Appendix A

Algebra Skills Refresher

A.1 Numbers and Operations

A.1.1 Order of Operations

Numerical calculations often involve more than one operation. So that everyone agrees on how such expressions should be evaluated, we follow the **order of operations**.

Order of Operations.

1 Simplify any expressions within grouping symbols (parentheses, brackets, square root bars, or fraction bars). Start with the innermost grouping symbols and work outward.

2 Evaluate all powers and roots.

3 Perform multiplications and divisions in order from left to right.

4 Perform additions and subtractions in order from left to right.

A.1.2 Parentheses and Fraction Bars

We can use parentheses to override the multiplication-first rule. Compare the two expressions below.

The sum of 4 times 6 and 10	$\mathbf{4 \cdot 6} + 10$
4 times the sum of 6 and 10	$4(\mathbf{6 + 10})$

In the first expression, we perform the multiplication 4×6 first, but in the second expression we perform the addition $6 + 10$ first, because it is enclosed in parentheses.

The location (or absence) of parentheses can drastically alter the meaning of an expression. In the following example, note how the location of the parentheses changes the value of the expression.

Example A.1.1

a $5 - 3 \cdot \mathbf{4^2} = 5 - 3 \cdot 16$

$= 5 - 48 = -43$

b $5-(\mathbf{3\cdot 4})^2 = 5-12^2$
$= 5-144 = -139$

c $(5-\mathbf{3\cdot 4})^2 = (5-12)^2$
$= (-7)^2 = 49$

d $(\mathbf{5-3})\cdot 4^2 = 2\cdot 4^2$
$= 2\cdot 16 = 32$

□

Caution A.1.2 In the expression $5-12^2$, which appears in Example A.1.1, p. 851, the exponent 2 applies only to 12, not to -12. Thus, $5-12^2 \neq 5+144$.

The order of operations mentions other grouping devices besides parentheses: fraction bars and square root bars. Notice how the placement of the fraction bar affects the expressions in the next example.

Example A.1.3

a $\dfrac{1+2}{3\cdot 4} = \dfrac{3}{12}$
$= \dfrac{1}{4}$

b $1+\dfrac{2}{3\cdot 4} = 1+\dfrac{2}{12}$
$= 1+\dfrac{1}{6} = \dfrac{7}{6}$

c $\dfrac{1+2}{3}\cdot 4 = \dfrac{3}{3}\cdot 4$
$= 1\cdot 4 = 4$

d $1+\dfrac{2}{3}\cdot 4 = 1+\dfrac{8}{3}$
$= \dfrac{3}{3}+\dfrac{8}{3} = \dfrac{11}{3}$

□

A.1.3 Radicals

You are already familiar with square roots. Every nonnegative number has two square roots, defined as follows:

$$\boldsymbol{s} \quad \text{is a square root of} \quad \boldsymbol{n} \quad \text{if} \quad \boldsymbol{s^2 = n}$$

There are several other kinds of roots, one of which is called the **cube root**, denoted by $\sqrt[3]{n}$. We define the cube root as follows.

Cube Roots.

$$b \quad \text{is a cube root of} \quad n \quad \text{if} \quad b \ \text{cubed equals} \quad n.$$

In symbols, we write

$$\boldsymbol{b = \sqrt[3]{n}} \quad \text{if} \quad \boldsymbol{b^3 = n}$$

Note A.1.4 Although we cannot take the square root of a *negative number*, we can take the *cube root* of *any* real number. For example,

$$\sqrt[3]{64} = 4 \quad \text{because} \quad 4^3 = 64$$

and

$$\sqrt[3]{-27} = -3 \quad \text{because} \quad (-3)^3 = -27$$

In the order of operations, simplifying radicals and powers comes after parentheses but before products and quotients.

Example A.1.5 Simplify each expression.

a $3\sqrt[3]{-8}$ b $2-\sqrt[3]{-125}$ c $\dfrac{6-\sqrt[3]{-27}}{2}$

Solution.

a $3\sqrt[3]{-8}=3(-2)=-6$

b $2-\sqrt[3]{-125}=2-(-5)=7$

c $\dfrac{6-\sqrt[3]{-27}}{2}=\dfrac{6-(-3)}{2}=\dfrac{9}{2}$

□

A.1.4 Scientific Notation

Scientists and engineers regularly encounter very large numbers such as

$$5,980,000,000,000,000,000,000,000$$

(the mass of the Earth in kilograms) and very small numbers such as

$$0.000\,000\,000\,000\,000\,000\,000\,001\,67$$

(the mass of a hydrogen atom in grams). These numbers can be written in a more compact and useful form by using powers of 10.

In our base 10 number system, multiplying a number by a positive power of 10 has the effect of moving the decimal place k places to the right, where k is the exponent in the power of 10. For example,

$$3.529\times 10^{2}=352.9 \quad \text{and} \quad 25\times 10^{4}=250,000$$

Multiplying by a power of 10 with a negative exponent moves the decimal place to the left. For example,

$$1728\times 10^{-3}=1.728 \quad \text{and} \quad 4.6\times 10^{-5}=0.000046$$

Using this property, we can write any number as the product of a number between 1 and 10 (including 1) and a power of 10. For example, the mass of the Earth and the mass of a hydrogen atom can be expressed as

$$5.98\times 1024 \text{ kilograms} \qquad \text{and} \qquad 1.67\times 10^{-24} \text{ gram}$$

respectively. A number written in this form is said to be expressed in **scientific notation**.

To Write a Number in Scientific Notation:.

1 Locate the decimal point so that there is exactly one nonzero digit to its left.

2 Count the number of places you moved the decimal point: This determines the power of 10.

   a If the original number is greater than 10, the exponent is positive.

   b If the original number is less than 1, the exponent is negative.

Example A.1.6 Write each number in scientific notation.

a $478,000 = 4.78000 \times 10^5$ Move the decimal 5 places.
$= 4.78 \times 10^5$

b $0.00032 = 00003.2 \times 10^{-4}$ Move the decimal 4 places.
$= 3.2 \times 10^{-4}$

□

Example A.1.7 The average American eats 110 kilograms of meat per year. It takes about 16 kilograms of grain to produce 1 kilogram of meat, and advanced farming techniques can produce about 6000 kilograms of grain on each hectare of arable land. (The hectare is $10,000$ square meters, or just under $2\frac{1}{2}$ acres.) Now, the total land area of the Earth is about 13 billion hectares, but only about 11% of that land is arable. Is it possible for each of the 7.6 billion people on Earth to eat as much meat as Americans do?

Solution. First we will compute the amount of meat necessary to feed every person on Earth 110 kilograms per year. In 2018 there are 7.6×10^9 people on Earth.

$$(5.5 \times 10^9 \text{ people}) \times (110 \text{ kg/person}) = 8.36 \times 10^{11} \text{ kg of meat}$$

Next we will compute the amount of grain needed to produce that much meat.

$$(16 \text{ kg of grain/kg of meat}) \times (8.36 \times 10^{11} \text{ kg of meat}) = 1.34 \times 10^{13} \text{ kg of grain}$$

Next we will see how many hectares of land are needed to produce that much grain.

$$(1.34 \times 10^{13} \text{ kg of grain}) \div (6000 \text{ kg/hectare}) = 2.23 \times 10^{11} \text{ hectares}$$

Finally, we will compute the amount of arable land available for grain production.

$$0.11 \times (13 \times 10^9 \text{ hectares}) = 1.43 \times 10^9 \text{ hectares}$$

Thus, even if we use every hectare of arable land to produce grain for livestock, we will not have enough to provide every person on Earth with 110 kilograms of meat per year. □

A.1.5 Section Summary

A.1.5.1 Vocabulary

Look up the definitions of new terms in the Glossary.

- Order of operations
- Radical
- Fraction bar
- Cube root
- Scientific notation
- Grouping symbol
- Square root bar

A.1.5.2 SKILLS

Practice each skill in the exercises listed.

1 Follow the order of operations: #1–26

2 Compute cube roots: #27–39

3 Use a calculator to simplify expressions: #31–42

4 Evaluate an expression: #43–50

5 Convert between standard and scientific notation: #51–54

6 Compute using scientific notation: #55–62

A.1.6 Exercises A.1

For Problems 1-26, simplify each expression according to the order of operations.

1. $\dfrac{3(6-8)}{-2} - \dfrac{6}{-2}$

Answer. 6

2. $\dfrac{5(3-5)}{2} - \dfrac{18}{-3}$

3. $6\left[3 - 2(4+1)\right]$

Answer. -42

4. $5\left[3 + 4(6-4)\right]$

5. $(4-3)\left[2 + 3(2-1)\right]$

Answer. 5

6. $(8-6)\left[5 + 7(2-3)\right]$

Answer. 6

7. $64 \div 8\left[4 - 2(3+1)\right]$

Answer. -2

8. $27 \div 3\left[9 - 3(4-2)\right]$

9. $5\left[3 + (8-1)\right] \div (-25)$

Answer. -2

10. $-3\left[-2 + (6-1)\right] \div 9$

Answer. 5

11. $\left[-3(8-2) + 3\right] \cdot [24 \div 6]$

Answer. -60

12. $\left[-2 + 3(5-8)\right] \cdot [-15 \div 3]$

13. -5^2

Answer. -25

14. $(-15)^2$

15. $(-3)^4$

Answer. 81

16. -3^4

17. -4^3

Answer. -64

18. $(-4)^3$

19. $(-2)^5$

Answer. -32

20. -2^5

21. $\dfrac{4 \cdot 2^3}{16} + 3 \cdot 4^2$

Answer. 50

22. $\dfrac{4 \cdot 3^2}{6} + (3 \cdot 4)^2$

23. $\dfrac{3^2 - 5}{6 - 2^2} - \dfrac{6^2}{3^2}$

Answer. -2

24. $\dfrac{3 \cdot 2^2}{4-1} + \dfrac{(-3)(2)^3}{6}$

25. $\dfrac{(-5)^2 - 3^2}{4-6} + \dfrac{(-3)^2}{2+1}$

Answer. -5

26. $\dfrac{7^2 - 6^2}{10+3} - \dfrac{8^2 \cdot (-2)}{(-4)^2}$

For Problems 27-28, compute each cube root. Round your answers to three decimal places if necessary. Verify your answers by cubing them.

27.

a $\sqrt[3]{512}$ b $\sqrt[3]{-125}$ c $\sqrt[3]{-0.064}$ d $\sqrt[3]{1.728}$

Answer.

a 8

b -5

c 0.4

d 1.2

28.

a $\sqrt[3]{9}$ b $\sqrt[3]{258}$ c $\sqrt[3]{-0.002}$ d $\sqrt[3]{-3.1}$

For Problems 29-30, simplify each expression according to the order of operations.

29.

a $\dfrac{4-3\sqrt[3]{64}}{2}$ b $\dfrac{4+\sqrt[3]{-216}}{8-8\sqrt[3]{8}}$

Answer.

a -4

b $\dfrac{-1}{3}$

30.

a $\sqrt[3]{3^3+4^3+5^3}$ b $\sqrt[3]{9^3+10^3-1^3}$

For Problems 31-42, use a calculator to simplify each expression.

31. $\dfrac{-8398}{26\cdot 17}$

Answer. -19

32. $\dfrac{-415.112}{8.58+18.73}$

33. $\dfrac{112.78+2599.124}{27.56}$

Answer. 98.4

34. $\dfrac{202,462-9510}{356}$

35. $\sqrt{24\cdot 54}$

Answer. 36

36. $\sqrt{\dfrac{1216}{19}}$

37. $\dfrac{116-35}{215-242}$

Answer. -3

38. $\dfrac{842-987}{443-385}$

39. $\sqrt{27^2+36^2}$

Answer. 45

40. $\sqrt{13^2-4\cdot 21\cdot 2}$

41. $\dfrac{-27-\sqrt{27^2-4(4)(35)}}{2\cdot 4}$

Answer. -5

42. $\dfrac{13+\sqrt{13^2-4(5)(-6)}}{2\cdot 5}$

For Problems 43-50, evaluate the expression for the given values of the variable. Use your calculator where appropriate.

43. $\dfrac{5(F-32)}{9}$; $F=212$

Answer. 100

44. $\dfrac{a-4s}{1-r}$; $r=2,\ s=12,$ and $a=4$

45. $P+Prt$; $P=1000,\ r=0.04,$ and $t=2$

Answer. 1080

46. $R(1+at)$; $R=2.5,\ a=0.05,$ and $t=20$

47. $\dfrac{1}{2}gt^2-12t$; $g=32$ and $t=\dfrac{3}{4}$

Answer. 0

48. $\dfrac{Mv^2}{g}$; $M = \dfrac{16}{3}$, $a = \dfrac{3}{2}$, and $g = 32$

49. $\dfrac{32(V-v)^2}{g}$; $V = 12.78$, $v = 4.26$, and $g = 32$

Answer. 72.5904

50. $\dfrac{32(V-v)^2}{g}$; $V = 38.3$, $v = -6.7$, and $g = 9.8$

For Problems 51-52, write each number in scientific notation.

51.

a 285

b $8,372,000$

c 0.024

d 0.000523

Answer.

(a) 2.85×10^2

(b) 8.372×10^6

(c) 2.4×10^{-2}

(d) 5.23×10^{-4}

52.

a $68,742$

b $481,000,000,000$

c 0.421

d 0.000004

For Problems 53-54, write each number in standard notation.

53.

a 2.4×10^2

b 6.87×10^{15}

c 5.0×10^{-3}

d 2.02×10^{-4}

Answer.

(a) 240

(b) $6,870,000,000,000,000$

(c) 0.005

(d) 0.000202

54.

a 4.8×10^3

b 8.31×10^{12}

c 8.0×10^{-1}

d 4.31×10^{-5}

For Problems 55-56, compute with the aid of a calculator. Write your answers in standard notation.

