20	30	40	50	60	70	80	90	100
C										

(c) Graph the function C for the cost per calculator. Use the window

$$\text{Xmin} = 0 \qquad \text{Xmax} = 150$$
$$\text{Ymin} = 4500 \qquad \text{Ymax} = 5500$$

Estimate the minimum possible value for C.

(d) How many computers should the manager order in each shipment so as to minimize the inventory cost? How many orders will she make during the year?

(e) Graph the function $y = 6x + 4500$ in the same window with the function C. What do you observe?

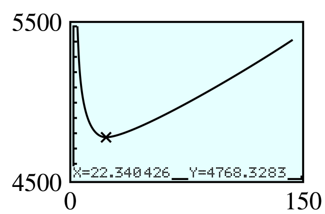
Answer.

(a) $4500 + \dfrac{3000}{x}$; $C(x) = 6x + 4500 + \dfrac{3000}{x}$

(b)

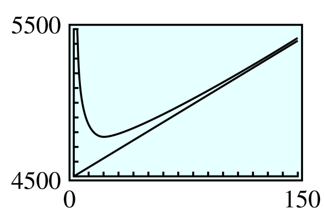
x	10	20	30	40	50	60	70	80	90	100
C	4860	4770	4780	4815	4860	4910	5018	5073	5130	

(c) \$4768.33



(d) 22; 14

(e)



The graph of C approaches the line as an asymptote.

8. A chain of electronics stores sells approximately 500 portable phones every year. The owner would like to minimize his annual inventory cost by ordering the optimal number of phones, x, at regular intervals. The cost of storing the phones will then be $2x$ dollars, and the cost of reordering will be $\dfrac{500}{x}(4x+10)$. The total annual inventory cost is the sum of the storage cost and the reordering cost.

(a) Use the distributive law to simplify the expression for the reordering cost. Then express the inventory cost, C, as a function of x.

(b) Complete the table of values for the inventory cost for various reorder sizes.

x	10	20	30	40	50	60	70	80	90	100
C										

(c) Graph the function C in the window

$$\text{Xmin} = 0 \qquad \text{Xmax} = 150$$
$$\text{Ymin} = 2000 \qquad \text{Ymax} = 2500$$

Estimate the minimum possible value for C.

(d) How many portable phones should the manager order in each shipment so as to minimize the inventory cost? How many orders will he make during the year?

(e) Graph the function $y = 2x + 2000$ in the same window with the function C. What do you observe?

9. Francine wants to make a rectangular box. In order to simplify construction and keep her costs down, she plans for the box to have a square base and a total surface area of 96 square centimeters. She would like to know the largest volume that such a box can have.

(a) If the square base has length x centimeters, show that the height of the box is $h = \dfrac{24}{x} - \dfrac{x}{2}$ centimeters. (*Hint*: The surface area of the box is the sum of the areas of the six sides of the box.)

(b) Write an expression for the volume, V, of the box as a function of the length, x, of its base.

(c) Complete the table showing the heights and volumes of the box for various base lengths.

x	1	2	3	4	5	6	7
h							
V							

Explain why the values of h and V are negative when $x = 7$.

(d) Graph your expression for volume $V(x)$ in an appropriate window. Approximate the maximum possible volume for a box of surface area 96 square centimeters.

(e) What value of x gives the maximum volume?

(f) Graph the height, $h(x)$, in the same window with $V(x)$. What is the height of the box with greatest volume? (Find the height directly from your graph and verify by using the formula given for $h(x)$.)

Answer.

(a) The surface area is $2x^2 + 4xh = 96$. Solving for h, $h = \dfrac{96 - 2x^2}{4x} = \dfrac{24}{x} - \dfrac{x}{2}$.

(b) $V = 24x - \dfrac{1}{2}x^3$

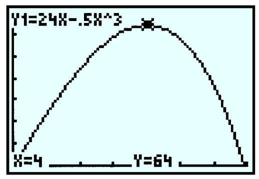
(c)

x	1	2	3	4	5	6	7
h	23.5	11	6.5	4	2.3	1	−0.07
V	23.5	44	58.5	64	57.5	36	−3.5

If the base is more than 7 cm, the top and bottom alone exceed the total area allowed.

(d)

70



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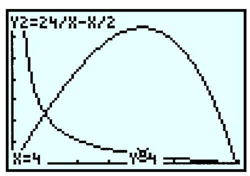
Maximum of 64 cu. cm

(e) 4 cm

(f) $h = 4$ cm

700



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10. Delbert wants to make a box with a square base and a volume of 64 cubic centimeters. He would like to know the smallest surface area that such a box can have.

(a) If the square base has length x centimeters, show that the height of the box is $h = \dfrac{64}{x^2}$ centimeters.

(b) Write an expression for the surfacd area, S, of the box as a function of the length, x, of its base. (*Hint*: The surface area of the box is the sum of the areas of the six sides of the box.)

(c) Complete the table showing the heights and surface areas of the box for various base lengths.

x	1	2	3	4	5	6	7	8
h								
S								

(d) Graph your expression for surface area $S(x)$ in an appropriate window. Approximate the minimum possible surface area for Delbert's box.

(e) What value of x gives the minimum surface area?

(f) Graph the height, $h(x)$, in the same window with $S(x)$. What is the height of the box with smallest surface area? (Find the height directly from your graph and verify by using the formula given for $h(x)$.)

11. A train whistle sounds higher when the train is approaching you than when it is moving away from you. This phenomenon is known as the Doppler effect. If the actual pitch of the whistle is 440 hertz (this is the A note below middle C), then the note you hear will have the pitch

$$P(v) = \frac{440(332)}{332 - v}$$

where the velocity, v, in meters per second is positive as the train approaches and is negative when the train is moving away. (The number 332 that appears in this expression is the speed of sound in meters per second.)

(a) Complete the table of values showing the pitch of the whistle at various train velocities.

v	-100	-75	-50	-25	0	25	50	75	100
P									

(b) Graph the function P. (Use the window Xmin $= -94$, Xmax $= 94$, and appropriate values of **Ymin** and **Ymax**.)

(c) What is the velocity of the train if the note you hear has a pitch of 415 hertz (corresponding to the note A-flat)? A pitch of $553.\overline{3}$ hertz (C-sharp)?

(d) For what velocities will the pitch you hear be greater than 456.5 hertz?

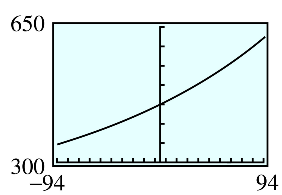
(e) The graph has a vertical asymptote (although it is not visible in the suggested window). Where is it and what is its significance in this context?

Answer.

(a)

v	-100	-75	-50	-25	0	25	50	75	100
P	338.15	358.92	382.41	409.19	440	475.83	518.01	568.4	629.66

(b)



(c) -20 m/sec; 68 m/sec

(d) $v > 12$ m/sec

(e) $v = 332$; As v approaches 332 m per sec, the pitch increases without bound.

12. The maximum altitude (in meters) attained by a projectile shot from the surface of the Earth is

$$h(v) = \frac{6.4 \times 10^6 v^2}{19.6 \cdot 6.4 \times 10^6 - v^2}$$

where v is the speed (in meters per second) at which the projectile was launched. (The radius of the Earth is 6.4×10^6 meters, and the constant

19.6 is related to the Earth's gravitational constant.)

(a) Complete the table of values showing the maximum altitude for various launch velocities.

v	100	200	300	400	500	600	700	800	900	1000
h										

(b) Graph the function h. (Use the window Xmin = 0, Xmax = 940, and appropriate values of **Ymin** and **Ymax**.)

(c) Approximately what speed is needed to attain an altitude of 4000 meters? An altitude of 16 kilometers?

(d) For what velocities will the projectile attain an altitude exceeding 32 kilometers?

(e) The graph has a vertical asymptote (although it is not visible in the suggested window). Where is it and what is its significance in this context?

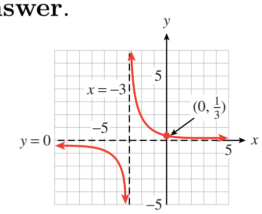
For Problems 13–30,

a Find the horizontal and vertical asymptotes for each function.

b Find the x- and y-intercepts for each function.

13. $y = \dfrac{1}{x+3}$

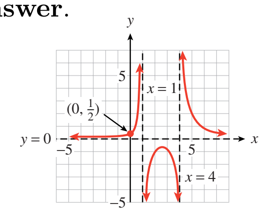
Answer.



14. $y = \dfrac{1}{x-3}$

15. $y = \dfrac{2}{x^2-5x+4}$

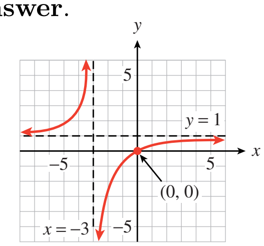
Answer.



16. $y = \dfrac{4}{x^2-x-6}$

17. $y = \dfrac{x}{x-3}$

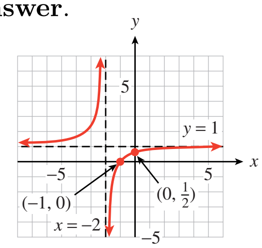
Answer.



18. $y = \dfrac{x}{x-2}$

19. $y = \dfrac{x+1}{x+2}$

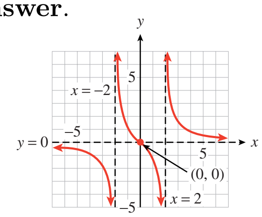
Answer.



20. $y = \dfrac{x-1}{x-3}$

21. $y = \dfrac{2x}{x^2-4}$

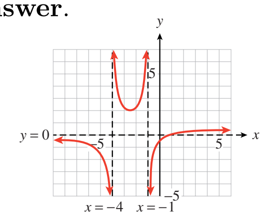
Answer.



22. $y = \dfrac{x}{x^2-9}$

23. $y = \dfrac{x-2}{x^2+5x+4}$

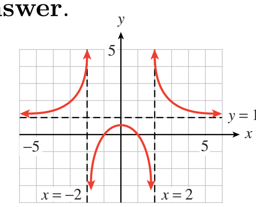
Answer.