55.

a $\dfrac{(2.4 \times 10^{-8})(6.5 \times 10^{32})}{5.2 \times 10^{18}}$

b $\dfrac{(7.5 \times 10^{-13})(3.6 \times 10^{-9})}{(1.5 \times 10^{-15})(1.6 \times 10^{-11})}$

Answer.

(a) $3,000,000$

(b) $112,500$

56.

a $\dfrac{(8.4 \times 10^{-22})(1.6 \times 10^{15})}{3.2 \times 10^{-11}}$ b $\dfrac{(9.4 \times 10^{24})(7.2 \times 10^{-18})}{(4.5 \times 10^{26})(6.4 \times 10^{-16})}$

57. In 2018, the public debt of the United States was over $20,620,000,000,000.

a Express this number in scientific notation.

b If the population of the United States in 2018 was 327,112,000, what was the per capita debt (the debt per person) in 2018?

Answer.

a 2.062×10^{13}

b $63,036.51

58. A light-year is the number of miles traveled by light in 1 year (365 days). The speed of light is approximately 186,000 miles per second.

a Compute the number of miles in 1 light-year, and express your answer in scientific notation.

b The star nearest to the Sun is Proxima Centauri, at a distance of 4.3 light-hears. How long would it take **Pioneer 10** (the first space vehicle to achieve escape velocity from the solar system), traveling at 32,114 miles per hour, to reach Proxima Centauri?

59. The diameter of the galactic disk is about 1.2×10^{18} kilometers, and our Sun lies about halfway from the center of the galaxy to the edge of the disk. The Sun orbits the galactic center once in 240 million years.

a What is the speed of the Sun in its orbit, in kilometers per year?

b What is its speed in meters per second?

Answer.

a 7.9×10^{9} km per year

b 250,000 meters per second

60. Lake Superior has an area of 31,700 square miles and an average depth of 483 feet.

a Find the approximate volume of Lake Superior in cubic feet.

b If 1 cubic foot of water is equivalent to 7.48 gallons, how many gallons of water are in Lake Superior?

61. The average distance from the Earth to the Sun is 1.5×10^{11} meters. The distance from the Sun to Proxima Centauri, the next closest star, is 3.99×10^{16} meters. The most distant star visible to the unaided eye are 2000 times as far away as Proxima Centauri.

a How many times farther is Proxima Centauri from the Sun than the Sun is from Earth?

b How far from the Sun are the most distant visible stars?

Answer.

a 250,000 times

b 8×10^{19} meters

62. The radius of the Earth is 6.37×10^6 meters, and the radius of the Sun is 6.96×10^8 meters. The radii of the other stars range from 1% of the solar radius to 1000 times the solar radius.

a What fraction of the solar radius is the Earth's radius?

b What is the range of stellar radii, in meters?

A.2 Linear Equations and Inequalities

An **equation** is just a mathematical statement that two expressions are equal. Equations relating two variables are particularly useful. If we know the value of one of the variables, we can find the corresponding value of the other variable by solving the equation.

Example A.2.1 The equation $w = 6h$ gives Loren's wages, w, in terms of the number of hours she works, h. How many hours does Loren need to work next week if she wants to earn $225?

Solution. We know that $w = 225$, and we would like to know the value of h. We substitute the value for w into our equation and then solve for h.

$$\begin{aligned} w &= 6h && \text{Substitute 225 for } \boldsymbol{w}. \\ \mathbf{225} &= 6h && \text{Divide both sides by } \mathbf{6}. \\ \frac{225}{\mathbf{6}} &= \frac{6h}{\mathbf{6}} && \text{Simplify.} \\ 375.5 &= h \end{aligned}$$

Loren must work 37.5 hours in order to earn $225. In reality, Loren will probably have to work for 38 hours, because most employers do not pay for portions of an hour's work. Thus, Loren needs to work for 38 hours. □

To solve an equation we can generate simpler equations that have the same solutions. Equations that have identical solutions are called **equivalent equations**. For example,

$$3x - 5 = x + 3$$

and

$$2x = 8$$

are equivalent equations because the solution of each equation is 4. Often we can find simpler equivalent equations by undoing in reverse order the operations performed on the variable.

A.2.1 Solving Linear Equations

Linear, or first-degree, equations can be written so that every term is either a constant or a constant times the variable. The equations above are examples of linear equations. Recall the following rules for solving linear equations.

To Generate Equivalent Equations.

1 We can add or subtract the *same* number on *both* sides of an equation.

2 We can multiply or divide *both* sides of an equation by the *same*

number (except zero).

Applying either of these rules produces a new equation equivalent to the old one and thus preserves the solution.

We use the rules to isolate the variable on one side of the equation.

Example A.2.2 Solve the equation $3x - 5 = x + 3$.

Solution. We first collect all the variable terms on one side of the equation, and the constant terms on the other side.

$$\begin{aligned} 3x - 5 - x &= x + 3 - x && \text{Subtract } x \text{ from both sides.} \\ 2x - 5 &= 3 && \text{Simplify.} \\ 2x - 5 + 5 &= 5 + 5 && \text{Add 5 to both sides.} \\ 2x &= 8 && \text{Simplify.} \\ \frac{2x}{2} &= \frac{8}{2} && \text{Divide both sides by 2.} \\ x &= 4 && \text{Simplify.} \end{aligned}$$

The solution is 4. (You can check the solution by substituting 4 into the original equation to show that a true statement results.) □

The following steps should enable you to solve any linear equation. Of course, you may not need all the steps for a particular equation.

To Solve a Linear Equation:.

1 Simplify each side of the equation separately.

  a Apply the distributive law to remove parentheses.

  b Collect like terms.

2 By adding or subtracting appropriate terms on both sides of the equation, get all the variable terms on one side and all the constant terms on the other.

3 Divide both sides of the equation by the coefficient of the variable.

Example A.2.3 Solve $3(2x - 5) - 4x = 2x - (6 - 3x)$.

Solution. We begin by simplifying each side of the equation.

$$\begin{aligned} 3(2x - 5) - 4x &= 2x - (6 - 3x) && \text{Apply the distributive law.} \\ 6x - 15 - 4x &= 2x - 6 + 3x && \text{Combine like terms on each side.} \\ 2x - 15 &= 5x - 6 \end{aligned}$$

Next, we collect all the variable terms on the left side of the equation, and all the constant terms on the right side.

$$\begin{aligned} 2x - 15 - 5x + 15 &= 5x - 6 - 5x + 15 && \text{Add } -5x + 15 \text{ to both sides.} \\ -3x &= 9 \end{aligned}$$

Finally, we divide both sides of the equation by the coefficient of the variable.

$$\begin{aligned} -3x &= 9 && \text{Divide both sides by } -3. \\ x &= -3 \end{aligned}$$

The solution is -3. □

A.2.2 Formulas

A **formula** is an equation that relates several variables. For example, the equation

$$P = 2l + 2w$$

gives the perimeter of a rectangle in terms of its length and width.

Suppose we have some wire fence to enclose an exercise area for rabbits, and we would like to see what dimensions are possible for different rectangles with that perimeter. In this case, it would be more useful to have a formula for the length of the rectangle in terms of its perimeter and its width. We can find such a formula by solving the perimeter formula for l in terms of P and w.

$$\begin{aligned} 2l + 2w &= P && \text{Subtract } \mathbf{2w} \text{ from both sides.} \\ 2l &= P - 2w && \text{Divide both sides by 2.} \\ l &= \frac{P - 2w}{2} \end{aligned}$$

The result is a new formula that gives the length of a rectangle in terms of its perimeter and its width.

Example A.2.4 The formula $5F = 9C + 160$ relates the temperature in degrees Fahrenheit, F, to the temperature in degrees Celsius, C. Solve the formula for C in terms of F.

Solution. We begin by isolating the term that contains C.

$$\begin{aligned} 5F &= 9C + 160 && \text{Subtract 160 from both sides.} \\ 5F - 160 &= 9C && \text{Divide both sides by 9.} \\ \frac{5F - 160}{9} &= C \end{aligned}$$

We can also write the formula for C in terms of F as $C = \frac{5}{9}F - \frac{160}{9}$. □

Example A.2.5 Solve $3x - 5y = 40$ for y in terms of x.

Solution. We isolate y on one side of the equation.

$$\begin{aligned} 3x - 5y &= 40 && \text{Subtract } \mathbf{3x} \text{ from both sides.} \\ -5y &= 40 - 3x && \text{Divide both sides by } \mathbf{-5}. \\ \frac{-5y}{-5} &= \frac{40 - 3x}{-5} && \text{Simplify both sides.} \\ y &= -8 + \frac{3}{5}x \end{aligned}$$

□

A.2.3 Linear Inequalities

The symbol $>$ is called an **inequality symbol**, and the statement $a > b$ is called an **inequality**. There are four inequality symbols:

$>$	is greater than
$<$	is less than
$\geq$	is greater than or equal to
$\leq$	is less than or equal to

Inequalities that include the symbols $>$ or $\leq$ are called **strict inequalities**; those that include $\geq$ or $\leq$ are called **nonstrict**.

If we multiply or divide both sides of an inequality by a negative number, the direction of the inequality must be reversed. For example, if we multiply both sides of the inequality

$$2 < 5$$

by -3, we get

$$\begin{aligned} \mathbf{-3}(2) &> \mathbf{-3}(5) && \text{Change inequality symbol from } < \text{ to } > . \\ -6 &> -15 \end{aligned}$$

Because of this property, the rules for solving linear equations must be revised slightly for solving linear inequalities.

To Solve a Linear Inequality:.

1 We may add or subtract the same number to both sides of an inequality without changing its solutions.

2 We may multiply or divide both sides of an inequality by a *positive* number without changing its solutions.

3 If we multiply or divide both sides of an inequality by a *negative* number, we must *reverse the direction of the inequality symbol.*

Example A.2.6 Solve the inequality $4 - 3x \geq -17$.

Solution. Use the rules above to isolate x on one side of the inequality.

$$\begin{aligned} 4 - 3x &\geq -17 && \text{Subtract 4 from both sides.} \\ -3x &\geq -21 && \text{Divide both sides by } \mathbf{-3.} \\ x &\leq 7 \end{aligned}$$

Notice that we reversed the direction of the inequality when we divided by -3. Any number less than or equal to 7 is a solution of the inequality. □

A **compound inequality** involves two inequality symbols.

Example A.2.7 Solve $4 \leq 3x + 10 \leq 16$.

Solution. We isolate x by performing the same operations on all three sides of the inequality.

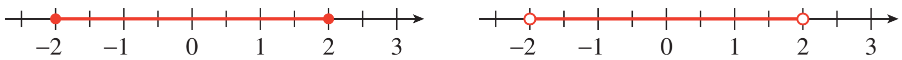
$$\begin{aligned} 4 \leq 3x + 10 &\leq 16 && \text{Subtract } \mathbf{10.} \\ -6 \leq 3x &\leq 6 && \text{Divide by } \mathbf{3.} \\ -2 \leq x &\leq 2 \end{aligned}$$

The solutions are all numbers between -2 and 2, inclusive. □

A.2.4 Interval Notation

The solutions of the inequality in Example A.2.7, p. 862 form an interval. An **interval** is a set that consists of all the real numbers between two numbers a and b.

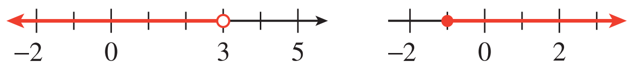
The set $-2 \leq x \leq 2$ includes its endpoints -2 and 2, so we call it a **closed interval**, and we denote it by $[-2, 2]$. Its graph is shown in figure (a). The square brackets tell us that the endpoints are included in the interval. An interval that does not include its endpoints, such as $-2 < x < 2$, is called an **open interval**, and we denote it with round brackets, $(-2, 2)$. Its graph is shown in figure (b).



(a) **(b)**

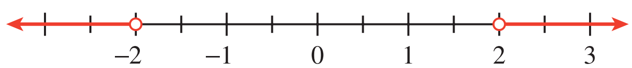
Caution A.2.8 Do not confuse the open interval $(-2, 2)$ with the point $(-2, 2)$! The notation is the same, so you must decide from the context whether an interval or a point is being discussed.

We can also discuss **infinite intervals**, such as $x < 3$ and $x \geq -1$, shown in the figure below. We denote the interval $x < 3$ by $(-\infty, 3)$, and the interval $x \geq -1$ by $[-1, \infty)$. The symbol ∞, for infinity, does not represent a specific real number; it indicates that the interval continues forever along the real line.



(a) **(b)**

Finally, we can combine two or more intervals into a larger set. For example, the set consisting of $x < -1$ or $x > 2$, shown below, is the **union** of two intervals and is denoted by $(-\infty, -2) \cup (2, \infty)$.



Many solutions of inequalities are intervals or unions of intervals.

Example A.2.9 Write each of the solution sets with interval notation and graph the solution set on a number line.

a $3 \leq x < 6$

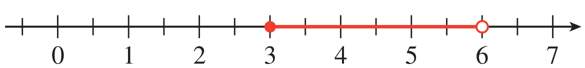
b $x \geq -9$

c $x \leq 1$ or $x > 4$

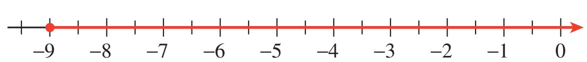
d $-8 < x \leq -5$ or $-1 \leq x < 3$

Solution.

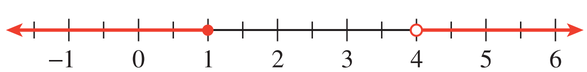
a $[3, 6)$. This is called a **half-open** or **half-closed** interval.