24. $y = \dfrac{x+1}{x^2-x-6}$

25. $y = \dfrac{x^2-1}{x^2-4}$

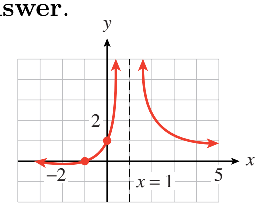
Answer.



26. $y = \dfrac{2x^2}{x^3-1}$

27. $y = \dfrac{x+1}{(x-1)^2}$

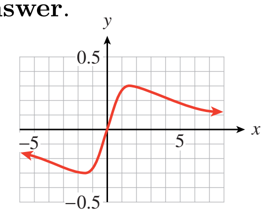
Answer.



28. $y = \dfrac{2(x^2-1)}{x^2+4}$

29. $y = \dfrac{x}{x^2+3}$

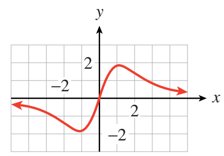
Answer.



30. $y = \dfrac{x^2+2}{x^2+4}$

31. Graph the curve known as Newton's Serpentine: $y = \dfrac{4x}{x^2+1}$.

Answer.



32. Graph the curve known as the Witch of Agnesi: $y = \dfrac{8}{x^2+4}$.

For Problems 33–38,

a Use polynomial division to write the fraction in the form $y = \dfrac{k}{p(x)} + c$, where k and c are constants.

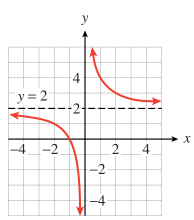
b Use transformations to sketch the graph.

33. $y = \dfrac{2x+2}{x}$

Answer.

a $y = \dfrac{2}{x} + 2$

b



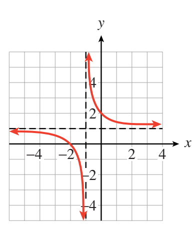
34. $y = \dfrac{4x^2+3}{x^2}$

35. $y = \dfrac{x+2}{x+1}$

Answer.

a $y = \dfrac{1}{x+1} + 1$

b



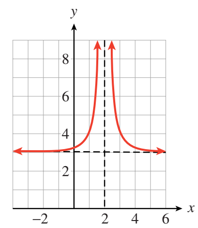
36. $y = \dfrac{7-2x}{x-3}$

37. $y = \dfrac{3x^2 - 12x + 13}{(x-2)^2}$

38. $y = \dfrac{-4x^2 + 8x - 3}{(x-1)^2}$

Answer.

a $y = \dfrac{1}{(x-2)^2} + 3$

b 

Problems 39–45 involve operations on algebraci fractions. To review operations on algebraic fractions, see Algebra Skills Refresher Section A.9, p. 918.

39. River Queen Tours offers a 50-mile round-trip excursion on the Mississippi River on a paddle wheel boat. The current in the Mississippi is 8 miles per hour.

(a) Express the time required for the downstream journey as a function of the speed of the paddle wheel boat in still water.

(b) Write a function for the time required for the return trip upstream.

(c) Write and simplify an expression for the time needed for the round trip as a function of the boat's speed.

Answer.

(a) $\dfrac{25}{s+8}$ (b) $\dfrac{25}{s-8}$ (c) $\dfrac{50s}{s^2-64}$

40. A rowing team can maintain a speed of 15 miles per hour in still water. The team's daily training session includes a 5-mile run up the Red Cedar River and the return downstream.

(a) Express the team's time on the upstream leg as a function of the speed of the current.

(b) Write a function for the team's time on the downstream leg.

(c) Write and simplify an expression for the total time for the training run as a function of the current's speed.

41. Two pilots for the Flying Express parcel service receive packages simultaneously. Orville leaves Boston for Chicago at the same time Wilbur leaves Chicago for Boston. Each selects an airspeed of 400 miles per hour for the 900-mile trip. The prevailing winds blow from east to west.

(a) Express Orville's flying time as a function of the windspeed.

(b) Write a function for Wilbur's flying time.

(c) Who reaches his destination first? By how much time (in terms of windspeed)?

Answer.

(a) $\dfrac{900}{400+w}$

(b) $\dfrac{900}{400-w}$

(c) Orville by $\dfrac{1800w}{160,000-w^2}$ hours

42. On New Year's Day, a blimp leaves its berth in Carson, California, and heads north for the Rose Bowl, 23 miles away. There is a breeze from the north at 6 miles per hour.

(a) Express the time required for the trip as a function of the blimp's airspeed.

(b) Write a function for the time needed for the return trip.

(c) Which trip takes longer? By how much time (in terms of the blimp's airspeed)?

43. The focal length of a lens is given by the formula

$$\frac{1}{f} = \frac{1}{p} + \frac{1}{q}$$

where f stands for the focal length, p is the distance from the object viewed to the lens, and q is the distance from the image to the lens. Suppose you estimate that the distance from your cat (the object viewed) to your camera lens is 60 inches greater than the distance from the lens to the film inside the camera, where the image forms.

(a) Express $1/f$ as a single fraction in terms of q.

(b) Write an expression for f as a function of q.

Answer.

(a) $\dfrac{1}{f} = \dfrac{2q+60}{q^2+60q}$ (b) $f = \dfrac{q^2+60q}{2q+60}$

44. If two resistors, R_1 and R_2, in an electrical circuit are connected in parallel, the total resistance R in the circuit is given by

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}$$

(a) Suppose that the second resistor, R_2, is 10 ohms greater than the first. Express $1/R$ as a single fraction in terms of R_1.

(b) Write an expression for R as a function of R_1.

45.

(a) Show that the equation $\dfrac{1}{y} - \dfrac{1}{x} = \dfrac{1}{k}$ is equivalent to $y = \dfrac{kx}{x+k}$ on their common domain.

(b) Graph the functions $y = \dfrac{kx}{x+k}$ for $k = 1, 2,$ and 3 in the window

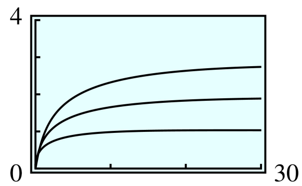
Xmin = 0	Xmax = 30
Ymin = 0	Ymax = 4

Describe the graphs.

Answer.

(a) $\dfrac{1}{y} = \dfrac{1}{x} + \dfrac{1}{k} = \dfrac{k+x}{xk}$, so by taking reciprocals, $y = \dfrac{kx}{x+k}$.

(b)



The graphs increase from the origin and approach a horizontal asymptote at $y = k$.

46. Consider the graph of $y = \dfrac{ax}{x+k}$, where a and k are positive constants.

(a) What is the horizontal asymptote of the graph?

(b) Show that for $x = k$, $y = \dfrac{a}{2}$.

(c) Sketch the graph of $y = \dfrac{ax}{x+k}$ for $a = 4$ and $k = 10$ in the window

Xmin = 0	Xmax = 60
Ymin = 0	Ymax = 5

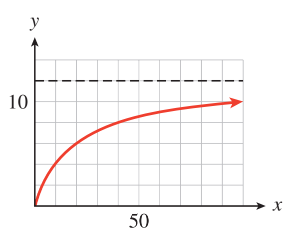
Illustrate your answers to parts (a) and (b) on the graph.

For Problems 47–48,

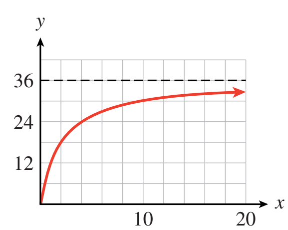
(a) Use your answers to Problem 46 to find equations of the form $y = \dfrac{ax}{x+k}$ for the graphs shown.

(b) Check your answer with a graphing calculator.

47.



48.



Answer. $\dfrac{12x}{x+20}$

49. The Michaelis-Menten equation is the rate equation for chemical reactions catalyzed by enzymes. The speed of the reaction v is a function of the initial concentration of the reactant s and is given by

$$v = f(s) = \frac{Vs}{s+K}$$

where V is the maximum possible reaction rate and K is called the Michaelis constant. (Source: Holme and Peck, 1993)

(a) What value does v approach as s increases?

(b) What is the value of v when $s = K$?

(c) The table gives data from reactions of the enzyme D-amino acid oxidase.

s	0.33	0.66	1.00	1.66	2.50	3.33	6.66
v	0.08	0.14	0.20	0.30	0.39	0.46	0.58

Plot the data and estimate the values of V and K from your graph.

(d) Graph the function $v = \dfrac{0.88s}{s + 3.34}$ on top of your data points.

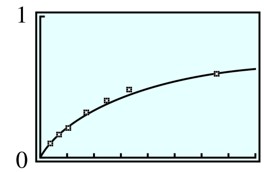
(e) For a fixed s and V, what happens to v if K is very big?

(f) For a fixed K and V, what happens to v if s is very big?

Answer.

(a) V

(b) $\dfrac{V}{2}$

(c) 

$V \approx 0.7$, $K \approx 2.2$ (many answers are possible)

(d) (See figure.)

50. Show that

$$\frac{1}{v} = \frac{1}{V} + \frac{K}{Vs}$$

is another form of the Michaelis-Menten equation. (See Problem 49.)

51.

(a) Refer to the Michaelis-Menten equation in Problem 49. Solve for $\dfrac{1}{v}$, then write your new equation in the form $\dfrac{1}{v} = a \cdot \dfrac{1}{s} + b$. Express a and b in terms of V and K.

(b) Use the data from part (c) of Problem 49 to make a table of values for $\left(\dfrac{1}{s}, \dfrac{1}{v}\right)$.

(c) Plot the points $\left(\dfrac{1}{s}, \dfrac{1}{v}\right)$, then use linear regression to find the line of best fit.

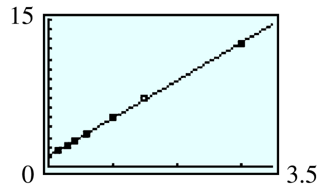
(d) Use your values for a and b to solve for V and K.

Answer.

(a) $\dfrac{1}{v} = \dfrac{K}{V} \cdot \dfrac{1}{s} + \dfrac{1}{V}$; Therefore, $a = \dfrac{K}{V}$ and $b = \dfrac{1}{V}$

(b)

$\frac{1}{s}$	3	1.5	1	0.6	0.4	0.3	0.15
$\frac{1}{v}$	12.5	7.1	5	3.3	2.6	2.2	1.7

(c) 

$$\frac{1}{v} = 3.8 \cdot \frac{1}{s} + 1.1$$

(d) $V \approx 0.89$, $K \approx 3.37$

52.

(a) Refer to the Michaelis-Menten equation in Problem 49. Write an equation for $\frac{s}{v}$ in the form $\frac{s}{v} = cs + d$. Express c and d in terms of V and K.

(b) Use the data from part (c) of Problem 49 to make a table of values for $\left(s, \frac{s}{v}\right)$.

(c) Plot the points $\left(s, \frac{s}{v}\right)$, then use linear regression to find the line of best fit.

(d) Use your values for c and d to solve for V and K.

Problems 53-56 give examples of functions whose graphs have holes.

a Find the domain of the function.

b Reduce the fraction to lowest terms.

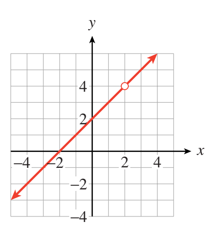
c Graph the function. (*Hint*: The graph of the original function is identical to the graph of the function in part (b) except that certain points are excluded from the domain.) Indicate a hole in the graph by an open circle.

53. $y = \dfrac{x^2 - 4}{x - 2}$

Answer.

(a) $x \neq 2$

(b) $x + 2$

(c) 

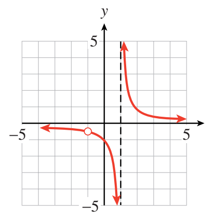
54. $y = \dfrac{x^2 - 1}{x + 1}$

55. $y = \dfrac{x+1}{x^2-1}$

Answer.

(a) $x \neq \pm 1$

(b) $\dfrac{1}{x-1}$

(c) 

56. $y = \dfrac{x-3}{x^2-9}$

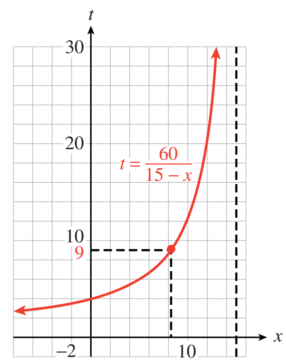
7.5 Equations That Include Algebraic Fractions

When working with rational functions, we often need to solve equations that involve algebraic fractions.

In Example 7.4.1, p. 749 of Section 7.4, p. 749, we wrote a function that gave the time Francine needs for a 60-mile training run on her cycle-plane in terms of the windspeed, x:

$$t = f(x) = \frac{60}{15-x}$$

If it takes Francine 9 hours to cover 60 miles, what is the speed of the wind? We can answer this question by reading values from the graph of f, as shown at right. When $t = 9$, the value of x is between 8 and 9, so the windspeed is between 8 and 9 miles per hour.



7.5.1 Solving Equations with Fractions Algebraically

If we need a more accurate value for the windspeed, we can solve the equation

$$\frac{60}{15-x} = 9$$

To solve an equation involving an algebraic fraction, we multiply each side of the equation by the denominator of the fraction. This has the effect of clearing the fraction, giving us an equivalent equation without fractions.

Example 7.5.1 Solve the equation $\dfrac{60}{15-x} = 9$

Solution. We multiply both sides of the equation by $\mathbf{15 - x}$ to obtain

$$(\mathbf{15-x})\frac{60}{15-x} = 9(\mathbf{15-x})$$

$$60 = 9(15-x) \qquad \text{Apply the distributive law.}$$

From here we can proceed as usual.

$$60 = 135 - 9x \qquad \text{Subtract 135 from both sides.}$$

$$-75 = -9x \qquad \text{Divide by } \mathbf{-9.}$$
$$8.\overline{3} = x$$

The windspeed was $8.\overline{3}$, or $8\frac{1}{3}$ miles per hour. □

Checkpoint 7.5.2 Solve $\dfrac{x^2}{x+4} = 2$

Answer. $x = -2,\ x = 4$

If the equation contains more than one fraction, we can clear all the denominators at once by multiplying both sides by the LCD of the fractions.

Example 7.5.3 Rani times herself as she kayaks 30 miles down the Derwent River with the help of the current. Returning upstream against the current, she manages only 18 miles in the same amount of time. Rani knows that she can kayak at a rate of 12 miles per hour in still water. What is the speed of the current?

Solution. If we let x represent the speed of the current, we can use the formula time $= \dfrac{\text{distance}}{\text{rate}}$ to fill in the following table.

	Distance	Rate	Time
Downstream	30	$12+x$	$\dfrac{30}{12+x}$
Upstream	18	$12-x$	$\dfrac{18}{12-x}$

Because Rani paddled for equal amounts of time upstream and downstream, we have the equation

$$\frac{30}{12+x} = \frac{18}{12-x}$$

The LCD for the fractions in this equation is $(12+x)(12-x)$. We multiply both sides of the equation by the LCD to obtain

$$\mathbf{(12+x)(12-x)}\frac{30}{12+x} = \frac{18}{12-x}\mathbf{(12+x)(12-x)}$$
$$30(12-x) = 18(12+x)$$

Solving this equation, we find

$$360 - 30x = 216 + 18x$$
$$144 = 48x$$
$$3 = x$$

The speed of the current is 3 miles per hour. □

Checkpoint 7.5.4 Solve $\dfrac{x}{6-x} = \dfrac{1}{2}$

Answer. $x = 2$

7.5.2 Extraneous Solutions

A rational function is undefined for any values of x that make its denominator equal zero. These values are not in the domain of the function, and they therefore cannot be solutions to equations involving the function. Consider the equation

$$\frac{x}{x-3} = \frac{3}{x-3} + 2$$

When we multiply both sides by the LCD, $x-3$, we obtain

$$\boldsymbol{(x-3)}\frac{x}{x-3} = \boldsymbol{(x-3)}\frac{3}{x-3} + \boldsymbol{(x-3)}\cdot 2$$

or

$$x = 3 + 2x - 6$$

whose solution is

$$x = 3$$

However, $x = 3$ is not a solution of the original equation. Both sides of the equation are undefined at $x = 3$. If you graph the two functions

$$Y_1 = \frac{x}{x-3} \qquad \text{and} \qquad Y_2 = \frac{3}{x-3} + 2$$

you will find that the graphs never intersect, which means that there is no solution to the original equation.

What went wrong with our method of solution? We multiplied both sides of the equation by $x-3$, which is zero when $x = 3$, so we really multiplied both sides of the equation by zero. Multiplying by zero does not produce an equivalent equation, and false solutions may be introduced.

An apparent solution that does not satisfy the original equation is called an **extraneous solution**. Whenever we multiply an equation by an expression containing the variable, we should check that the solution obtained is not excluded from the domain of the rational functions involved.

When solving an equation with fractions algebraically, we must be careful to multiply *each* term of the equation by the LCD, no matter whether each term involves fractions.

Example 7.5.5

a Solve the equation $\dfrac{6}{x} + 1 = \dfrac{1}{x+2}$ algebraically.

b Solve the same equation graphically.

Solution.

a To solve the equation algebraically, we multiply both sides by the LCD, $x(x+2)$. Notice that we multiply each term on the left side by the LCD, to get

$$\boldsymbol{x(x+2)}\left(\frac{6}{x} + 1\right) = \boldsymbol{x(x+2)}\frac{1}{x+2}$$

or

$$6(x+2) + x(x+2) = x$$

We use the distributive law to remove the parentheses and write the result in standard form:

$$\begin{aligned} 6x + 12 + x^2 + 2x &= x \\ x^2 + 7x + 12 &= 0 \end{aligned}$$

This is a quadratic equation that we can solve by factoring.

$$(x+3)(x+4) = 0$$

so the solutions are $x = -3$ and $x = -4$. Neither of these values causes either denominator to equal zero, so they are not extraneous solutions.

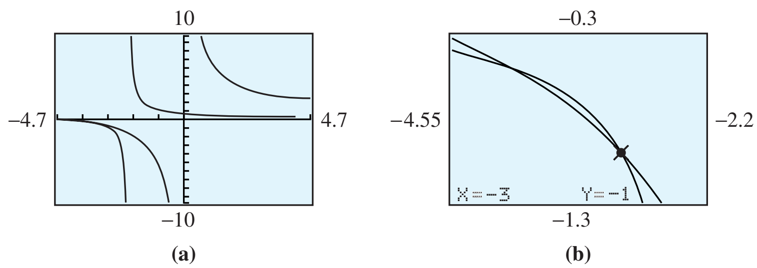
b To solve the equation graphically, graph the two functions

$$Y_1 = \frac{6}{x} + 1 \qquad \text{and} \qquad Y_2 = \frac{1}{x+2}$$

in the window

$$\text{Xmin} = -4.7 \qquad \text{Xmax} = 4.7$$
$$\text{Ymin} = -10 \qquad \text{Ymax} = 10$$

as shown in figure (a).



We see that the first graph has an asymptote at $x = 0$, and the second graph has one at $x = -2$. It appears that the two graphs may intersect in the third quadrant, around $x = -3$. To investigate further, we change the window settings to

$$\text{Xmin} = -4.55 \qquad \text{Xmax} = -2.2$$
$$\text{Ymin} = -1.3 \qquad \text{Ymax} = -0.3$$

to obtain the close-up view shown in figure (b). In this window, we can see that the graphs intersect in two distinct points, and by using the Trace we find that their x-coordinates are $x = -3$ and $x = -4$.

□

Checkpoint 7.5.6 Solve $\dfrac{9}{x^2+x-2} + \dfrac{1}{x^2-4} = \dfrac{4}{x-1}$

Answer. $x = \dfrac{-1}{2}$

7.5.3 Formulas

Algebraic fractions may appear in formulas that relate several variables. If we want to solve for one variable in terms of the others, we may need to clear the fractions.

Example 7.5.7 Solve the formula $p = \dfrac{v}{q+v}$ for v.