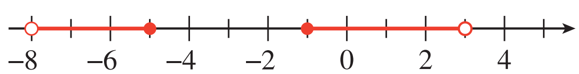
b $[-9, \infty)$. We always use round brackets next to the symbol ∞ because ∞ is not a specific number and is not included in the set.



c $(-\infty, 1] \cup (4, \infty)$. The word *or* describes the union of two sets.



d $(-8, -5] \cup [-1, 3)$.



□

A.2.5 Section Summary

A.2.5.1 Vocabulary

Look up the definitions of new terms in the Glossary.

- Equation
- Closed interval
- Compound inequality
- Inequality
- Linear equation
- Equivalent equation
- Open interval
- Interval
- Strict inequality
- Formula
- Solve an equation
- Union

A.2.5.2 SKILLS

Practice each skill in the exercises listed.

1 Solve a linear equation: #1–10

2 Solve a formula for one variable in terms of the others: #11–28

3 Solve a linear inequality: #26–34

4 Solve a compound inequality: #35–41

5 Write solutions to inequalities in interval notation: #41–50

A.2.6 Exercises A.2

For Problems 1-10, solve the linear equation.

1. $3x + 5 = 26$

Answer. 7

2. $2 + 5x = 37$

3. $3(z + 2) = 37$

Answer. $\dfrac{31}{3}$

4. $2(z - 3) = 15$

5. $3y - 2(y - 4) = 12 - 5y$

Answer. $\dfrac{2}{3}$

6. $5y - 3(y + 1) = 14 + 2y$

Answer. 6

7. $0.8w - 2.6 = 1.4w + 0.3$

Answer. $-4.8\overline{3}$

8. $4.8 - 1.3w = 0.7w + 2.1$

9. $0.25t + 0.10(t - 4) = 11.60$

Answer. 34.29

10. $0.12t + 0.08(t + 10,000) = 12,000$

For problems 11-20, solve for y in terms of x.

11. $4x + 3y = -2$

Answer. $y = \dfrac{-2}{3} - \dfrac{4x}{3}$

12. $x - 2y = -7$

13. $\dfrac{x}{8} - \dfrac{y}{2} = 1$

Answer. $y = \dfrac{x}{4} - 2$

14. $\dfrac{x}{5} + \dfrac{y}{7} = 1$

15. $3x + \frac{2}{7}y = 1$

Answer. $y = \frac{7}{2} - \frac{21x}{2}$

16. $\frac{5}{6}x + 8y = 1$

17. $-(x-1) = 6(y-3)$

Answer. $y = \frac{19}{6} - \frac{x}{6}$

18. $2y - 4 = 3(x+5)$

19. $\frac{y+8}{x-1} = \frac{-7}{4}$

Answer. $y = \frac{-25}{4} - \frac{7x}{4}$

20. $\frac{2}{3} = \frac{y-5}{x+2}$

For Problems 21-28, solve the formula for the specified variable.

21. $v = k + gt$, for t

Answer. $t = \frac{v-k}{g}$

22. $S = 3\pi d + \pi a$, for d

23. $S = 2w(w+2h)$, for h

Answer. $h = \frac{S - 2w^2}{4w}$

24. $A = P(1+rt)$, for r

25. $P = a + (n-1)d$, for n

Answer. $n = \frac{P-a+d}{d}$

26. $R = 2d + h(a+b)$, for b

27. $A = \pi rh + \pi r^2$, for h

Answer. $h = \frac{A - \pi r^2}{\pi r}$

28. $A = 2w^2 + 4lw$, for l

For Problems 29-40, solve the inequality.

29. $3x - 2 > 1 + 2x$

Answer. $x > 3$

30. $2x + 3 \le x - 1$

31. $\frac{-2x-6}{-3} > 2$

Answer. $x > 0$

32. $\frac{-2x-3}{2} \le -5$

33. $\frac{2x-3}{3} \le \frac{3x}{-2}$

Answer. $x \le \frac{6}{13}$

34. $\frac{3x-4}{-2} > \frac{-2x}{5}$

35. $-6 < 4x + 10 < 20$

Answer. $-4 < x < \frac{5}{2}$

36. $3 < -2x - 5 < 15$

37. $-9 \le -3x + 6 < 2$

Answer. $\frac{4}{3} \le x \le 5$

38. $4 < 8x + 12 \le 16$

39. $5 < \frac{8-2x}{4} \le 7$

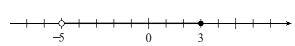
Answer. $-10 \le x \le -6$

40. $-1 \le \frac{4x-6}{-3} \le 0$

For Problems 41-50, write the set with interval notation, and graph the set on a number line.

41. $-5 < x \le 3$

Answer. $(-5, 3]$



42. $0 \le x < 4$

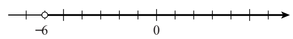
43. $0 \geq x \geq -4$

Answer. $[-4, 0]$

44. $8 > x > 5$

45. $x > -6$

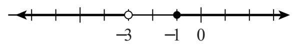
Answer. $(-6, \infty)$



46. $x \leq 1$

47. $x < -3$ or $x \geq -1$

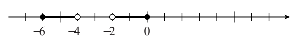
Answer.
$(-\infty, -3) \cup [-1, \infty)$



48. $x \geq 3$ or $x \leq -3$

49. $-6 \leq x < -4$ or $-2 < x \leq 0$

Answer. $[-6, -4) \cup (-2, 0]$



50. $x < 2$ or $2 < x < 3$

A.3 Algebraic Expressions and Problem Solving

You are familiar with the use of letters, or **variables**, to stand for unknown numbers in equations or formulas. Variables are also used to represent numerical quantities that change over time or in different situations. For example, p might stand for the atmospheric pressure at different heights above the Earth's surface. Or N might represent the number of people infected with cholera t days after the start of an epidemic.

An **algebraic expression** is any meaningful combination of numbers, variables, and symbols of operation. Algebraic expressions are used to express relationships between variable quantities.

Example A.3.1 Loren makes $6 an hour working at the campus bookstore.

a Choose a variable for the number of hours Loren works per week.

b Write an algebraic expression for the amount of Loren's weekly earnings.

Solution.

a Let h stand for the number of hours Loren works per week.

b The amount Loren earns is given by

$$\mathbf{6} \times (\text{number of hours Loren worked})$$

or $6 \cdot h$. Loren's weekly earnings can be expressed as $6h$.

□

The algebraic expression $6h$ represents the amount of money Loren earns *in terms of* the number of hours she works. If we substitute a specific value for the variable in an expression, we find a numerical value for the expression. This is called **evaluating** the expression.

Example A.3.2 If Loren from Example A.3.1, p. 866 works for 16 hours in the bookstore this week, how much will she earn?

Solution. Evaluate the expression $6h$ for $h = \mathbf{16}$.

$$6h = 6(\mathbf{16}) = 96$$

Loren will make \$96. □

Example A.3.3 April sells environmentally friendly cleaning products. Her income consists of \$200 per week plus a commission of 9% of her sales.

a Choose variables to represent the unknown quantities and write an algebraic expression for April's weekly income in terms of her sales.

b Find April's income for a week in which she sells \$350 worth of cleaning products.

Solution.

a Let I represent April's total income for the week, and let S represent the total amount of her sales. We translate the information from the problem into mathematical language as follows:

Her income consists of	**\$200...** plus	**...9%**	of her sales
I	$= 200$	$+ 0.09$	S

Thus, $I = 200 + 0.09S$.

b We want to evaluate our expression from part (a) with $S = 350$. We substitute **350** for S to find

$$I = 200 + 0.09(\mathbf{350})$$

Following the order of operations, we perform the multiplication before the addition. Thus, we begin by computing 0.09(350).

$$\begin{aligned} I &= 200 + 0.09(350) && \text{Multiply } \mathbf{0.09(350)} \text{ first.} \\ &= 200 + 31.5 \\ &= 231.50 \end{aligned}$$

April's income for the week is \$231.50.

□

Remark A.3.4 Calculator Tip. On a scientific or a graphing calculator, we can enter the expression from Example A.3.3, p. 867 just as it is written:

$$200 + 0.09 \boxed{\times} 350 \boxed{\text{ENTER}}$$

The calculator will perform the operations in the correct order -- multiplication first.

Example A.3.5 Economy Parcel Service charges \$2.80 per pound to deliver a package from Pasadena to Cedar Rapids. Andrew wants to mail a painting that weighs 8.3 pounds, plus whatever packing material he uses.

a Choose variables to represent the unknown quantities and write an expression for the cost of shipping Andrew's painting.

b Find the shipping cost if Andrew uses 2.9 pounds of packing material.

Solution.

a Let C stand for the shipping cost and let w stand for the weight of the packing material. Andrew must find the total weight of his package first, then multiply by the shipping charge.

The total weight of the package is $8.3 + w$ pounds. We use parentheses around this expression to show that it should be computed first, and the sum should be multiplied by the shipping charge of \$2.80 per pound. Thus,

$$C = 2.80(8.3 + w)$$

b Evaluate the formula from part (a) with $w = \mathbf{2.9}$.

$$\begin{aligned} C &= 2.80(8.3 + \mathbf{2.9}) && \text{Add inside parentheses.} \\ &= 2.80(11.2) && \text{Multiply.} \\ &= 31.36 \end{aligned}$$

The cost of shipping the painting is \$31.36.

□

Remark A.3.6 Calculator Tip. On a calculator, we enter the expression for C in the order it appears, including the parentheses. (Experiment to see whether your calculator requires you to enter the × symbol after 2.80.) The keying sequence

$$2.80 \times (8.3 + 2.9) \;\; \boxed{\text{ENTER}}$$

gives the correct result, 31.36.

Caution A.3.7 If we omit the parentheses, the calculator will perform the multiplication before the addition. Thus, the keying sequence

$$2.80 \times 8.3 + 2.9$$

gives an incorrect result for Example A.3.5, p. 867. (The sequence

$$8.3 + 2.9 \times 2.80$$

does not work either!)

A.3.1 Problem Solving

Problem solving often involves translating a real-life problem into a computer programming language, or, in our case, into algebraic expressions. We can then use algebra to solve the mathematical problem and interpret the solution in the context of the original problem. Here are some guidelines for problem solving with algebraic equations.

Guidelines for Problem Solving.

1. Identify the unknown quantity and assign a variable to represent it.
2. Find some quantity that can be expressed in two different ways and write an equation.
3. Solve the equation.
4. Interpret your solution to answer the question in the problem.

In step 1, begin by writing an English phrase to describe the quantity you are looking for. Be as specific as possible -- if you are going to write an equation about this quantity, you must understand its properties! Remember that your variable must represent a numerical quantity. For example, x can represent the *speed* of a train, but not just ``the train.''

Writing an equation is the hardest part of the problem. Note that the quantity mentioned in step 2 will probably *not* be the same unknown quantity you are looking for, but the algebraic expressions you write *will* involve your variable. For example, if your variable represents the *speed* of a train, your equation might be about the *distance* the train traveled.

A.3.2 Supply and Demand

The law of supply and demand is fundamental in economics. If you increase the price of a product, the supply increases because its manufacturers are willing to provide more of the product, but the demand decreases because consumers are not willing to buy as much at a higher price. The price at which the demand for a product equals the supply is called the **equilibrium price**.

Example A.3.8 The Coffee Connection finds that when it charges p dollars for a pound of coffee, it can sell $800 - 60p$ pounds per month. On the other hand, at a price of p dollars a pound, International Food and Beverage will supply the Connection with $175 + 40p$ pounds of coffee per month. What price should the Coffee Connection charge for a pound of coffee so that its monthly inventory will sell out?

Solution.

1. We are looking for the equilibrium price, p.

2. The Coffee Connection would like the demand for its coffee to equal its supply. We equate the expressions for supply and for demand to obtain the equation
$$800 - 60p = 175 + 40p$$

3. Solve the equation. To get all terms containing the variable, p, on one side of the equation, we add $60p$ to both sides and subtract 175 from both sides to obtain
$$\begin{aligned} 800 - 60p + \mathbf{60p - 175} &= 175 + 40p + \mathbf{60p - 175} \\ 625 &= 100p && \text{Divide both sides by } \mathbf{100.} \\ 6.25 &= p \end{aligned}$$

4. The Coffee Connection should charge \$6.25 per pound for its coffee.

□

A.3.3 Percent Problems

Recall the basic formula for computing percents.

Percent Formula.
$P = rW$
the **P**art (or percent) = the percentage **r**ate × the **W**hole Amount

A **percent increase** or **percent decrease** is calculated as a fraction of the *original* amount. For example, suppose you make \$16.00 an hour now, but next month you are expecting a 5% raise. Your new salary should be

$$\overset{\text{Original salary}}{\$16.00} + \overset{\text{Increase}}{0.05(\$16.00)} = \overset{\text{New Salary}}{\$16.80}$$

Example A.3.9 The price of housing in urban areas increased 4% over the past year. If a certain house costs $100,000 today, what was its price last year?

Solution.

1. Let c represent the cost of the house last year.
2. Express the current price of the house in two different ways. During the past year, the price of the house increased by 4%, or $0.04c$. Its current price is thus

$$\overset{\text{Original cost}}{(1)c} + \overset{\text{Price increase}}{0.04c} = c(1+0.04) = 1.04c$$

This expression is equal to the value given for current price of the house:

$$1.04c = 100,000$$

3. To solve this equation, we divide both sides by 1.04 to find

$$c = \frac{100,000}{1.04} = 96,153.846$$

4. To the nearest cent, the cost of the house last year was $96,153.85.

□

Caution A.3.10 In Example A.3.9, p. 870, it would be incorrect to calculate last year's price by subtracting 4% of $100,000 from $100,000 to get $96,000. (Do you see why?)

A.3.4 Weighted Averages

We find the **average**, or **mean**, of a set of values by adding up the values and dividing the sum by the number of values. Thus, the average, $\overline{x}$, of the numbers $x_1, x_2, \ldots, x_n$ is given by

$$\overline{x} = \frac{x_1 + x_2 + \cdots + x_n}{n}$$

In a **weighted average**, the numbers being averaged occur with different frequencies or are weighted differently in their contribution to the average value. For instance, suppose a biology class of 12 students takes a 10-point quiz. Of the 12 students, 2 receive 10s, three receive 9s, 5 receive 8s, and 2 receive scores of 6. The average score earned on the quiz is then

$$\overline{x} = \frac{\mathbf{2}(10) + \mathbf{3}(9) + \mathbf{5}(8) + \mathbf{2}(6)}{12} = 8.25$$

The numbers in color are called the weights -- in this example they represent the number of times each score was counted. Note that n, the total number of scores, is equal to the sum of the weights:

$$12 = 2 + 3 + 5 + 2$$

Example A.3.11 Kwan's grade in his accounting class will be computed as follows: Tests count for 50% of the grade, homework counts for 20%, and the final exam counts for 30%. If Kwan has an average of 84 on tests and 92 on homework, what score does he need on the final exam to earn a grade of 90?

Solution.

1. Let x represent the final exam score Kwan needs.

2. Kwan's grade is the weighted average of his test, homework, and final exam scores.
$$\frac{\mathbf{0.50}(84) + \mathbf{0.20}(92) + \mathbf{0.30}x}{1.00} = 90$$
(The sum of the weights is 1.00, or 100% of Kwan's grade.) Multiply both sides of the equation by 1.00 to get
$$0.50(84) + 0.20(92) + 0.30x = 1.00(90)$$

3. Solve the equation. Simplify the left side first.
$$\begin{aligned} 60.4 + 0.30x &= 90 && \text{Subtract 60.4 from both sides.} \\ 0.30x &= 29.6 && \text{Divide both sides by 0.30.} \\ x &= 98.7 \end{aligned}$$

4. Kwan needs a score of 98.7 on the final exam to earn a grade of 90.

□

In step 2 of Example A.3.11, p. 871, we rewrote the formula for a weighted average in a simpler form.

Weighted Average.

The sum of the weighted values equals the sum of the weights times the average value. In symbols,
$$w_1x_1 + w_2x_2 + \cdots + w_nx_n = W\,\overline{x}$$
where W is the sum of the weights.

This form is particularly useful for solving problems involving mixtures.

Example A.3.12 The vet advised Delbert to feed his dog Rollo with kibble that is no more than 8% fat. Rollo likes JuicyBits, which are 15% fat. LeanMeal is more expensive, but it is only 5% fat. How much LeanMeal should Delbert mix with 50 pounds of JuicyBits to make amixture that is 8% fat?

Solution.

1. Let p represent the number of pounds of LeanMeal needed.

2. In this problem, we want the weighted average of the fat contents in the two kibbles to be 8%. The weights are the number of pounds of each kibble we use. It is often useful to summarize the given information in a table.

	% fat	Total pounds	Pounds of fat
Juicy Bits	15%	50	0.15(50)
LeanMeal	5%	p	$0.05p$
Mixture	8%	$50 + p$	$0.08(50 + p)$

The amount of fat in the mixture must come from adding the amounts of fat in the two ingredients. This gives us an equation,

$$0.15(50) + 0.05p = 0.08(50 + p)$$

This equation is an example of the formula for weighted averages.

3. Simplify each side of the equation, using the distributive law on the right side, then solve.

$$\begin{aligned} 7.5 + 0.05p &= 4 + 0.08p && \text{Subtract } \mathbf{4 + 0.05}\boldsymbol{p} \text{ from both sides.} \\ 3.5 &= 0.03p && \text{Divid both sides by } \mathbf{0.03.} \\ p &= 116.\overline{6} \end{aligned}$$

4. Delbert should mix $116\frac{2}{3}$ pounds of LeanMeal with 50 pounds of JuicyBits to make a mixture that is 8% fat.

□

A.3.5 Section Summary

A.3.5.1 Vocabulary

Look up the definitions of new terms in the Glossary.

- Variable
- Weighted average
- Demand
- Equilibrium price
- Supply
- Algebraic expression
- Evaluate an expression

A.3.5.2 SKILLS

Practice each skill in the exercises listed.

1 Write an algebraic expression: #1–12

2 Evaluate an algebraic expression: #1–12

3 Write and solve an equation to solve a problem: #13–28

A.3.6 Exercises A.3

For Problems 1-12, write algebraic expressions to describe the situation and then evaluate for the given values.

1. Jim was 27 years old when Ana was born.

a Write an expression for Jim's age in terms of Ana's age.

b Use your expression to find Jim's age when ana is 22 years old.

Answer.

a $j = a + 27$ b 49

2. Rani wants to replace the wheels of her in-line skates. New wheels cost \$6.59 each.

a Write an expression for the total cost of new wheels in terms of the number of wheels Rani must replace.

b Use your expression to find the total cost if Rani must replace 8 wheels.

3. Helen decides to drive to visit her father. The trip is a distance of 1260 miles.

a Write an expression for the total number of hours Helen must drive in terms of her average driving speed.

b Use your expression to find how long Helen must drive if she averages 45 miles per hour.

Answer.

a $h = \dfrac{1260}{r}$ b 28 hours

4. Ben will inherit one million dollars on his twenty-first birthday.

a Write an expression for the number of years before Ben gets his inheritance in terms of his present age.

b Use your expression to find how many more years Ben must wait after he turns 13 years old.

5. The area of a circle is equal to π times the square of its radius.

a Write an expression for the area of a circle in terms of its radius.

b Find the area of a circle whose radius is 5 centimeters.

Answer.

a $A = \pi r^2$ b 78.54 sq cm

6. The volume of a sphere is equal to $\frac{4}{3}\pi$ times the cube of the radius.

a Write an expression for the volume of a sphere in terms of its radius.

b Find the volume of a sphere whose radius is 5 centimeters.

7. The sales tax in the city of Preston is 7.9%.

a Write an expression for the total bill for an item (price plus tax) in terms of the price of the item.

b Find the total bill for an item whose price is \$490.

Answer.

a $b = 1.079p$ b \$528.71

8. A savings account pays 6.4% annual interest on the amount deposited.

a Write an expression for the balance (initial deposit plus interest) in the account after one year in terms of the amount deposited.

b Find the total amount in the account after one year if \$350 was deposited.

9. Your best friend moves to another state. To call her, a long-distance phone call costs \$1.97 plus \$0.39 for each minute.

a Write an expression for the cost of a long-distance phone call in terms of the number of minutes of the call.

b Find the cost of a 27-minute phone call.

Answer.

a $C = 1.97 + 0.39m$ b \$12.50

10. Arenac Airplines charges 47 cents per pound on its flight from Omer to Pinconning, both for passengers and for luggage. Mr. Owsley wants to take the flight with 15 pounds of luggage.

a Write an expression for the cost of the flight in terms of Mr. Owlsley's weight.

b Find the cost if Mr. Owsley weights 162 pounds.

11. Juan buys a 50-pound bag of rice and consumes about 0.4 pound per week.

a Write an expression for the amount of rice Juan has consumed in terms of the number of weeks since he bought the bag.

b Write an expression for the amount of rice Juan has left in terms of the number of weeks since he bought the bag.

c Find the amount of rice Juan has left after 6 weeks.

Answer.

a $c = 0.4w$ c 47.6 lbs

b $r = 50 - 0.4w$

12. Trinh is bicycling down a mountain road that loses 500 feet in elevation for each 1 mile of road. She started at an elevation of 6300 feet.

a Write an expression for the elevation that Trinh has lost in terms of the distance she has cycled.

b Write an expression for Trinh's elevation in terms of the number of miles she has cycled.

c Find Trinh's elevation after she has cycled 9 miles.

For Problems 13-28, write and solve an equation to answer the question.

13. Celine's boutique carries a line of jewelry made by a local artists' co-op. If Celine charges p dollars for a pair of earrings, she finds that she can sell $200 - 5p$ pairs per month. On the other hand, the co-op will provide her with $56 + 3p$ pairs of earrings when she charges p dollars per pair. What price should Celine charge so that the demand for earrings will equal her supply?

Answer. \$18

14. Curio Electronics sells garage door openers. If it charges p dollars per unit, it sells $120 - p$ openers per month. The manufacturer will supply $20 + 2p$ openers at a price of p dollars each. What price should Curio Electronics charge so that its monthly supply will meet its demand?

15. Roger sets out on a bicycle trip at an average speed of 16 miles per hour. Six hours later, his wife finds his patch kit on the dining room table. If she heads after him in the car at 45 miles per hour, how long will it be before she catches him?

a What are we asked to find in this problem? Assign a variable to represent it.

b Write an expression in terms of your variable for the distance Roger's wife drives.

c Write an expression in terms of your variable for the distance Roger has cycled.

d Write an equation and solve it.

Answer.

a $t =$ time wife drives

b $45t$

c $16(t + 6)$

d 3.3 hours

16. Kate and Julie set out in their sailboat on a straight course at 9 miles per hour. Two hours later, their mother becomes worried and sends their father after them in the speedboat. If their father travels at 24 miles per hour, how long will it be before he catches them?

a What are we asked to find in this problem? Assign a variable to represent it.

b Write an expression in terms of your variable for the distance Kate and Julie sailed.

c Write an expression in terms of your variable for the distance their father traveled.

d Write an equation and solve it.

17. The reprographics department has a choice of 2 new copying machines. One sells for $20,000 and costs $0.02 per copy to operate. The other sells for $17,500, but its operating costs are $0.025 per copy. The repro department decides to buy the more expensive machine. How many copies must the repro department make before the higher price is justified?

a What are we asked to find in this problem? Assign a variable to represent it.

b Write an expression in terms of your variable for the total cost incurred by each machine.

c Write an equation and solve it.

Answer.

a $n =$ number of copies

b $20,000 + 0.02n$, $17,500 + 0.025m$

c 500,000 copies

18. Annie needs a new refrigerator and can choose between two models of the same size. One model sells for $525 and costs $0.08 per hour to run. A more energy-efficient model sells for $700 but runs for $0.05

per hour. If Annie buys the more expensive model, how long will it be before she starts saving money?

a What are we asked to find in this problem? Assign a variable to represent it.

b Write an expression in terms of your variable for the total cost incurred by each refrigerator.

c Write an equation and solve it.

19. The population of Midland has been growing at an annual rate of 8% over the past 5 years. Its present population is 135,000.

a Assuming the same rate of growth, what do you predict for the population of Midland next year?

b What was the population of Midland last year?

Answer.

a 145,800 b 125,000

20. For the past 3 years, the annual inflation rate has been 6%. This year, a steak dinner at Benny's costs $12.

a Assuming the same rate of inflation, what do you predict for the price of a steak dinner next year?

b What did a steak dinner cost last year?

21. Virginia took a 7% pay cut when she changed jobs last year. What percent pay increase must she receive this year in order to match her old salary of $24,000?

Hint. What was Virginia's salary after the pay cut?

Answer. 7.53%

22. Clarence W. Networth took a 16% loss in the stock market last year. What percent gain must he realize this year in order to restore his original holdings of $85,000?

Hint. What was the value of Clarence's stock holdings after the loss?

23. Delbert's test average in algebra is 77. If the final exam counts for 30% of the grade and the test average counts for 70%, what must Delbert score on the final exam to have a term average of 80?

Answer. 87

24. Harold's batting average for the first 8 weeks of the baseball season is 0.385. What batting average must he maintain over the last 18 weeks so that his season average will be 0.350 (assuming he continues the same number of at-bats per week)?

25. A horticulturist needs a fertilizer that is 8% potash, but she can find only fertilizers that contain 6% and 15% potash. How much of each should she mix to obtain 10 pounds of 8% potash fertilizer?

Pounds of fertilizer	% potash	Pounds of potash

a What are we asked to find in this problem? Assign a variable to represent it.

b Write algebraic expressions in terms of your variable for the amounts of each fertilizer the horticulturist uses. Use the table.

c Write expressions for the amount of potash in each batch of fertilizer.

d Write two different expressions for the amount of potash in the mixture. Now write an equation and solve it.

Answer.

a $x =$ amount of 6% fertilizer

b x; $10 - x$

c $0.06x$; $0.15(10 - x)$

d $0.06x + 0.15(10 - x) = 0.08(10)$; $7.\overline{7}$ lbs of 6%, $2.\overline{2}$ lbs of 15%

26. A sculptor wants to cast a bronze statue from an alloy that is 60% copper. He has 30 pounds of a 45% alloy. How much 80% copper alloy should he mix with it to obtain the 60% copper alloy?

Pounds of alloy	% copper	Pounds of copper

a What are we asked to find in this problem? Assign a variable to represent it.

b Write algebraic expressions in terms of your variable for the amounts of each alloy the sculptor uses. Use the table.

c Write expressions for the amount of copper in each batch of alloy.

d Write two different expressions for the amount of copper in the mixture. Now write an equation and solve it.

27. Lacy's Department Stores wants to keep the average salary of its employees under \$19,000 per year. If the downtown store pays its 4 managers \$28,000 per year and its 12 department heads \$22,000 per year, how much can it pay its 30 clerks?

a What are we asked to find in this problem? Assign a variable to represent it.

b Write algebraic expressions for the total amounts Lacy's pays its managers, its department heads, and its clerks.

c Write two different expressions for the total amount Lacy's pays in salaries each year.

d Write an equation and solve it.