Solution. Because the variable we want appears in the denominator, we must first multiply both sides of the equation by that denominator, $q + v$.

$$\begin{aligned}(q+v)p &= (q+v)\frac{v}{q+v}\\ (q+v)p &= v\end{aligned}$$

We apply the distributive law on the left side, then collect all terms that involve v on one side of the equation.

$$qp + vp = v \qquad \text{Subtract } \boldsymbol{vp} \text{ from both sides.}$$

$$qp = v - vp$$

We cannot combine the two terms containing v because they are not like terms. However, we can factor out v, so that the right side is written as a single term containing the variable v. We can then complete the solution.

$$\begin{aligned} qp &= v(1-p) && \text{Divide both sides by } \mathbf{1-p}. \\ \frac{qp}{1-p} &= v \end{aligned}$$

□

Checkpoint 7.5.8 Solve for a: $\dfrac{2ab}{a+h} = H$

Answer. $a = \dfrac{bH}{2b-H}$

7.5.4 Section Summary

7.5.4.1 Vocabulary

Look up the definitions of new terms in the Glossary.

- Extraneous solution

7.5.4.2 CONCEPTS

1 To solve an equation involving an algebraic fraction, we multiply each side of the equation by the denominator of the fraction. This has the effect of clearing the fraction, giving us an equivalent equation without fractions.

2 Whenever we multiply an equation by an expression containing the variable, we should check that the solutions obtained are not extraneous.

7.5.4.3 STUDY QUESTIONS

1 What is the first step in solving an equation that includes algebraic fractions?

2 If the equation also contains terms without fractions, should you multiply those terms by the LCD?

3 What are extraneous solutions, and when might they arise?

4 If you are solving a formula and two or more terms contain the variable you are solving for, what should you do?

7.5.4.4 SKILLS

Practice each skill in the Homework 7.5.5, p. 781 problems listed.

1 Solve a fractional equation by clearing denominators: #1–14, 47–54

2 Write and solve proportions: #25–36

3 Solve equations by graphing: #15–22

4 Solve formulas that involve fractions: #39–48

5 Solve problems that involve algebraic fractions: #55–58

7.5.5 Equations that include Algebraic Fractions (Homework 7.5)

For Problems 1-8, solve the equation algebraically.

1. $\dfrac{6}{w+2} = 4$

Answer. $\dfrac{-1}{2}$

2. $\dfrac{12}{r-7} = 3$

3. $9 = \dfrac{h-5}{h-2}$

Answer. $\dfrac{13}{8}$

4. $-3 = \dfrac{v+1}{v-6}$

5. $\dfrac{15}{s^2} = 8$

Answer. $\pm\sqrt{\dfrac{15}{8}}$

6. $\dfrac{3}{m^2} = 5$

7. $4.3 = \sqrt{\dfrac{18}{y}}$

Answer. $\dfrac{1800}{1849} \approx 0.97$

8. $6.5 = \dfrac{52}{\sqrt{z}}$

9. The total weight, S, that a beam can support is given in pounds by

$$S = \frac{182.6wh^2}{l}$$

where w is the width of the beam in inches, h is its height in inches, and l is the length of the beam in feet. A beam over the doorway in an interior wall of a house must support 1600 pounds. If the beam is 4 inches wide and 9 inches tall, how long can it be?

Answer. 37 ft

10. If two appliances are connected in parallel in an electrical circuit, the total resistance, R, in the circuit is given by

$$R = \frac{ab}{a+b}$$

where a and b are the resistances of the two appliances. If one appliance has a resistance of 18 ohms, and the total resistance in the circuit is measured at 12 ohms, what is the resistance of the second appliance?

11. A flock of eider ducks is making a 150-mile flight at an average airspeed of 50 miles per hour against a moderate headwind.

(a) Express the ducks' travel time, t, as a function of the windspeed, v, and graph the function in the window

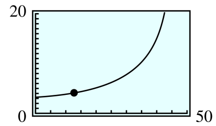
Xmin = 0	Xmax = 50
Ymin = 0	Ymax = 20

(See Problem 7.4.7.1, p. 760 of Homework Section 7.4 7.4.7, p. 760.)

(b) Write and solve an equation to find the windspeed if the flock makes its trip in 4 hours. Label the corresponding point on your graph.

Answer.

(a) $t = \dfrac{150}{50 - v}$



(b) $4 = \dfrac{150}{50 - v}$; $v = 12.5$ mph

12. Bluefin tuna swim at average speed of 36 miles per hour in still water. A school of tuna is making a 200-mile trip against a current.

(a) Express the tuna's travel time, t, as a function of the current speed, v, and graph the function in the window

Xmin = 0	Xmax = 36
Ymin = 0	Ymax = 50

(See Problem 7.4.7.2, p. 761 of Homework Section 7.4 7.4.7, p. 760.)

(b) Write and solve an equation to find the current speed if the school makes its trip in 8 hours. Label the corresponding point on your graph.

13. The cost, in thousands of dollars, for immunizing p percent of the residents of Emporia against a dangerous new disease is given by the function

$$C(p) = \frac{72p}{100 - p}$$

Write and solve an equation to determine what percent of the population can be immunized for \$168, 000.

Answer. $168 = \dfrac{72p}{100 - p}$; $p = 70\%$

14. The cost, in thousands of dollars, for extracting p percent of a precious ore from a mine is given by the function

$$C(p) = \frac{360p}{100 - p}$$

Write and solve an equation to determine what percentage of the ore can be extracted for \$390, 000.

For Problems 15–18,

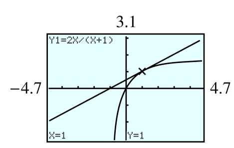
a Solve the equation graphically by graphing two functions, one for each side of the equation.

b Solve the equation algebraically.

15. $\dfrac{2x}{x+1} = \dfrac{x+1}{2}$

Answer.

(a)



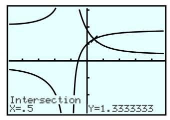
(b) $x = 1$

16. $\dfrac{3}{2x+1} = \dfrac{2x-3}{x}$

17. $\dfrac{2}{x+1} = \dfrac{x}{x+1} + 1$

Answer.

(a)



(b) $x = \dfrac{1}{2}$

18. $\dfrac{5}{x-3} = \dfrac{x+2}{x-3} + 3$

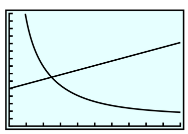
19. The manager of Joe's Burgers discovers that he will sell $\dfrac{160}{x}$ burgers per day if the price of a burger is x dollars. On the other hand, he can afford to make $6x + 49$ burgers if he charges x dollars apiece for them.

(a) Graph the **demand function**, $D(x) = \dfrac{160}{x}$, and the **supply function**, $S(x) = 6x + 49$. At what price x does the demand for burgers equal the number that Joe can afford to supply? This value for x is called the **equilibrium price**.

(b) Write and solve an equation to verify your equilibrium price.

Answer.

(a)



$2.50

(b) $\dfrac{160}{x} = 6x + 49$; $x = 2.50$

20. A florist finds that she will sell $\dfrac{300}{x}$ dozen roses per week if she charges x dollars for a dozen. Her suppliers will sell her $5x - 55$ dozen roses if she sells them at x dollars per dozen.

(a) Graph the demand function, $D(x) = \dfrac{300}{x}$, and the supply function, $S(x) = 5x - 55$, in the same window. At what equilibrium price x will the florist sell all the roses she purchases?

(b) Write and solve an equation to verify your equilibrium price.

21. Francine wants to fence a rectangular area of 3200 square feet to grow vegetables for her family of three.

(a) Express the length of the garden as a function of its width.

(b) Express the perimeter, P, of the garden as a function of its width.

(c) Graph your function for perimeter and find the coordinates of the lowest point on the graph. Interpret those coordinates in the context of the problem.

(d) Francine has 240 feet of chain link to make a fence for the garden, and she would like to know what the width of the garden should be. Write an equation that describes this situation.

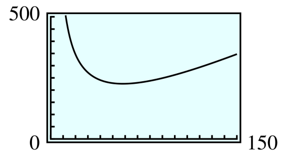
(e) Solve your equation and find the dimensions of the garden.

Answer.

(a) $L = \dfrac{3200}{w}$

(b) $P = \dfrac{6400}{w} + 2w$

(c)



Lowest point: $(56.6, 226)$; The minimum perimeter is 226 ft for a width of 56.6 ft.

(d) $240 = \dfrac{6400}{w} + 2w$

(e) 40 ft by 80 ft

22. The cost of wire fencing is \$7.50 per foot. A rancher wants to enclose a rectangular pasture of 1000 square feet with this fencing.

(a) Express the length of the pasture as a function of its width.

(b) Express the cost of the fence as a function of its width.

(c) Graph your function for the cost and find the coordinates of the lowest point on the graph. Interpret those coordinates in the context of the problem.

(d) The rancher has \$1050 to spend on the fence, and she would like to know what the width of the pasture should be. Write an equation to describe this situation.

(e) Solve your equation and find the dimensions of the pasture.

23. A proportion is an equation in which each side is a ratio: $\dfrac{a}{b} = \dfrac{c}{d}$. Show that this equation may be rewritten as $ad = bc$.

Answer. Multiply both sides of the equation by bd and simplify.

$$\frac{a}{b} \cdot \frac{bc}{1} = \frac{c}{d} \cdot \frac{bc}{1}, \text{ so } ac = bd$$

24. Suppose that y varies directly with x, and (a, b) and (c, d) are two points on the graph of y in terms of x. Show that $\dfrac{b}{a} = \dfrac{d}{c}$.

For Problems 25-28, solve the proportion using your result from Problem 23.

25. $\dfrac{3}{4} = \dfrac{y+2}{12-y}$

Answer. 4

26. $\dfrac{-3}{4} = \dfrac{y-7}{y+14}$

27. $\dfrac{50}{r} = \dfrac{75}{r+20}$

Answer. 40

28. $\dfrac{30}{r} = \dfrac{20}{r-10}$

For Problems 29-36, use your result from Problem 24 to write and solve a proportion for the problem.

29. Property taxes on a house vary directly with the value of the house. If the taxes on a house worth \$120,000 are \$2700, what would the taxes be on a house assessed at \$275,000?

Answer. \$6187.50

30. The cost of electricity varies directly with the number of units (BTUs) consumed. If a typical household in the Midwest uses 83 million BTUs of electricity annually and pays \$1236, how much will a household that uses 70 million BTUs annually spend for energy?