Answer.

a $x =$ clerk's salary

b $4 \times 28,000$; $12 \times 22,000$; $19,000(4 + 12 + x)$

c $4 \times 28,000 + 12 \times 22,000 + 30x$; $19,000(4 + 12 + x)$

d $4 \times 28,000 + 12 \times 22,000 + 30x = 19,000(4 + 12 + x)$; \$16,600

28. Federal regulations require that 60% of all vehicles manufactured next year comply with new emission standards. Major Motors can bring 85% of its small trucks in line with the standards, but only 40% of its automobiles. If Major Motors plans to manufacture 20,000 automobiles next year, how many trucks will it have to produce in order to comply with the federal regulations?

a What are we asked to find in this problem? Assign a variable to represent it.

b Write algebraic expressions for the number of trucks and the number of cars that will meet emission standards.

c Write two different expressions for the total number of vehicles that will meet the standards.

d Write an equation and solve it.

A.4 Graphs and Equations

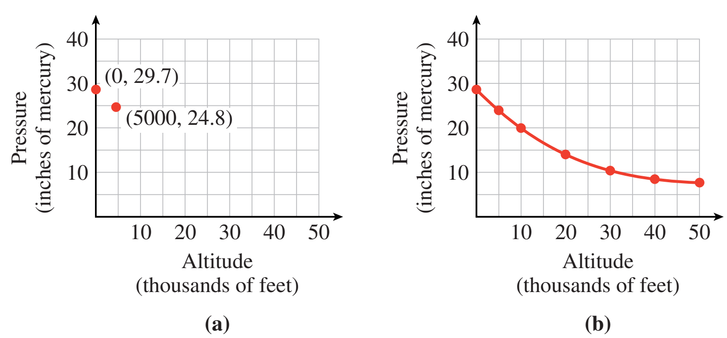
Graphs are useful tools for studying mathematical relationships. A graph provides an overview of a quantity of data, and it helps us identify trends or unexpected occurrences. Interpreting the graph can help us answer questions about the data.

For example, here are some data showing the atmospheric pressure at different altitudes. Altitude is given in feet, and atmospheric pressure is given in inches of mercury.

Altitude (ft)	0	5000	10,000	20,000	30,000	40,000	50,000
Pressure (in. Hg)	29.7	24.8	20.5	14.6	10.6	8.5	7.3

We observe a generally decreasing trend in pressure as the altitude increases, but it is difficult to say anything more precise about this relationship. A clearer picture emerges if we plot the data. To do this, we use two perpendicular number lines called axes. We use the horizontal axis for the values of the first variable, altitude, and the vertical axis for the values of the second variable, pressure.

The entries in the table are called **ordered pairs**, in which the **first component** is the altitude and the **second component** is the atmospheric pressure measured at that altitude. For example, the first two entries can be represented by $(0, 29.7)$ and $(5000, 24.8)$. We plot the points whose **coordinates** are given by the ordered pairs, as shown in the figure on the left.



We can connect the data points with a smooth curve as shown in the figure on the right. In doing this, we are assuming that one variable changes smoothly with respect to the other, and in fact this is true for many physical situations. Thus, a smooth curve will thus serve as a good model.

A.4.1 Reading a Graph

Once we have constructed a graph, we can use it to estimate values of the variables between the known data points.

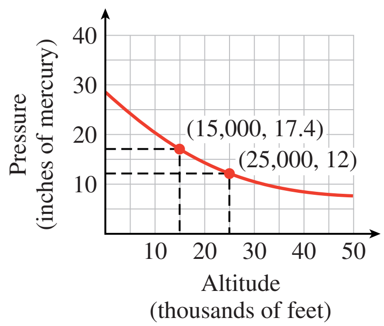
Example A.4.1 From the graph of atmospheric pressure, estimate the following:

a The atmospheric pressure measured at an altitude of $15,000$ feet

b The altitude at which the pressure is 12 inches of mercury

Solution.

a The point with first coordinate $15,000$ on the graph at right has second coordinate approximately 17.4. We estimate the pressure at $15,000$ feet to be 17.4 inches of mercury.



b The point on the graph with second coordinate 12 has first coordinate approximately $25,000$, so an atmospheric pressure of 12 inches of mercury occurs at about $25,000$ feet.

□

We can also use the graph to obtain information about the relationship between altitude and pressure that would be difficult to see from the data alone.

Example A.4.2

a For what altitudes is the pressure less than 18 inches of mercury?

b How much does the pressure decrease as the altitude increases from $15,000$ feet to $25,000$ feet?

c For which $10,000$-foot increase in altitude does the pressure change most rapidly?

Solution.

a From the graph we see that the pressure has dropped to 18 inches of mercury at about $14,000$ feet, and that it continues to decrease as the altitude increases. Therefore, the pressure is less than 18 inches of mercury for altitudes greater than $14,000$ feet.

b The pressure at $15,000$ feet is approximately 17.4 inches of mercury, and at $25,000$ feet it is 12 inches. This represents a decrease in pressure of $17.4 - 12$, or 5.4, inches of mercury.

c By studying the graph we see that the pressure decreases most rapidly at low altitudes, so we conclude that the greatest drop in pressure occurs between 0 and $10,000$ feet.

□

A.4.2 Graphs of Equations

In Example A.4.1, p. 879, we used a graph to illustrate data given in a table. Graphs can also help us analyze models given by equations. Let's first review some facts about solutions of equations in two variables.

An equation in two variables, such as y = 2x + 3, is said to be satisfied if the variables are replaced by a pair of numbers that make the statement true. The pair of numbers is called a **solution** of the equation and is usually written as an ordered pair (x, y). (The first number in the pair is the value of x and the second number is the value of y.)

To find a solution of a given equation, we can assign a number to one of the variables and then solve for the second variable.

Example A.4.3 Find solutions to the equation $y = 2x + 3$.

Solution. We choose some values for x, say, -2, 0, and 1. Substitute these x-values into the equation to find a corresponding y-value for each.

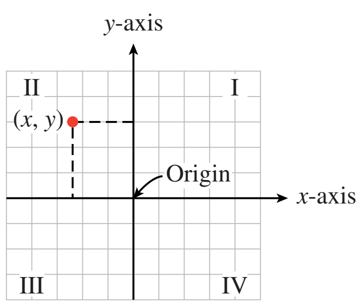
When $x = -2$, $\quad y = 2(\mathbf{-2}) + 3 = -1$
When $x = 0$, $\quad y = 2(\mathbf{0}) + 3 = 3$
When $x = 1$, $\quad y = 2(\mathbf{1}) + 3 = 5$

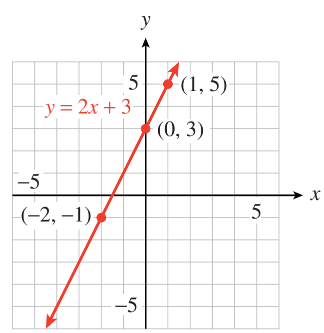
Thus, the ordered pairs $(-2, -1)$, $(0, 3)$, and $(1, 5)$ are three solutions of $y = 2x + 3$. We can also substitute values for y. For example, if we let $y = \mathbf{10}$, we have

$$\mathbf{10} = 2x + 3$$

Solving this equation for x, we find $7 = 2x$, or $x = 3.5$. This means that the ordered pair $(3.5, 10)$ is another solution of the equation $y = 2x + 3$. ☐

An equation in two variables may have infinitely many solutions, so we cannot list them all. However, we can display the solutions on a graph. For this we use a **Cartesian** (or **rectangular**) **coordinate system**, as shown below left.





The **graph of an equation** is a picture of its solutions. A point is included in the graph if its coordinates satisfy the equation, and if the coordinates do not satisfy the equation, the point is not part of the graph. A graph of $y = 2x + 3$ is shown above right.

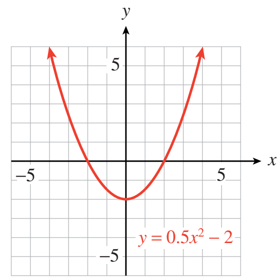
This graph does not display *all* the solutions of the equation, but it shows important features such as the intercepts on the x- and y-axes. Because there is a solution corresponding to every real number x, the graph extends infinitely in either direction, as indicated by the arrows.

Example A.4.4

Use the graph of $y = 0.5x^2 - 2$ shown at right to decide whether the given ordered pairs are solutions of the equation. Verify your answers algebraically.

a $(-4, 6)$

b $(3, 0)$



Solution.

a Because the point $(-4, 6)$ does lie on the graph, the ordered pair $x = -4, y = 6$ is a solution of $y = 0.5x^2 - 2$. We can verify this by substituting $\mathbf{-4}$ for x and $\mathbf{6}$ for y:

$$\begin{aligned} 0.5(\mathbf{-4})^2 - 2 &= 0.5(16) - 2 \\ &= 8 - 2 = \mathbf{6} \end{aligned}$$

b Because the point $(3, 0)$ does not lie on the graph, the ordered pair $x = 3, y = 0$ is not a solution of $y = 0.5x^2 - 2$. We substitute $\mathbf{3}$ for x and $\mathbf{0}$ for y to verify this.

$$\begin{aligned} 0.5(\mathbf{3})^2 - 2 &= 0.5(9) - 2 \\ &= 4.5 - 2 = 2.5 \neq \mathbf{0} \end{aligned}$$

□

A.4.3 Section Summary

A.4.3.1 Vocabulary

Look up the definitions of new terms in the Glossary.

- Ordered pair
- Component
- Cartesian coordinate system
- Solution
- Equation in two variables
- Satisfy an equation
- Coordinate
- Axis
- Graph

A.4.3.2 SKILLS

Practice each skill in the exercises listed.

1 Read values from a graph: #1–4

2 Find solutions to an equation in two variables: #5–8

3 Make a table of values from an equation: #9–12

4 Make a table of values from a graph: #13–16

5 Estimate values from a graph: #17–20

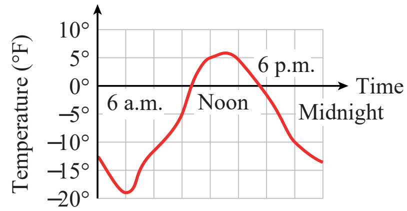
6 Use the Trace feature on a calculator: #21–24

7 Find solutions to an equation in two variables from a graph: #25–32

A.4.4 Exercises A.4

For Problems 1-4, answer the questions about the graph.

1. The graph shows the temperatures recorded during a winter day in Billings, Montana.



a What were the high and low temperatures recorded during the day?

b During what time intervals is the temperature above 5°F? Below −5°F?

c Estimate the temperatures at 7 a.m. and 2 p.m. At what time(s) is the temperature approximately 0°F? Approximately −12°F?

d How much did the temperature increase between 3 a.m. and 6 a.m.? Between 9 a.m. and noon? How much did the temperature decrease between 6 p.m. and 9 p.m.?

e During which 3-hour interval did the temperature increase most rapidly? Decrease most rapidly?

Answer.

a High: 7°F; Low: −19°F

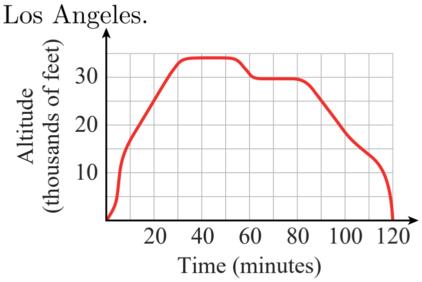
b Above 5°F from noon to 3 p.m.; Below −5°F from midnight to 9 a.m. and from 7 p.m. to midnight

c 7 a.m.: −10°F; 2 p.m.: 6°F; 10 a.m. and 5 p.m.: 0°F; 6 a.m. and 10 p.m.: −12°F

d Between 3 a.m. and 6 a.m.: 6°F; Between 9 a.m. and noon: 10°F; Between 6 p.m. and 9 p.m.: 9°F

e Increased most rapidly: 9 a.m. to noon; Decreased most rapidly: 6 p.m. to 9 p.m.

2. The graph shows the altitude of a commercial jetliner during its flight from Denver to Los Angeles.



a What was the highest altitude the jet achieved? At what time(s) was this altitude recorded?

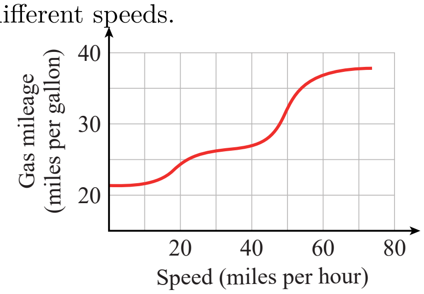
b During what time intervals was the altitude greater than 10,000 feet? Below 20,000 feet?

c Estimate the altitudes 15 minutes into the flight and 35 minutes into the flight. At what time(s) was the altitude approximately 16,000 feet? 32,000 feet?

d How many feet did the jet climb during the first 10 minutes of flight? Between 20 minutes and 30 minutes? How many feet did the jet descend between 100 minutes and 120 minutes?

e During which 10-minute interval did the jet ascend most rapidly? Descend most rapidly?

3. The graph shows the gas mileage achieved by an experimental model automobile at different speeds.



a Estimate the gas mileage achieved at 43 miles per hour.

b Estimate the speed at which a gas mileage of 34 miles per gallon is achieved.

c At what speed is the best gas mileage achieved? Do you think that the gas mileage will continue to improve as the speed increases? Why or why not?

d The data illustrated by the graph were collected under ideal test conditions. What factors might affect the gas mileage if the car were driven under more realistic conditions?

Answer.

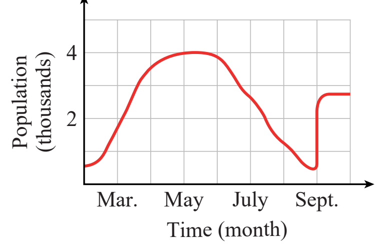
a 28 mpg

b 50 mph

c Best gas mileage at 70 mph. The graph seems to be leveling off for higher speeds; any improvement in mileage probably would not be significant, and the mileage might in fact deteriorate.

d Road condition, weather conditions, traffic, weight in the car

4. The graph shows the fish population of a popular fishing pond.



a During what months do the young fish hatch?

b During what months is fishing allowed?

c When does the park service restock the pond?

For Problems 5-8, find five solutions (ordered pairs) for the equation.

5. $y = 4 - \dfrac{x}{3}$

Answer.
$(-6, 6),\ (-3, 5),\ (0, 4),\ (3, 3),\ (6, 2)$

6. $\dfrac{x-5}{2} + 1 = y$

7. $3x^2 - 1 = y$

Answer.
$(-2, -11),\ (-1, 2),\ (0, -1),\ (1, 2),\ (2, 11)$

8. $y = 9 - (x-2)^2$

For Problems 9-12, fill in the table of values for the given equation.

9. $3x + 2y = 1$

x	y
-3	
0	
	0
1	
	-2
	-4

Answer.

x	y
-3	**5**
0	**$\frac{1}{2}$**
$\frac{1}{3}$	0
1	**-1**
$\frac{5}{3}$	-2
3	-4

10. $5y - 3x = 1$

x	y
-2	
	-1
	0
0	
1	
	3

11. $y = 1 - \dfrac{x}{4}$

x	y
-4	
	1
3	
	0
5	
	-1

Answer.

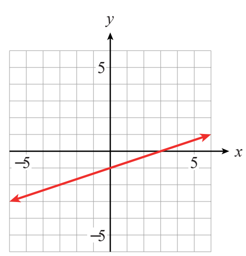
x	y
-4	**2**
0	1
3	**$\frac{1}{4}$**
4	0
5	**$\frac{-1}{4}$**
8	-1

12. $\dfrac{x+7}{3} = y$

x	y
-2	
0	
	3
5	
	5
	7

For Problems 13-16, fill in the table of values for the graph.

13.

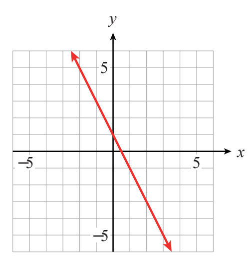


x	y
-6	
	-1
0	
	0
	1

Answer.

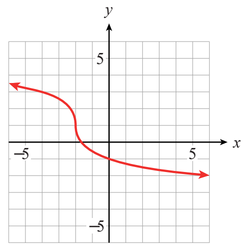
x	y
-6	**-3**
0	-1
0	**-1**
3	0
6	1

14.



x	y
	5
-1	
0	
	-1
	-5

15.

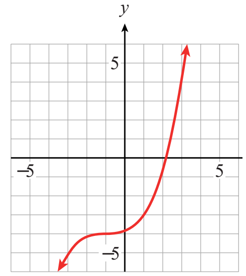


x	y
	3
	1
0	
5	

Answer.

x	y
−4	3
−2	1
0	**−1**
5	**−2**

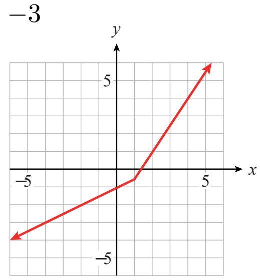
16.



x	y
−3	
−1	
	−3
	4

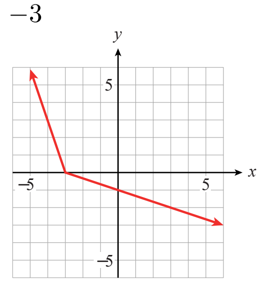
For Problems 17-20, estimate from the graph, any values of x with the given value of y.

17. $y = -3$

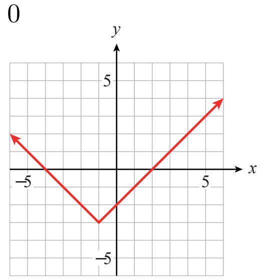


Answer. -4

18. $y = -3$

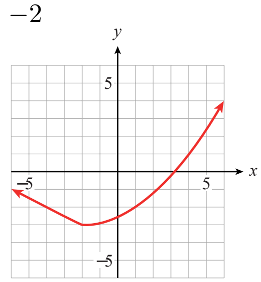


19. $y = 0$



Answer. $-4,\ 2$

20. $y = -2$



For Problems 21-24, graph the equation in the given friendly window. Use the calculator's *Trace* feature to make a table of values. (See Appendix B, p. 977 for help with entering expressions.) Round y-values to three decimal places.

21. $y = \left||x+2| - |x-2|\right|$
Xmin $= -4.7$; Xmax $= 4.7$
Ymin $= -6.2$; Ymax $= 6.2$

x	−3.2	−1.5	0.1	1.9	
y					

Answer.

x	−3.2	−1.5	0.1	1.9	**2.5**	**3**
y	**4**	**3**	**0.2**	**3.8**	**4**	**4**

22. $y = \left|x^2 - x - 2\right|$
Xmin $= -4.7$; Xmax $= 4.7$
Ymin $= -9.3$; Ymax $= 9.3$

x	−3.1	−1.5	0.5	1.5		
y						

23. $y = \dfrac{x-2}{x+2}$
Xmin $= -4.7$; Xmax $= 4.7$
Ymin $= -9.3$; Ymax $= 9.3$

x	-3	-2.2	-2	4		
y						

Answer.

x	-3	-2.2	-2	4	**6**	**8**
y	**2**	**21**	undefined	$\mathbf{0.3\overline{3}}$	**0.5**	**0.6**

24. $y = \sqrt{x^2 - 1.96}$
Xmin $= -4.7$; Xmax $= 4.7$
Ymin $= -6.2$; Ymax $= 6.2$

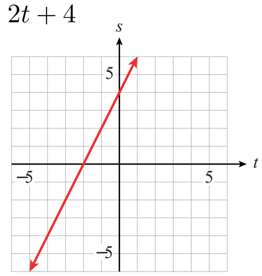
x	-3.0	-1.4	0.1	1.9		
y						

For Problems 25–32,

a Use the graph to find the missing component in each solution of the equation.

b Verify your answers algebraically.

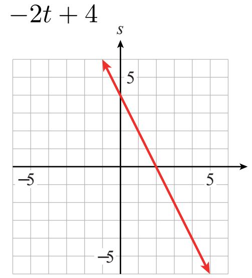
25. $s = 2t + 4$



$(-3,\ ?)\quad(1,\ ?)\quad(?,\ 0)\quad(?,\ 4)$

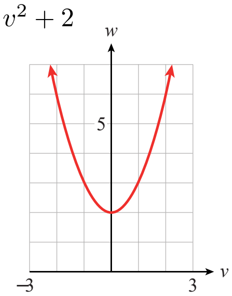
Answer.
$(-3,-2),\ (1,6),\ (-2,0),\ (0,4)$

26. $s = -2t + 4$



$(-2,\ ?)\quad(3,\ ?)\quad(?,\ 0)\quad(?,\ 4)$

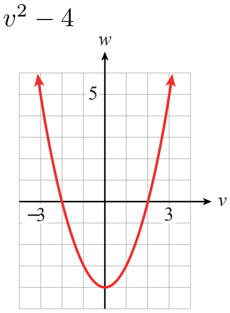
27. $w = v^2 + 2$



$(-2,\ ?)\quad(2,\ ?)\quad(?,\ 3)\quad(?,\ 2)$

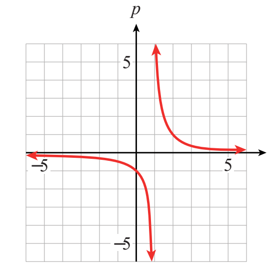
Answer.
$(-2,6),\ (2,6),\ (1,3)$ or $(-1,3),\ (0,2)$

28. $w = v^2 - 4$



$(-1,\ ?)\quad(3,\ ?)\quad(?,\ 0)\quad(?,\ -4)$

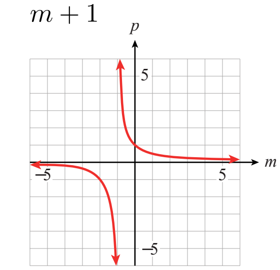
29. $p = \dfrac{1}{m-1}$



$(-1,\ ?)$ $\left(\dfrac{1}{2},\ ?\right)$ $\left(?,\ \dfrac{1}{3}\right)$ $(?, -1)$

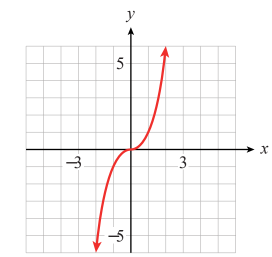
Answer.
$\left(-1, -\dfrac{1}{2}\right)$, $\left(\dfrac{1}{2}, -2\right)$, $\left(4,\ \dfrac{1}{3}\right)$ $(0, -1)$

30. $p = \dfrac{1}{m+1}$



$\left(\dfrac{-3}{2},\ ?\right)$ $(3,\ ?)$ $(?, -1)$ $(?,\ 2)$

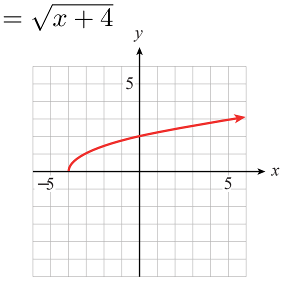
31. $y = x^3$



$(-2,\ ?)$ $\left(\dfrac{1}{2},\ ?\right)$ $(?,\ 0)$ $(?, -1)$

Answer.
$(-2, -8)$ $\left(\dfrac{1}{2}, \dfrac{1}{8}\right)$ $(0,\ 0)$ $(-1, -1)$

32. $y = \sqrt{x+4}$



$(0,\ ?)$ $(5,\ ?)$ $(?,\ 0)$ $(?,\ 1)$

A.5 Linear Systems in Two Variables

A 2×2 **system** of equations is a set of 2 equations in the same 2 variables. A **solution** of a 2×2 system is an ordered pair that makes each equation in the system true. In this section, we review two algebraic methods for solving 2×2 linear systems: substitution and elimination.

A.5.1 Solving Systems by Substitution

The basic strategy for the **substitution** method can be described as follows.

Steps for Solving a 2×2 System by Substitution.

1 Solve one of the equations for one of the variables in terms of the other.

2 Substitute this expression into the second equation; doing so yields an equation in one variable.

3 Solve the new equation.

4 Use the result of step 1 to find the other variable.

Example A.5.1 Staci stocks two kinds of sleeping bags in her sporting goods store, a standard model and a down-filled model for colder temperatures. From past experience, she estimates that she will sell twice as many of the standard variety as of the down filled. She has room to stock 60 sleeping bags at a time. How many of each variety should Staci order?

Solution.

1.

Number of standard sleeping bags: x
Number of down-filled sleeping bags: y

2. Write two equations about the variables. Staci needs twice as many standard model as down filled, so

$$x = 2y \qquad (1)$$

Also, the total number of sleeping bags is 60, so

$$x + y = 60 \qquad (2)$$

3. We will solve this system using substitution. Notice that Equation (1) is already solved for x in terms of y: $x = 2y$. Substitute $\mathbf{2y}$ for x in Equation (2) to obtain

$$\begin{aligned} \mathbf{2y} + y &= 60 \\ 3y &= 60 \end{aligned}$$

Solving for y, we find $y = \mathbf{20}$. Finally, substitute this value into Equation (1) to find

$$x = 2(\mathbf{20}) = 40$$

The solution to the system is $x = 40, y = 20$.

4. Staci should order 40 standard sleeping bags and 20 down-filled bags.

□

A.5.2 Solving Systems by Elimination

The method of substitution is convenient if one of the variables in the system has a coefficient of 1 or -1, because it is easy to solve for that variable. If none of the coefficients is 1 or -1, then a second method, called **elimination**, is usually more efficient.

The method of elimination is based on the following properties of linear equations.

Properties of Linear Systems.

1 Multiplying a linear equation by a (nonzero) constant does not change its solutions. That is, any solution of the equation

$$ax + by = c$$

is also a solution of the equation

$$kax + kby = kc$$

2 Adding (or subtracting) two linear equations does not change their common solutions. That is, any solution of the system

$$\begin{aligned} a_1x + b_1y &= c_1 \\ a_2x + b_2y &= c_2 \end{aligned}$$

is also a solution of the equation

$$(a_1 + a_2)x + (b_1 + b_2)y = c_1 + c_2$$

Example A.5.2 Solve the system by the method of elimination.

$$\begin{aligned} 2x + 3y &= 8 \qquad (1) \\ 3x - 4y &= -5 \qquad (2) \end{aligned}$$

Solution. We first decide which variable to eliminate, x or y. We can choose whichever looks easiest. In this problem, we choose to eliminate x.

We next look for the smallest number that both coefficients, 2 and 3, divide into evenly. This number is 6. We want the coefficients of x to become 6 and -6, so we will multiply Equation (1) by 3 and Equation (2) by -2 to obtain

$$\begin{aligned} 6x + 9y &= 24 \qquad (1a) \\ -6x + 8y &= 10 \qquad (2a) \end{aligned}$$

Now we add the corresponding terms of (1a) and (2a). The x-terms are eliminated, yielding an equation in one variable.

$$\begin{aligned} 6x + 9y &= 24 \qquad (1a) \\ -6x + 8y &= 10 \qquad (2a) \\ 17y &= 34 \qquad (3) \end{aligned}$$

We solve this equation for y to find $y = \mathbf{2}$. We can substitute this value of y into any of our equations involving both x and y. If we choose Equation (1), then

$$2x + 3(\mathbf{2}) = 8$$

and solving this equation yields $x = 1$. The ordered pair $(1, 2)$ is a solution to the system. You should verify that these values satisfy both original equations.