31. Distances on a map vary directly with actual distances. The scale on a map of Michigan uses $\frac{3}{8}$ inch to represent 10 miles. If Isle Royale is $1\frac{11}{16}$ inches long on the map, what is the actual length of the island?

Answer. 45 mi

32. The dimensions of an enlargement vary directly with the dimensions of the original. A photographer plans to enlarge a photograph that measures 8.3 centimeters by 11.2 centimeters to produce a poster that is 36 centimeters wide. How long will the poster be?

33. The Forest Service tags 200 perch and releases them into Spirit Lake. One month later, it captures 80 perch and finds that 18 of them are tagged. What is the Forest Service's estimate of the original perch population of the lake?

Answer. 689

34. The Wildlife Commission tags 30 Canada geese at one of its migratory feeding grounds. When the geese return, the commission captures 45 geese, of which 4 are tagged. What is the commission's estimate of the number of geese that use the feeding ground?

35. The highest point on Earth is Mount Everest in Tibet, with an elevation of 8848 meters. The deepest part of the ocean is the Challenger Deep in the Mariana Trench, near Indonesia, $11,034$ meters below sea level.

(a) What is the total height variation in the surface of the Earth?

(b) What percentage of the Earth's radius, 6400 kilometers, is this variation?

(c) If the Earth were shrunk to the size of a basketball, with a radius of 4.75 inches, what would be the corresponding height of Mount Everest?

Answer.

(a) $19,882$ m (b) 0.3% (c) 0.00657 in

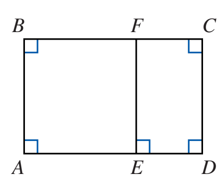
36. Shortly after the arrival of human beings at the Hawaiian islands around 400 A.D., many species of birds became extinct. Fossils of 29 different species have been found, but some species may have left no fossils for us to find. We can estimate the total number of extinct species using a proportion. Of 9 species that are still alive, biologists have found fossil evidence of 7. (Source: Burton, 1998)

(a) Assuming that the same fraction of extinct species have left fossil records, calculate the total number of extinct species

(b) Give two reasons why this estimate may not be completely accurate.

37. In the figure, the rectangle $ABCD$ is divided into a square and a smaller rectangle, $CDEF$. The two rectangles $ABCD$ and $CDEF$ are similar (their corresponding sides are proportional.) A rectangle $ABCD$ with this property is called a **golden rectangle**, and the ratio of its length to its width is called the golden ratio.

The golden ratio appears frequently in art and nature, and it is considered to give the most pleasing proportions to many figures. We will compute the golden ratio as follows.



(a) Let $AB = 1$ and $AD = x$. What are the lengths of AE, DE, and CD?

(b) Write a proportion in terms of x for the similarity of rectangles $ABCD$ and $CDEF$. Be careful to match up the corresponding sides.

(c) Solve your proportion for x. Find the golden ratio, $\dfrac{AD}{AB} = \dfrac{x}{1}$.

Answer.

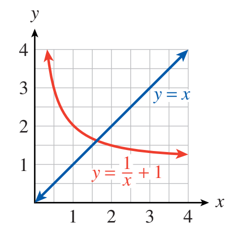
(a) $AE = 1,\ DE = x - 1,\ CD = 1$

(b) $\dfrac{1}{x} = \dfrac{x-1}{x}$

(c) $\dfrac{1+\sqrt{5}}{2}$

38.

The figure shows the graphs of two equations, $y = x$ and $y = \dfrac{1}{x} + 1$.



(a) Find the x-coordinate of the intersection point of the two graphs.

(b) Compare your answer to the golden ratio you computed in Problem 37.

For Problems 39-46, solve the formula for the specified variable and answer any additional questions.

39. $S = \dfrac{a}{1-r}$, for r

Answer. $r = \dfrac{S-a}{S}$

40. $I = \dfrac{E}{r+R}$, for R

41. $H = \dfrac{2xy}{x+y}$, for x

Answer. $x = \dfrac{Hy}{2y-H}$

42. $M = \dfrac{ab}{a+b}$, for b

43. $F = \dfrac{Gm_1m_2}{d^2}$, for d. What happens to F as $d>$ gets big?

Answer. $d = \pm\sqrt{\dfrac{Gm_1m_2}{F}}$

44. $F = \dfrac{kq_1q_2}{r^2}$, for q_2

45. $\dfrac{1}{Q} + \dfrac{1}{I} = \dfrac{2}{r}$, for r

Answer. $r = \dfrac{2QI}{I+Q}$

46. $\dfrac{1}{R} = \dfrac{1}{A} + \dfrac{1}{B}$, for B

47. The sidereal period of a planet is the time for the planet to make one trip around the Sun (as seen from the Sun itself). The synodic period is the time between two successive conjunctions of the planet and the Sun, as seen from Earth. The relationship among the sidereal period, P, of a planet, the synodic period, S, of the planet, and the sidereal period of Earth, E, is given by

$$\frac{1}{P}=\frac{1}{S}+\frac{1}{E}$$

when the planet is closer to the Sun than the Earth is. Solve for P in terms of S and E.

Answer. $P=\dfrac{ES}{E+S}$

48. When a planet is farther from the Sun than Earth is,

$$\frac{1}{P}=\frac{1}{E}-\frac{1}{S}$$

where P, E, and S are as defined in Problem 47. Solve for P in terms of S and E.

For Problems 49-56, solve the equation algebraically.

49. $\dfrac{3}{x-2}=\dfrac{1}{2}+\dfrac{2x-7}{2x-4}$

Answer. 5

50. $\dfrac{2}{x+1}+\dfrac{1}{3x+3}=\dfrac{1}{6}$

51. $\dfrac{4}{x+2}-\dfrac{1}{x}=\dfrac{2x-1}{x^2+2x}$

Answer. 1

52. $\dfrac{1}{x-1}+\dfrac{2}{x+1}=\dfrac{x-2}{x^2-1}$

53. $\dfrac{x}{x+2}-\dfrac{3}{x-2}=\dfrac{x^2+8}{x^2-4}$

Answer. $\dfrac{-14}{5}$

54. $\dfrac{4}{2x-3}+\dfrac{4x}{4x^2-9}=\dfrac{1}{2x+3}$

55. $\dfrac{4}{3x}+\dfrac{3}{3x+1}+2=0$

Answer. $\dfrac{-1}{6},\dfrac{-4}{3}$

56. $-3=\dfrac{-10}{x+2}+\dfrac{10}{x+5}$

57. A chartered sightseeing flight over the Grand Canyon is scheduled to return to its departure point in 3 hours. The pilot would like to cover a distance of 144 miles before turning around, and he hears on the Weather Service that there will be a headwind of 20 miles per hour on the outward journey.

(a) Express the time it takes for the outward journey as a function of the airspeed of the plane.

(b) Express the time it takes for the return journey as a function of the speed of the plane.

(c) Graph the sum of the two functions and find the point on the graph with y-coordinate 3. Interpret the coordinates of the point in the context of the problem.

(d) The pilot would like to know what airspeed to maintain in order to complete the tour in 3 hours. Write an equation to describe this situation.

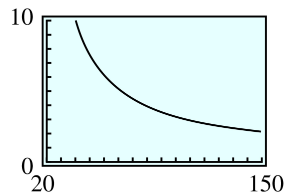
(e) Solve your equation to find the appropriate airspeed.

Answer.

(a) $t_1 = \dfrac{144}{s-20}$

(b) $t_2 = \dfrac{144}{s+20}$

(c)



If the airspeed is 100 mph, the round trip will take 3 hours.

(d) $\dfrac{144}{s-20} + \dfrac{144}{s+20} = 3$

(e) 100 mph

58. Two student pilots leave the airport at the same time. They both fly at an airspeed of 180 miles per hour, but one flies with the wind and the other flies against the wind.

(a) Express the time it takes the first pilot to travel 500 miles as a function of the windspeed.

(b) Express the time it takes the second pilot to travel 400 miles as a function of the windspeed.

(c) Graph the two functions in the same window, and find the coordinates of the intersection point. Interpret those coordinates in the context of the problem.

(d) Both pilots check in with their instructors at the same time, and the first pilot has traveled 500 miles while the second pilot has gone 400 miles. Write an equation to describe this situation.

(e) Solve your equation to find the speed of the wind.

59. Andy drives 300 miles to Lake Tahoe at 70 miles per hour and returns home at 50 miles per hour. What is his average speed for the round trip? (It is not 60 miles per hour!)

(a) Write expressions for the time it takes for each leg of the trip if Andy drives a distance, d, at speed r_1 and returns at speed r_2.

(b) Write expressions for the total distance and total time for the trip.

(c) Write an expression for the average speed for the entire trip.

(d) Write your answer to part (c) as a simple fraction.

(e) Use your formula to answer the question stated in the problem.

Answer.

(a) $t_1 = \dfrac{d}{r_1}$, $t_2 = \dfrac{d}{r_2}$

(b) Total distance is $2d$; total time $\dfrac{d}{r_1} + \dfrac{d}{r_2}$.

(c) $\dfrac{2d}{\dfrac{d}{r_1} + \dfrac{d}{r_2}}$

(d) $\dfrac{2r_1 r_2}{r_1 + r_2}$

(e) $58\frac{1}{3}$ mph

60. The owner of a print shop volunteers to produce flyers for his candidate's campaign. His large printing press can complete the job in 4 hours, and the smaller model can finish the flyers in 6 hours. How long will it take to print the flyers if he runs both presses simultaneously?

(a) Suppose that the large press can complete a job in t_1 hours and the smaller press takes t_2 hours. Write expressions for the fraction of a job that each press can complete in 1 hour.

(b) Write an expression for the fraction of a job that can be completed in 1 hour with both presses running simultaneously.

(c) Write an expression for the amount of time needed to complete the job with both presses running.

(d) Write your answer to part (c) as a simple fraction.

(e) Use your formula to answer the question stated in the problem.

7.6 Chapter Summary and Review

7.6.1 Key Concepts

1 The degree of a product of nonzero polynomials is the sum of the degrees of the factors.

2 **Cube of a Binomial.**

1 $(x + y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$

2 $(x - y)^3 = x^3 - 3x^2y + 3xy^2 - y^3$

3 **Factoring the Sum or Difference of Two Cubes.**

1 $x^3 + y^3 = (x + y)(x^2 - xy + y^2)$

2 $x^3 - y^3 = (x - y)(x^2 + xy + y^2)$

4 The graphs of all polynomials are smooth curves without breaks or holes.

5 The graph of a polynomial of degree n (with positive lead coefficient) has the same long-term behavior as the power function of the same degree.