□

We summarize the strategy for solving a linear system by elimination.

Steps for Solving a 2×2 Linear System by Elimination.

1 Choose one of the variables to eliminate. Multiply each equation by a suitable factor so that the coefficients of that variable are opposites.

2 Add the two new equations termwise.

3 Solve the resulting equation for the remaining variable.

4 Substitute the value found in step 3 into either of the original equations and solve for the other variable.

In Example A.5.2, p. 889, we added 3 times the first equation to -2 times the second equation. The result from adding a constant multiple of one equation to a constant multiple of another equation is called a **linear combination** of the two equations. The method of elimination is also called the method of linear combinations.

If either equation in a system has fractional coefficients, it is helpful to clear the fractions before applying the method of linear combinations.

Example A.5.3 Solve the system by linear combinations.

$$\begin{aligned} \frac{2}{3}x - \quad y &= 2 && (1) \\ x + \frac{1}{2}y &= 7 && (2) \end{aligned}$$

Solution. Multiply each side of Equation (1) by 3 and each side of Equation (2) by 2 to clear the fractions:

$$\begin{aligned} 2x - 3y &= \ \ 6 && (1a) \\ 2x + \ \ y &= 14 && (2a) \end{aligned}$$

To eliminate the variable x, multiply Equation (2a) by -1 and add the result to Equation (1a) to get

$$\begin{aligned} -4y &= -8 && \text{Divide both sides by -4.} \\ y &= 2 \end{aligned}$$

Substitute **2** for y in one of the original equations and solve for x. We use Equation (2).

$$\begin{aligned} x + \frac{1}{2}(\mathbf{2}) &= 7 && \text{Subtract 1 from both sides.} \\ x &= 6 \end{aligned}$$

Verify that $x = 6$ and $y = 2$ satisfy both Equations (1) and (2). The solution to the system is the ordered pair $(6, 2)$. □

A.5.3 Section Summary

A.5.3.1 Vocabulary

Look up the definitions of new terms in the Glossary.

- System of equations
- Dependent
- Linear combination
- Inconsistent
- Elimination
- Solution of a system
- Substitution

A.5.3.2 SKILLS

Practice each skill in the exercises listed.

1 Solve a system by substitution: #1–4

2 Solve a system by elimination: #5–8

3 Choose a method and solve the system: #9–18, 29–32

4 Solve problems by writing and solving a system: #19–28

A.5.4 Exercises A.5

For Problems 1-4, solve the system by substitution.

1. $a + 2b = -6$
$2a - 3b = 16$
Answer. $(2, -4)$

2. $7r - 4s = 1$
$3r + s = 14$

3. $2x - 3y = 6$
$x + 3y = 3$
Answer. $(3, 0)$

4. $2r = s + 7$
$2s = 14 - 3r$

For Problems 5-8, solve the system by elimination.

5. $5x - 2y = -4$
$-6x + 3y = 5$
Answer. $\left(\dfrac{-2}{3}, \dfrac{1}{3}\right)$

6. $2p + 3q = 38$
$6p - 5q = 2$

7. $3x - 4y = -11$
$2x + 6y = -3$
Answer. $\left(-3, \dfrac{1}{2}\right)$

8. $2u - 3v = -4$
$5u + 2v = 9$

For Problems 9–12, solve the system by substitution or by linear combinations.

9. $3m + n = 7$
$2m = 5n - 1$
Answer. $(2, 1)$

10. $3x + 5y = 1$
$2x - 3y = 7$

11. $3y = 2x - 8$
$4y + 11 = 3x$
Answer. $(1, -2)$

12. $4L - 3 = 3W$
$25 + 5L = -2W$

In Problems 13–18, clear the fractions in each equation first, then solve the system by substitution or by linear combinations.

13. $\dfrac{2}{3}A - B = 4$
$A - \dfrac{3}{4}B = 6$
Answer. $(6, 0)$

14. $\dfrac{1}{8}w - \dfrac{3}{8}z = 1$
$\dfrac{1}{2}w - \dfrac{1}{4}z = -1$

15. $\dfrac{M}{4} = \dfrac{N}{3} - \dfrac{5}{12}$
$\dfrac{N}{5} = \dfrac{1}{2} - \dfrac{M}{10}$
Answer. $(1, 2)$

16. $\dfrac{R}{3} = \dfrac{S}{3} + 2$
$\dfrac{S}{3} = \dfrac{R}{6} - 1$

17. $\dfrac{s}{2} = \dfrac{7}{6} - \dfrac{t}{3}$
$\dfrac{s}{4} = \dfrac{3}{4} - \dfrac{t}{4}$
Answer. $(1, 2)$

18. $\dfrac{2p}{3} + \dfrac{8q}{9} = \dfrac{4}{3}$
$\dfrac{p}{3} = 2 + \dfrac{q}{2}$

In Problems 19–28, write a system of equations for the problem, then solve algebraically.

19. Francine has \$2000, part of it invested in bonds paying 10%, and the rest in a certificate account at 8%. Her annual income from the two investments is \$184. How much did Francine invest at each rate?

a Choose variables for the unknown quantities, and fill in the table.

	Principal	Interest rate	Interest
Bonds			
Certificate			
Total		——	

b Write one equation about the amount Francine invested.

c Write a second equation about Francine's annual interest.

d Solve the system and answer the question in the problem.

Answer.

a

	Principal	Interest rate	Interest
Bonds	x	0.10	$0.10x$
Certificate	y	0.08	$0.08y$
Total	$x+y$	——	$0.10x+0.08y$

b $x+y=2000$

c $0.10x+0.08y=184$

d \$800 at 8%, \$1200 at 10%

20. Carmella has \$1200 invested in two stocks; one returns 8% per year, and the other returns 12% per year. The income from the 8% stock is \$3 more than the income from the 12% stock. How much did Carmella invest in each stock?

a Choose variables for the unknown quantities, and fill in the table.

	Principal	Interest rate	Interest
First stock			
Second stock			
Total		——	——

b Write one equation about the amount Carmella invested.

c Write a second equation about Carmella's annual interest.

d Solve the system and answer the question in the problem.

21. Paul needs 40 pounds of 48% silver alloy to finish a collection of jewelry. How many pounds of 45% silver alloy should he melt with 60% silver alloy to obtain the alloy he needs?

a Choose variables for the unknown quantities, and fill in the table.

	Pounds	% silver	Amount of silver
First alloy			
Second alloy			
Mixture			

b Write one equation about the amount of alloy Paul needs.

c Write a second equation about the amount of silver in the alloys.

d Solve the system and answer the question in the problem.

Answer.

a

	Pounds	% silver	Amount of silver
First alloy	x	0.45	$0.45x$
Second alloy	y	0.60	$0.60y$
Mixture	$x+y$	——	$0.45x+0.60y$

b $x+y=40$

c $0.45x+0.60y=0.48(40)$

d $32 lb

22. Amal plans to make 10 liters of a 17% acid solution by mixing a 20% acid solution with a 15% acid solution. How much of each should she use?

a Choose variables for the unknown quantities, and fill in the table.

	Liters	% acid	Amount of acid
First solution			
Second solution			
Mixture			

b Write one equation about the amount of solution Amal needs.

c Write a second equation about the acid in the solution.

d Solve the system and answer the question in the problem.

23. Delbert answered 13 true-false and 9 fill-in questions correctly on his last test and got a score of 71. If he had answered 9 true-false and 13 fill-ins correctly, he would have made an 83. How many points was each type of problem worth?

Answer. True-false: 2 points; fill-ins: 5 points

24. In a recent election, 7179 votes were cast for the two candidates. If 6 votes had been switched from the winner to the loser, the loser would have won by 1 vote. How many votes were cast for each candidate?

25. Because of prevailing winds, a flight from Detroit to Denver, a distance of 1120 miles, takes 4 hours on Econoflite, while the return trip takes 3.5 hours. What were the speed of the airplane and the speed of the wind?

a Choose variables for the unknown quantities, and fill in the table.

	Rate	Time	Distance
Detroit to Denver			
Denver to Detroit			

b Write one equation about the trip from Detroit to Denver.

c Write a second equation about the return trip.

d Solve the system and answer the question in the problem.

Answer.

a

	Rate	Time	Distance
Detroit to Denver	$x+y$	4	1120
Denver to Detroit	$x+y$	3.5	1120

b $4(x-y)=1120$

c $3.5(x+y)=1120$

d Airplane: 300 mph; wind: 20 mph

26. On a breezy day, Bonnie propelled her human-powered aircraft 100 meters in 15 seconds going into the wind and made the return trip in 10 seconds with the wind. What were the speed of the wind and Bonnie's speed in still air?

a Choose variables for the unknown quantities, and fill in the table.

	Rate	Time	Distance
Against the wind			
With the wind			

b Write one equation about Bonnie's initial flight.

c Write a second equation about Bonnie's return trip.

d Solve the system and answer the question in the problem.

27. A cup of rolled oats provides 310 calories. A cup of rolled wheat flakes provides 290 calories. A new breakfast cereal combines wheat and oats to provide 302 calories per cup. How much of each grain does 1 cup of the cereal include?

a Choose variables for the unknown quantities, and fill in the table.

	Cups	Calories per cup	Calories
Oat flakes			
Wheat flakes			
Mixture		——	

b Write one equation about the amounts of each grain.

c Write a second equation about the number of calories.

d Solve the system and answer the question in the problem.

Answer.

a

	Cups	Calories per cup	Calories
Oat flakes	x	310	$310x$
Wheat flakes	y	290	$290y$
Mixture	$x+y$	——	$310x+290y$

b $x+y=1$

c $310x+290y=302$

d 0.6 cup oats, 0.4 cup wheat

28. Acme Motor Company is opening a new plant to produce chassis for two of its models, a sports coupe and a wagon. Each sports coupe requires a riveter for 3 hours and a welder for 4 hours; each wagon requires a riveter for 4 hours and a welder for 5 hours. The plant has available 120 hours of riveting and 155 hours of welding per day. How many of each model of chassis can it produce in a day?

a Choose variables for the unknown quantities, and fill in the table.

	Sports coupes	Wagons	Total
Hours of riveting			
Hours of welding			

b Write one equation about the hours of riveting.

c Write a second equation about the hours of welding.

d Solve the system and answer the question in the problem.

For Problems 19-28, use a calculator to solve the system.

29. $4.8x - 3.5y = 5.44$
$2.7x + 1.3y = 8.29$
Answer. $(2.3, 1.6)$

30. $6.4x + 2.3y = -14.09$
$-5.2x - 3.7y = -25.37$

31. $0.9x = 25.78 + 1.03y$
$0.25x + 0.3y = 85.7$
Answer. $(182, 134)$

32. $0.02x = 0.6y - 78.72$
$1.1y = -0.4x + 108.3$

A.6 Laws of Exponents

In this section, we review the rules for performing operations on powers.

A.6.1 Product of Powers

Consider a product of two powers with the same base.

$$(a^3)(a^2) = aaa \cdot aa = a^5$$

because a occurs as a factor five times. The number of a's in the product is the *sum* of the number of a's in each factor.

First Law of Exponents: Product of Powers.

To multiply two powers with the same base, add the exponents and leave the base unchanged.

$$a^m \cdot a^n = a^{m+n}$$

Example A.6.1

a. $5^3 \cdot 5^4 = 5^{3+4} = 5^7$

Same base — Add exponents.

b. $x^4 \cdot x^2 = x^{4+2} = x^6$

Same base — Add exponents.

□

Here are some mistakes to avoid.

Caution A.6.2

1 Note that we do not *multiply* the exponents when simplifying a product.

For example,

$$b^4 \cdot b^2 \neq b^8$$

You can check this with your calculator by choosing a value for b, for instance, $b = 3$:

$$3^4 \cdot 3^2 \neq 3^8$$

2 In order to apply the first law of exponents, the bases must be the same. For example,

$$2^3 \cdot 3^5 \neq 6^8$$

(Check this on your calculator.)

3 We do not multiply the bases when simplifying a product. In Example A.6.1, p. 895a, note that

$$5^3 \cdot 5^4 \neq 25^7$$

4 Although we can simplify the product x^2x^3 as x^5, we cannot simplify the sum $x^2 + x^3$, because x^2 and x^3 are not like terms.

Example A.6.3 Multiply $(-3x^4z^2)(5x^3z)$.

Solution. Rearrange the factors to group the numerical coefficients and the powers of each base. Apply the first law of exponents.

$$\begin{aligned}(-3x^4z^2)(5x^3z) &= (-3)(5)x^4x^3z^2z\\ &= -15x^7z^3\end{aligned}$$

□

A.6.2 Quotients of Powers

To reduce a fraction, we divide both numerator and denominator by any common factors.

$$\frac{x^7}{x^4} = \frac{xxx\cancel{x}\cancel{x}\cancel{x}\cancel{x}}{\cancel{x}\cancel{x}\cancel{x}\cancel{x}} = \frac{x^3}{1} = x^3$$

We can obtain the same result more quickly by *subtracting* the exponent of the denominator from the exponent of the numerator.

$$\frac{x^7}{x^4} = x^{7-4} = x^3$$

What if the larger power occurs in the denominator of the fraction?

$$\frac{x^4}{x^7} = \frac{\cancel{x}\cancel{x}\cancel{x}\cancel{x}}{xxx\cancel{x}\cancel{x}\cancel{x}\cancel{x}} = \frac{1}{x^3}$$

In this case, we subtract the exponent of the numerator from the exponent of the denominator.

$$\frac{x^4}{x^7} = \frac{1}{x^{7-4}} = \frac{1}{x^3}$$

These examples suggest the following law.

Second Law of Exponents: Quotient of Powers.