6 **Factor Theorem.**

Let $P(x)$ be a polynomial with real number coefficients. Then $(x - a)$ is a factor of $P(x)$ if and only if $P(a) = 0$.

7 A polynomial of degree n can have at most n x-intercepts.

8 At a zero of multiplicity 2, the graph of a polynomial has a turning point. At a zero of multiplicity 3, the graph of a polynomial has an inflection point.

9 The square root of a negative number is an imaginary number.

10 A complex number is the sum of a real number and an imaginary number.

11 We can perform the four arithmetic operations on complex numbers

12 The product of a nonzero complex number and its conjugate is always a positive real number.

13 **Fundamental Theorem of Algebra.**

Let $p(x)$ be a polynomial of degree $n \geq 1$. Then $p(x)$ has exactly n complex zeros.

14 We can graph complex numbers in the complex plane

15 Multiplying a complex number by i rotates its graph by $90°$ around the origin.

16 **Rational Function.**

A **rational function** is one of the form

$$f(x) = \frac{P(x)}{Q(x)}$$

where $P(x)$ and $Q(x)$ are polynomials and $Q(x)$ is not the zero polynomial.

17 A rational function $f(x) = \dfrac{P(x)}{Q(x)}$ is undefined for any value $x = a$ where $Q(a) = 0$. These x-values are not in the domain of the function.

18 **Vertical Asymptotes.**

If $Q(a) = 0$ but $P(a) \neq 0$, then the graph of the rational function $f(x) = \dfrac{P(x)}{Q(x)}$ has a **vertical asymptote** at $x = a$.

19 **Horizontal Asymptotes.**

Suppose $f(x) = \dfrac{P(x)}{Q(x)}$ is a rational function, where the degree of $P(x)$ is m and the degree of $Q(x)$ is n.

1 If $m < n$, the graph of f has a horizontal asymptote at $y = 0$.

2 If $m = n$, the graph of f has a horizontal asymptote at $y = \dfrac{a}{b}$, where a is the lead coefficient of $P(x)$ and b is the lead coefficient of $Q(x)$.

3 If $m > n$, the graph of f does not have a horizontal asymptote.

20 To solve an equation involving an algebraic fraction, we multiply each side of the equation by the denominator of the fraction. This has the effect of clearing the fraction, giving us an equivalent equation without fractions.

21 Whenever we multiply an equation by an expression containing the variable, we should check that the solutions obtained are not extraneous.

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7.6.2 Chapter 7 Review Problems

For Problems 1–4, multiply.

1. $(2x-5)(x^2-3x+2)$

Answer.
$2x^3-11x^2+19x-10$

2. $(b^2-2b-3)(2b^2+b-5)$

3. $(t+4)(t^2-t-1)$

Answer. t^3+3t^2-5t-4

4. $(b+3)(2b-1)(2b+5)$

For Problems 5–8, find the indicated term.

5. $(1-3x+5x^2)(7+x-x^2)$; x^2

Answer. $31x^2$

6. $(-3+x-4x^2)(4+3x-2x^3)$; x^3

7. $(4x-x^2+3x^3)(1+4x-3x^2)$; x^3

Answer. $-13x^3$

8. $(3-2x+2x^3)(5+3x-2x^2+4x^4)$; x^4

For Problems 9–12, factor.

9. $8x^3-27z^3$

Answer.
$(2x-3z)(4x^2+6xz+9z^2)$

10. $1+125a^3b^3$

11. y^3+27x^3

Answer.
$(y+3x)(y^2-3xy+9x^2)$

12. x^9-8

For Problems 13–14, write as a polynomial.

13. $(v-10)^3$

Answer.
$v^3-30v^2+300v-1000$

14. $(a+2b^2)^3$

15. The expression $\dfrac{n}{6}(n-1)(n-2)$ gives the number of different 3-item pizzas that can be created from a list of n toppings.

(a) Write the expression as a polynomial.

(b) If Mitch's Pizza offers 12 different toppings, how many different combinations for 3-item pizzas can be made?

(c) Use a table or graph to determine how many different toppings are needed in order to be able to have more than 1000 possible combinations for 3-item pizzas.

Answer.

(a) $\dfrac{1}{6}n^3-\dfrac{1}{2}n^2+\dfrac{1}{3}n$

(b) 220

(c) 20

16. The expression $n(n-1)(n-2)$ gives the number of different triple-scoop ice cream cones that can be created from a list of n flavors.

(a) Write the expression as a polynomial.

(b) If Zanner's Ice Cream Parlor offers 21 flavors, how many different triple-scoop ice cream cones can be made?

(c) Use a table or graph to determine how many different flavors are needed in order to be able to have more than 10,000 possible triple-scoop ice cream cones.

For Problems 17–18,

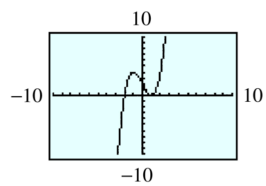
a Graph each polynomial in the standard window.

b Find the range of the function on the domain $[-10, 10]$.

17. $f(x) = x^3 - 3x + 2$

Answer.

(a)



(b) $[-968, 972]$

18. $g(x) = -0.1(x^4 - 6x^3 + x^2 + 24x + 16)$

For Problems 19–28,

a Find the zeros of the polynomial.

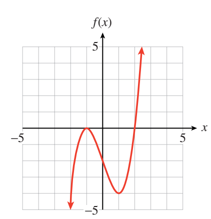
b Sketch the graph by hand.

19. $f(x) = (x - 2)(x + 1)^2$

Answer.

(a) $2, -1$

(b)



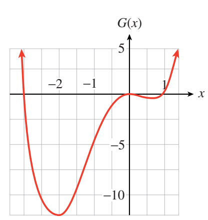
20. $g(x) = (x - 3)^2(x + 2)$

21. $G(x) = x^2(x - 1)(x + 3)$

Answer.

(a) $0, 1, -3$

(b)



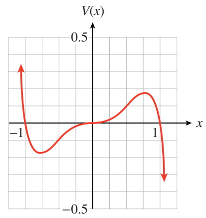
22. $F(x) = (x + 1)^2(x - 2)^2$

23. $V(x) = x^3 - x^5$

Answer.

(a) $0, 1, -1$

(b)



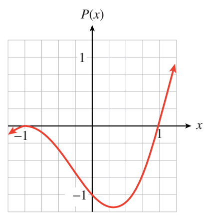
24. $H(x) = x^4 - 9x^2$

25. $P(x) = x^3 + x^2 - x - 1$

Answer.

(a) $-1, 1$

(b)



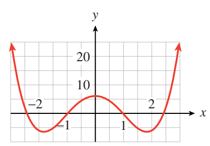
26. $y = x^3 + x^2 - 2x$

27. $y = x^4 - 7x^2 + 6$

Answer.

(a) $-1, 1, \pm\sqrt{6}$

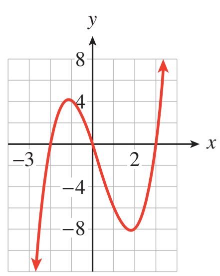
(b)



28. $y = x^4 + x^3 - 3x^2 - 3x$

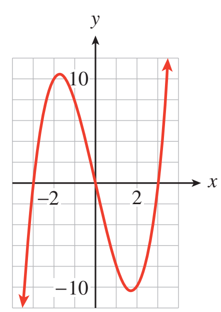
For Problems 29–34, find a possible formula for the polynomial, in factored form.

29.

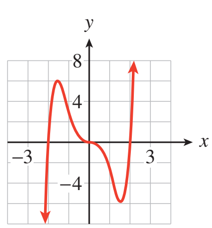


Answer. $x(x+2)(x-3)$

30.

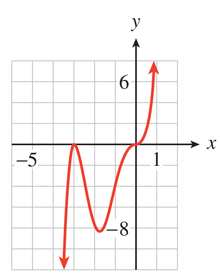


31.

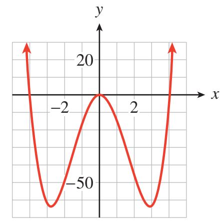


Answer. $x^3(x+2)(x-2)$

32.

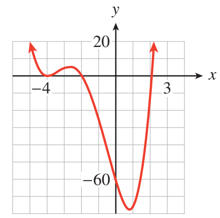


33.



Answer. $x^2(x+4)(x-4)$

34.



For Problems 35–36,

a Verify that the given value is a zero of the polynomial.

b Find the other zeros. (*Hint*: Use polynomial division to write $P(x) = (x-a)Q(x)$, then factor $Q(x)$.)

35. $P(x) = x^3 - x^2 - 7x - 2$; $a = -2$

Answer.
(a) $P(-2) = 0$ (b) $\dfrac{3 \pm \sqrt{13}}{2}$

36. $P(x) = 3x^3 - 11x^2 - 5x + 4$; $a = 4$

For Problems 37–40,

a Solve the quadratic equation, and write the solutions in the form $a + bi$.

b Check your solutions.

37. $x^2 + 4x + 10 = 0$

Answer. $-2 \pm i\sqrt{6}$

38. $x^2 - 2x + 7 = 0$

39. $3x^2 - 6x + 5 = 0$

Answer. $1 \pm \dfrac{\sqrt{6}}{3}i$

40. $2x^2 + 5x + 4 = 0$

For Problems 41–42, evaluate the polynomial for the given values of the variable.

41. $z^2 - 6z + 5$

(a) $z = 3 + 2i$

(b) $z = 3 - 2i$

Answer.

(a) -8 (b) -8

42. $w^2 + 4w + 7$

(a) $w = -1 - 3i$

(b) $w = -1 + 3i$

For Problems 43–44, find the quotient.

43. $\dfrac{2-5i}{3-i}$

Answer. $\dfrac{11}{10} - \dfrac{13}{10}i$

44. $\dfrac{1+i}{1-i}$

For Problems 45–46, find a fourth-degree polynomial with the given zeros.

45. $3i,\ 1-2i$

Answer.
$x^4 - 2x^3 + 14x^2 - 18x + 45$

46. $2 - \sqrt{3},\ 2 + 3i$

For Problems 45–46, plot each complex number as a point on the complex plane.

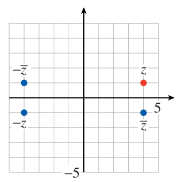
47.

(a) $z = 4 + i,\ \overline{z},\ -z,\ -\overline{z}$

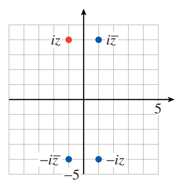
(b) $iz,\ i\overline{z},\ -iz,\ -i\overline{z}$

Answer.

(a)



(b)



48.

(a) $w = -2 + 3i,\ \overline{w},\ -w,\ -\overline{w}$

(b) $iw,\ i\overline{w},\ -iw,\ -i\overline{w}$

49. The radius, r, of a cylindrical can should be one-half its height, h.

(a) Express the volume, V, of the can as a function of its height.

(b) What is the volume of the can if its height is 2 centimeters? 4 centimeters?

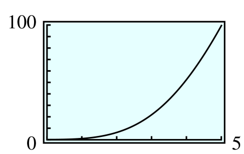
(c) Graph the volume as a function of the height and verify your results of part (b) graphically. What is the approximate height of the can if its volume is 100 cubic centimeters?

Answer.

(a) $V = \dfrac{\pi h^3}{4}$

(b) $2\pi\ \text{cm}^3 \approx 6.28\ \text{cm}^3$; $16\pi\ \text{cm}^3 \approx 50.27\ \text{cm}^3$

(c)



50. The Twisty-Freez machine dispenses soft ice cream in a cone-shaped peak with a height 3 times the radius of its base. The ice cream comes in a round bowl with base diameter d.

(a) Express the volume, V, of Twisty-Freez in the bowl as a function of d.

(b) How much Twisty-Freez comes in a 3-inch diameter dish? A 4-inch dish?

(c) Graph the volume as a function of the diameter and verify your results of part (b) graphically. What is the approximate diameter of a Twisty-Freez if its volume is 5 cubic inches?

51. A new health club opened up, and the manager kept track of the number of active members over its first few months of operation. The equation below gives the number, N, of active members, in hundreds, t months after the club opened.

$$N = \frac{44t}{40 + t^2}$$

(a) Use your calculator to graph the function N on a suitable domain.

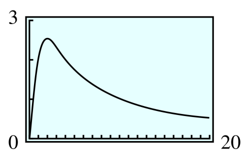
(b) How many active members did the club have after 8 months?

(c) In which months did the club have 200 active members?

(d) When does the health club have the largest number of active members? What happens to the number of active members as time goes on?

Answer.

(a)



(b) 338

(c) Months 2 and 20

(d) During month 6. The number of members eventually decreases to zero.

52. A small lake in a state park has become polluted by runoff from a factory upstream. The cost for removing p percent of the pollution from the lake is given, in thousands of dollars, by

$$C = \frac{25p}{100 - p}$$

(a) Use your calculator to graph the function C on a suitable domain.

(b) How much will it cost to remove 40% of the pollution?

(c) How much of the pollution can be removed for $\$100,000$?

(d) What happens to the cost as the amount of pollution to be removed increases? How much will it cost to remove all the pollution?

For Problems 53–54, state the domain of the function.

53. $h(x) = \dfrac{x^2 - 9}{x(x^2 - 4)}$

Answer. All numbers except $-2, 0, 2$.

54. $f(x) = \dfrac{x^2 - 3x + 10}{x^2(x^2 + 1)}$

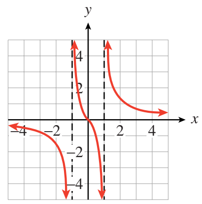
For Problems 55–56,

a Sketch the horizontal and vertical asymptotes for each function.

b Use the asymptotes to help you sketch the graph.

55. $F(x) = \dfrac{2x}{x^2 - 1}$

Answer.



56. $G(x) = \dfrac{2}{x^2 - 1}$

For Problems 57–62,

(a) Identify all asymptotes and intercepts.

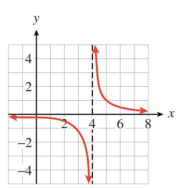
(b) Sketch the graph.

57. $y = \dfrac{1}{x - 4}$

Answer.

(a) Horizontal asymptote $y = 0$; Vertical asymptote $x = 4$; y-intercept $(0, \frac{-1}{4})$

(b)



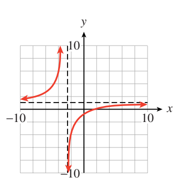
58. $y = \dfrac{2}{x^2 - 3x - 10}$

59. $y = \dfrac{x - 2}{x + 3}$

Answer.

(a) Horizontal asymptote $y = 1$; Vertical asymptote $x = -3$; x-intercept $(2, 0)$; y-intercept $(0, \frac{-2}{3})$

(b)

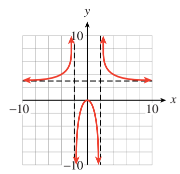


60. $y = \dfrac{x - 1}{x^2 - 2x - 3}$

61. $y = \dfrac{3x^2}{x^2 - 4}$

Answer.

(a) Horizontal asymptote $y = 3$; Vertical asymptote $x = \pm 2$; x-intercept $(0, 0)$; y-intercept $(0, 0)$

(b) 

62. $y = \dfrac{2x^2 - 2}{x^2 - 9}$

For Problems 63–66,

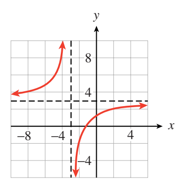
a Use polynomial division to write the fraction in the form $y = \dfrac{k}{p(x)} + c$, where k and c are constants.

b Use transformations to sketch the graph.

63. $y = \dfrac{3x + 4}{x + 3}$

Answer.

(a) $y = \dfrac{-5}{x + 3} + 3$

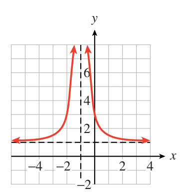
(b) 

64. $y = \dfrac{5x + 1}{x - 2}$

65. $y = \dfrac{x^2 + 2x + 3}{(x + 1)^2}$

Answer.

(a) $y = \dfrac{2}{(x + 1)^2} + 1$

(b) 

66. $y = \dfrac{x^2 - 4x + 3}{(x - 2)^2}$

67. The Explorer's Club is planning a canoe trip to travel 90 miles up the Lazy River and return in 4 days. Club members plan to paddle for 6 hours each day, and they know that the current in the Lazy River is 2 miles per hour.

(a) Express the time it will take for the upstream journey as a function of their paddling speed in still water.

(b) Express the time it will take for the downstream journey as a function of their paddling speed in still water.

(c) Graph the sum of the two functions and find the point on the graph with y-coordinate 24. Interpret the coordinates of the point in the context of the problem.

(d) The Explorer's Club would like to know what average paddling speed members must maintain in order to complete their trip in 4 days. Write an equation to describe this situation.

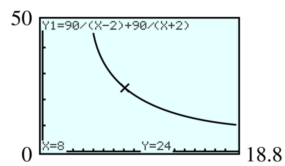
(e) Solve your equation to find the required paddling speed.

Answer.

(a) $t_1 = \dfrac{90}{v-2}$

(b) $t_2 = \dfrac{90}{v+2}$

(c)



(d) $\dfrac{90}{v-2} + \dfrac{90}{v+2} = 24$

(e) 8 mph

68. Pam lives on the banks of the Cedar River and makes frequent trips in her outboard motorboat. The boat travels at 20 miles per hour in still water.

(a) Express the time it takes Pam to travel 8 miles upstream to the gas station as a function of the speed of the current.

(b) Express the time it takes Pam to travel 12 miles downstream to Marie's house as a function of the speed of the current.

(c) Graph the two functions in the same window, then find the coordinates of the intersection point. Interpret those coordinates in the context of the problem.

(d) Pam traveled to the gas station in the same time it took her to travel to Marie's house. Write an equation to describe this situation.

(e) Solve your equation to find the speed of the current in the Cedar River.

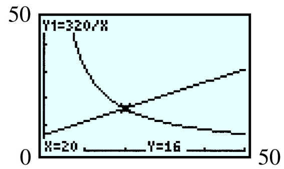
69. Mikala sells $\dfrac{320}{x}$ bottles of bath oil per week if she charges x dollars per bottle. Her supplier can manufacture $\dfrac{1}{2}x + 6$ bottles per week if she sells it at x dollars per bottle.

(a) Graph the demand function, $D(x) = \dfrac{320}{x}$, and the supply function, $S(x) = \dfrac{1}{2}x + 6$, in the same window.

(b) Write and solve an equation to find the equilibrium price, that is, the price at which the supply equals the demand for bath oil. Label this point on your graph.

Answer.

(a)



(b) $\frac{320}{x} = \frac{1}{2}x + 6$; \$20

70. Tomoko sells $\frac{4800}{x}$ exercise machines each month if the price of a machine is x dollars. On the other hand, her supplier can manufacture $2.5x + 20$ machines if she charges x dollars apiece for them.

(a) Graph the demand function, $D(x) = \frac{4800}{x}$, and the supply function, $S(x) = 2.5x + 20$, in the same window.

(b) Write and solve an equation to find the equilibrium price, that is, the price at which the supply equals the demand for exercise machines. Label this point on your graph.

For Problems 71–72, write and solve a proportion.

71. A polling firm finds that 78 of the 300 randomly selected students at Citrus College play some musical instrument. Based on the poll, how many of the college's 1150 students play a musical instrument?

Answer. 299

72. Claire wants to make a scale model of Salem College. The largest building on campus, Lausanne Hall, is 60 feet tall, and her model of Lausanne Hall will be 8 inches tall. How tall should she make the model of Willamette Hall, which is 48 feet tall?

For Problems 73–80, solve.

73. $\frac{y+3}{y+5} = \frac{1}{3}$

Answer. -2

74. $\frac{z^2+2}{z^2-2} = 3$

75. $\frac{x}{x-2} = \frac{2}{x-2} + 7$

Answer. No solution

76. $\frac{3x}{x+1} - \frac{2}{x^2+x} = \frac{4}{x}$

77. $\frac{2}{a+1} + \frac{1}{a-1} = \frac{3a-1}{a^2-1}$

Answer. All a except -1 and 1

78. $\frac{2b-1}{b^2+2b} = \frac{4}{b+2} - \frac{1}{b}$

79. $\frac{-10}{u-2} = \frac{u-4}{u^2-u-2} + \frac{3}{u+1}$

Answer. 0

80. $\frac{1}{t^2+t} + \frac{1}{t} = \frac{3}{t+1}$

For Problems 81–84, solve for the indicated variable.

81. $V = C\left(1 - \frac{t}{n}\right)$, for n

Answer. $n = \frac{Ct}{C-V}$

82. $r = \frac{dc}{1-ec}$, for c

83. $\frac{p}{q} = \frac{r}{q+r}$, for q

Answer. $q = \frac{pr}{r-p}$

84. $I = \frac{E}{R + \frac{r}{n}}$, for R

7.7 Projects for Chapter 7

Project 49 Solving cubics: Part I. In this project, we solve cubic equations of the form

$$x^3 + mx = n$$

Note that there is no quadratic term. This special form was first solved by the Italian mathematicians Scipione del Ferro and Niccolò Fontana Tartaglia early in the sixteenth century. Tartaglia revealed the secret to solving the special cubic equation in a poem. He first found values u and v to satisfy the system

$$\begin{aligned} u - v &= n \\ uv &= \left(\frac{m}{3}\right)^3 \end{aligned}$$

a We will use Tartaglia's method to solve

$$x^3 + 6x = 7$$

What are the values of m and n?

b Substitute the values of m and n into Tartaglia's system, then use substitution to solve for u and v. You should find two possible solutions.

c For each solution of the system, compute $x = \sqrt[3]{u} - \sqrt[3]{v}$. You should get the same value of x for each (u, v).

d Check that your value for x is a solution of $x^3 + 6x = 7$.

Project 50 Solving cubics: Part II. Tartaglia's method always works to solve the special cubic equation, even when u and v are not convenient values. We will show why in this project.

a Expand the expression $(a - b)^3 + 3ab(a - b)$ and complete the identity.

$$(a - b)^3 + 3ab(a - b) = \underline{\hspace{3cm}}$$

b Your answer to part (a) is actually Tartaglia's special cubic in disguise. Substitute $x = a - b$, $m = 3ab$, and $n = a^3 - b^3$ to see this. Therefore, if we can find numbers a and b that satisfy

$$\begin{aligned} 3ab &= m \\ a^3 - b^3 &= n \end{aligned}$$

then the solution to Tartaglia's cubic is $x = a - b$.

c Compare the system in part (b) to the system from Project 49, p. 801,

$$\begin{aligned} u - v &= n \\ uv &= \left(\frac{m}{3}\right)^3 \end{aligned}$$

to show that $u = a^3$ and $v = b^3$.

d Use your answer to part (c) to show that Tartaglia's value, $x = \sqrt[3]{u} - \sqrt[3]{v}$, is a solution of $x^3 + mx = n$.

Project 51 Solving cubics: Part III. Use Tartaglia's method to solve the equation

$$x^3 + 3x = 2 \tag{7.1}$$

by carrying out the following steps.

a Identify the values of m and n from (7.1) and write two equations for u and v.

b Solve for values of u and v. You will need to use the quadratic formula.

c Take the positive values of u and v. Write the solution $x = \sqrt[3]{u} - \sqrt[3]{v}$. Do not try to simply the radical expression; instead, use your calculator to check the solution numerically.

Project 52 Solving cubics: Part IV. We can solve any cubic equation by first using a substitution to put the equation in Tartaglia's special form.

a Consider the equation $X^3 + bX^2 + cX + d = 0$. Make the substitution $X = x - \dfrac{b}{3}$, and expand the left side of the equation.

b What is the coefficient of x^2 in the resulting equation? What are the values of m and n?

c If you solve the special form in part (a) for x, how can you find the value of X that solves the original equation?

Project 53 Duration of eclipse. The time, T, it takes for the Moon to eclipse the Sun totally is given (in minutes) by the formula

$$T = \frac{1}{v}\left(\frac{rD}{R} - d\right)$$

where d is the diameter of the Moon, D is the diameter of the Sun, r is the distance from the Earth to the Moon, R is the distance from the Earth to the Sun, and v is the speed of the Moon.

a Solve the formula for v in terms of the other variables.

b It takes 2.68 minutes for the Moon to eclipse the Sun. Calculate the speed of the Moon given the following values:

$$d = 3.48 \times 10^3 \text{ km} \qquad D = 1.41 \times 10^6 \text{ km}$$
$$r = 3.82 \times 10^5 \text{ km} \qquad R = 1.48 \times 10^8 \text{ km}$$

Project 54 Optimal traffic flow. The stopping distance, s, for a car traveling at speed v meters per second is given (in meters) by

$$s = vT + \frac{v^2}{2a}$$

where T is the reaction time of the driver and a is the average deceleration as the car brakes. Suppose that all the cars on a crowded motorway maintain the appropriate spacing determined by the stopping distance for their speed. What speed allows the maximum flow of cars along the road per unit time? Using the formula time $= \dfrac{\text{distance}}{\text{speed}}$, we see that the time interval, t, between cars is

$$t = \frac{s}{v} + \frac{L}{v}$$

where L is the length of the car. To achieve the maximum flow of cars, we would like t to be as small as possible. (Source: Bolton, 1974)

a Substitute the expression for s into the formula for t, then simplify.

b A typical reaction time is $T = 0.7$ seconds, a typical car length is $L = 5$ meters, and $a = 7.5$ meters per second squared. With these values, graph t as a function of v in the window

Xmin = 0	Xmax = 20
Ymin = 0	Ymax = 3

c To one decimal place, what value of v gives the minimum value of t? Convert your answer to miles per hour.

Project 55 Effective population of endangered species. Many endangered species have fewer than 1000 individuals left. To preserve the species, captive breeding programs must maintain a certain effective population, N, given by

$$N = \frac{4FM}{F + M}$$

where F is the number of breeding females and M the number of breeding males. (Source: Chapman and Reiss, 1992)

a What is the effective population if there are equal numbers of breeding males and females?

b In 1972, a breeding program for Speke's gazelle was established with just three female gazelle. Graph the effective population, N, as a function of the number of males.

c What is the largest effective population that can be created with three females? How many males are needed to achieve the maximum value?

d With three females, for what value of M is $N = M$?

e The breeding program for Speke's gazelle began with only one male. What was the effective population?

Project 56 Biological half-life. When a drug or chemical is injected into a patient, biological processes begin removing that substance. If no more of the substance in introduced, the body removes a fixed fraction of the substance each hour. The amount of substance remaining in the body at time t is an exponential decay function, so there is a **biological half-life** to the substance denoted by T_b. If the substance is a radioisotope, it undergoes radioactive decay and so has a physical half-life as well, denoted T_p.

The **effective half-life**, denoted by T_e, is related to the biological and physical half-lives by the equation

$$\frac{1}{T_e} = \frac{1}{T_b} + \frac{1}{T_p}$$

The radioisotope ^{131}I is used as a label for the human serum albumin. The physical half-life of ^{131}I is 8 days. (Source: Pope, 1989)

a If ^{131}I is cleared from the body with a half-life of 21 days, what is the effective half-life of ^{131}I?

b The biological half-life of a substance varies considerably from person to person. If the biological half-life of ^{131}I is x days, what is the effective half life?

c Let $f(x)$ represent the effective half-life of ^{131}I when the biological half-life is x days. Graph $y = f(x)$.

d What would the biological half-life of ^{131}I need to be to produce an effective half-life of 6 days? Label the corresponding point on your graph.

e For what possible biological half-lives of ^{131}I will the effective half-life be less than 4 days?

Project 57 Rate of eating. Animals spend most of their time hunting or foraging for food to keep themselves alive. Knowing the rate at which an animal (or population of animals) eats can help us determine its metabolic rate or its impact on its habitat. The rate of eating is proportional to the availability of food in the area, but it has an upper limit imposed by mechanical considerations, such as how long it takes the animal to capture and ingest its prey. (Source: Burton, 1998)

a Sketch a graph of eating rate as a function of quantity of available food. This will be a qualitative graph only; you do not have enough information to put scales on the axes.

b Suppose that the rate at which an animal catches its prey is proportional to the number of prey available, or $r_c = ax$, where a is a constant and x is the number of available prey. The rate at which it handles and eats the prey is constant, $r_h = b$. Write expressions for T_c and T_h, the times for catching and handling N prey.

c Show that the rate of food consumption is given by

$$y = \frac{abx}{b + ax} = \frac{bx}{b/a + x}$$

Hint: $y = \dfrac{N}{T}$, where N is the number of prey consumed in the interval T, where $T = T_c + T_h$.

d In a study of ladybirds, it was discovered that larvae in their second stage of development consumed aphids at a rate y_2 aphids per day, given by

$$y_2 = \frac{20x}{x + 16}$$

where x is the number of aphids available. Larvae in the third stage ate at rate y_3, given by

$$y_3 = \frac{90x}{x + 79}$$

Graph both of these functions on the domain $0 \le x \le 140$.

e What is the maximum rate at which ladybird larvae in each stage of development can consume aphids?

Project 58 Buoyancy. A person will float in fresh water if his or her density is less than or equal to 1 kilogram per liter, the density of water. (Density is given by the formula density $= \dfrac{\text{weight}}{\text{volume}}$.) Suppose a swimmer weighs $50 + F$ kilograms, where F is the amount of fat her body contains. (Source: Burton, 1998)

a Calculate the volume of her nonfat body mass if its density is 1.1 kilograms per liter.

b Calculate the volume of the fat if its density is 0.901 kilograms per liter.

c The swimmer's lungs hold 2.6 liters of air. Write an expression for the total volume of her body, including the air in her lungs.

d Write an expression for the density of the swimmer's body.

e Write an equation for the amount of fat needed for the swimmer to float in fresh water.

f Solve your equation. What percent of the swimmer's weight is fat?

g Suppose the swimmer's lungs can hold 4.6 liters of air. What percent body fat does she need to be buoyant?