To divide two powers with the same base, subtract the smaller exponent from the larger one, keeping the same base.

a If the larger exponent occurs in the numerator, put the power in the numerator.

$$\text{If } m > n, \quad \text{then } \frac{a^m}{a^n} = a^{m-n} \qquad (a \neq 0)$$

b If the larger exponent occurs in the denominator, put the power in the denominator.

$$\text{If } m < n, \quad \text{then } \frac{a^m}{a^n} = \frac{1}{a^{n-m}} \qquad (a \neq 0)$$

Example A.6.4

1. $\dfrac{3^8}{3^2} = 3^{8-2} = 3^6$ Subtract exponents: $\mathbf{8 > 2.}$

2. $\dfrac{w^3}{w^6} = \dfrac{1}{w^{6-3}} = \dfrac{1}{w^3}$ Subtract exponents: $\mathbf{3 < 6.}$

□

Example A.6.5 Divide $\dfrac{3x^2y^4}{6x^3y}$

Solution. Consider the numerical coefficients and the powers of each variable separately. Use the second law of exponents to simplify each quotient of powers.

$$\begin{aligned} \frac{3x^2y^4}{6x^3y} &= \frac{3}{6} \cdot \frac{x^2}{x^3} \cdot \frac{y^4}{y} && \text{Subtract exponents.} \\ &= \frac{1}{2} \cdot \frac{1}{x^{3-2}} \cdot y^{4-1} \\ &= \frac{1}{2} \cdot \frac{1}{x} \cdot y^3 = \frac{y^3}{2x} \end{aligned}$$

□

A.6.3 Power of a Power

Consider the expression $\left(a^4\right)^3$, the third power of a^4.

$$\left(a^4\right)^3 = \left(a^4\right)\left(a^4\right)\left(a^4\right) = a^{4+4+4} = a^{12} \qquad \text{Add exponents.}$$

We can obtain the same result by multiplying the exponents together.

$$\left(a^4\right)^3 = a^{4 \cdot 3} = a^{12}$$

Third Law of Exponents: Power of a Power.

To raise a power to a power, keep the same base and multiply the exponents.

$$(a^m)^n = a^{mn}$$

Example A.6.6

a. $(4^3)^5 = 4^{3 \cdot 5} = 4^{15}$ Multiply exponents.

b. $(y^5)^2 = y^{5 \cdot 2} = y^{10}$ Multiply exponents.

□

Caution A.6.7 Notice the difference between the expressions

$$(x^3)(x^4) = x^{3+4} = x^7$$

and

$$\left(x^3\right)^4 = x^{3\cdot 4} = x^{12}$$

The first expression is a product, so we add the exponents. The second expression raises a power to a power, so we multiply the exponents.

A.6.4 Power of a Product

To simplify the expression $(5a)^3$, we use the associative and commutative laws to regroup the factors as follows.

$$\begin{aligned}(5a)^3 &= (5a)(5a)(5a)\\ &= 5\cdot 5\cdot 5\cdot a\cdot a\cdot a\\ &= 5^3a^3\end{aligned}$$

Thus, to raise a product to a power, we can simply raise each factor to the power.

Fourth Law of Exponents: Power of a Product.

A power of a product is equal to the product of the powers of each of its factors.

$$(ab)^n = a^nb^n$$

Example A.6.8

a $(5a)^3 = 5^3a^3 = 125a^3$ Cube each factor.

b $\left(-xy^2\right)^4 = (-x)^4\left(y^2\right)^4$ Raise each factor to the fourth power.

$= x^4y^8$ Apply the third law of exponents.

□

Caution A.6.9

1 Compare the two expressions $3a^2$ and $(3a)^2$; they are not the same. In the expression $3a^2$, only the factor a is squared. But in $(3a)^2$, both 3 and a are squared. Thus,

$$3a^2 \quad \text{cannot be simplified}$$

but

$$(3a)^2 = 3^2a^2 = 9a^2$$

2 Compare the two expressions $(3a)^2$ and $(3+a)^2$. The fourth law of exponents applies to the *product* $3a$, but not to the *sum* $3+a$. Thus,

$$(3+a)^2 \neq 3^2 + a^2$$

In order to simplify $(3+a)^2$, we must expand the binomial product:

$$(3+a)^2 = (3+a)(3+a) = 9 + 6a + a^2$$

A.6.5 Power of a Quotient

To simplify the expression $\left(\frac{x}{3}\right)^4$, we multiply together 4 copies of the fraction $\frac{x}{3}$.

$$\left(\frac{x}{3}\right)^4 = \frac{x}{3}\cdot\frac{x}{3}\cdot\frac{x}{3}\cdot\frac{x}{3} = \frac{x\cdot x\cdot x\cdot x}{3\cdot 3\cdot 3\cdot 3}$$
$$= \frac{x^4}{3^4} = \frac{x^4}{81}$$

In general, we have the following rule.

Fifth Law of Exponents: Power of a Quotient.

To raise a quotient to a power, raise both the numerator and denominator to the power.

$$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$$

For reference, we state all of the laws of exponents together. All the laws are valid when a and b are not equal to zero and when the exponents m and n are whole numbers.

Laws of Exponents.

I $a^m \cdot a^n = a^{m+n}$

II a $\frac{a^m}{a^n} = a^{m-n}$ $\quad m > n$

b $\frac{a^m}{a^n} = \frac{1}{a^{n-m}}$ $\quad m < n$

III $(a^m)^n = a^{m+n}$

IV $(ab)^n = a^n b^n$

V $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$

Example A.6.10 Simplify $5x^2y^3\left(2xy^2\right)^4$

Solution. According to the order of operations, we should perform any powers before multiplications. Thus, we begin by simplifying $(2xy^2)^4$. We apply the fourth law.

$$5x^2y^3\left(2xy^2\right)^4 = 5x^2y^3\cdot 2^4x^4\left(y^2\right)^4 \qquad \text{Apply the fourth law.}$$
$$= 5x^2y^3\cdot 2^4x^4y^8$$

Finally, multiply powers with the same base. Apply the first law.

$$5x^2y^3\cdot 2^4x^4y^8 = 5\cdot 2^4x^2x^4y^3y^8 = 80x^6y^{11}$$

□

Example A.6.11 Simplify $\left(\frac{2x}{z^2}\right)^3$

Solution. Begin by applying the fifth law.

$$\left(\frac{2x}{z^2}\right)^3 = \frac{(2x)^3}{(z^2)^2}$$

Apply the fourth law to the numerator and the third law to the denominator.

$$= \frac{2^3x^3}{z^6} = \frac{8x^3}{z^6}$$

□

A.6.6 Section Summary

A.6.6.1 Vocabulary

Look up the definitions of new terms in the Glossary.

- Exponent
- Power

A.6.6.2 SKILLS

Practice each skill in the exercises listed.

1 Apply the laws of exponents: #1–8

2 Simplify expressions: #9–16, 25–32

3 Multiply and divide power: #17–24

A.6.7 Exercises A.6

For Problems 1–8, simplify by applying the appropriate law of exponents.

1.

a $b^4 \cdot b^5$

b $b^2 \cdot b^8$

c $(q^3)(q)(q^5)$

d $(p^2)(p^4)(p^4)$

Answer.

a b^9 b b^{10} c q^9 d p^{10}

2.

a $\frac{w^6}{w^3}$ b $\frac{c^{12}}{c^4}$ c $\frac{z^6}{z^9}$ d $\frac{b^4}{b^8}$

3.

a $2^7 \cdot 2^2$ b $6^5 \cdot 6^3$ c $\frac{2^9}{2^4}$ d $\frac{8^6}{8^2}$

Answer.

a 2^9 b 6^8 c 2^5 d 8^4

4.

a $(d^3)^5$ b $(d^4)^2$ c $(5^4)^3$ d $(4^3)^3$

5.

a $(6x)^3$ b $(3y)^4$ c $(2t^3)^5$ d $(6s^2)^2$

Answer.

a $216x^3$ b $81y^4$ c $32t^{15}$ d $36s^4$

6.

a $\left(\frac{w}{2}\right)^6$ b $\left(\frac{5}{u}\right)^4$ c $\left(\frac{-4}{p^5}\right)^3$ d $\left(\frac{-3}{q^4}\right)^5$

7.

a $\left(\frac{h^2}{m^3}\right)^4$ b $\left(\frac{n^3}{k^4}\right)^8$ c $(-4a^2b^4)^4$ d $(-5ab^8)^3$

Answer.

a $\frac{h^8}{m^{12}}$ b $\frac{n^{24}}{k^{32}}$ c $2565a^8b^{16}$ d $-125a^3b^{24}$

8.

a $\frac{ab^2}{(ab)^2}$ b $\frac{(x^2y)^2}{x^2y^2}$ c $\frac{(2mp)^3}{2m^3p}$ d $\frac{4^2rt^4}{2^4r^4t}$

For Problems 9–15, simplify if possible.

9.

a $w+w$ b $w(w)$

Answer.

a $2w$ b w^2

10.

a m^2-m^2 b $m^2(-m^2)$

11.

a $4z^2-6z^2$ b $4z^2(-6z^2)$

Answer.

a $-2z^2$ b $m-24z^4$

12.

a t^3+3t^3 b $t^3(3t^3)$

13.

a $4p^2+3p^3$ b $4p^2(3p^3)$

Answer.

a Cannot be simplified b $12p^5$

14.

a $2w^2-5w^4$ b $(2w^2)(-5w^4)$

15.

a $3^9\cdot 3^8$ b 3^9+3^8

Answer.

a 3^{17} b Cannot be simplified

16.

a $(-2)^7(-2)^5$ b -2^7-2^5

F mor Problems 17–20, multiply.

17.

a $(4y)(-6y)$ b $(-4z)(-8z)$

Answer.

a $-24y^2$ b $32z^2$

18.

a $(2wz^3)(-8z)$ b $(4wz)(-9w^2z^2)$

19.

a $-4x(3xy)(xy^3)$ b $(-5x^2)(2xy)(5x^2)$

Answer.

a $-12x^3y^4$ b $-50x^5y$

20.

a $-7ab^2(-3ab^3)$ b $-4a^2b(-3a^3b^2)$

F dor Problems 21–22, divide.

21.

a $\dfrac{2a^3b}{8a^4b^5}$ b $\dfrac{8a^2b}{12a^5b^3}$

Answer.

a $\dfrac{1}{4ab^4}$ b $\dfrac{2}{3a^3b^2}$

22.

a $\dfrac{-12qw^4}{8qw^2}$ b $\dfrac{-12rz^6}{20rz}$

For Problems 23–24, multiply or divide.

23.

a $\dfrac{-15bc(b^2c)}{-3b^3c^4}$ b $\dfrac{-25c(c^2d^2)}{-5c^8d^2}$

Answer.

a $\dfrac{5}{c^2}$ b $\dfrac{5}{c^5}$

24.

a $-2x^3(x^2y)(-4y^2)$ b $3xy^3(-x^4)(-2y^2)$

For Problems 25–28, simplify by applying the laws of exponents.

25.

a $b^3(b^2)^5$ b $b(b^4)^6$

Answer.

a b^{13} b b^{25}

26.

a $(p^2q)^3(pq^3)$ b $(p^3)^4(p^3q^4)$

27.

a $(2x^3y)^2(xy^3)^4$ b $(3xy^2)^3(2x^2y^2)^2$

Answer.

a $4x^{10}y^{14}$ b $108x^7y^{10}$

28.

a $-a^2(-a)^2$ b $-a^3(-a)^3$

For Problems 29–32, simplify by applying the laws of exponents.

29.

a $\left(\dfrac{-2x}{3y^2}\right)^3$ b $\left(\dfrac{-x^2}{2y}\right)^4$

Answer.

a $\dfrac{-8x^3}{27y^6}$ b $\dfrac{x^8}{16y^4}$

30.

a $\dfrac{(4x)^3}{(-2x^2)^2}$ b $\dfrac{(5x)^2}{(-3x^2)^3}$

31.

a $\dfrac{(xy)^2(-x^2y)^3}{(x^2y^2)^2}$ b $\dfrac{(-x^2)(-x^2)^4}{(x^2)^3}$

Answer.

a $-x^4y$ b x^4

32.

a $\left(\dfrac{-2x}{y^2}\right)\left(\dfrac{y^2}{3x}\right)^2$ b $\left(\dfrac{x^2z}{2}\right)^3\left(\dfrac{-2}{x^2z}\right)^3$

A.7 Polynomials and Factoring

In Section A.6, p. 895, we used the first law of exponents to multiply two or more monomials. In this section, we review techniques for multiplying and factoring polynomials of several terms.

A.7.1 Polynomials

A **polynomial** is a sum of terms in which all the exponents on the variables are whole numbers and no variables appear in the denominator or under a radical. The expressions

$$0.1R^4, \quad d^2 + 32d - 21, \quad \text{and} \quad 128x^3 - 960x^2 + 8000$$

are all examples of polynomials in one variable.

An algebraic expression consisting of one term of the form cx^n, where c is a constant and n is a whole number, is called a **monomial**. For example,

$$y3, \quad -3x^8, \quad \text{and} \quad 0.1R^4$$

are monomials. A polynomial is just a sum of one or more monomials.

A polynomial with exactly two terms, such as $\dfrac{1}{2}n^2 + \dfrac{1}{2}n$, is called a **binomial**. A polynomial with exactly three terms, such as $d^2 + 32d - 21$ or $128x^3 - 960x^2 + 8000$, is called a **trinomial**. We have no special names for polynomials with more than three terms.

Example A.7.1 Which of the following expressions are polynomials?

a πr^2

b $23.4s^6 - 47.9s^4$

c $\frac{2}{3}w^3 - \frac{7}{3}w^2 + \frac{1}{3}w$

d $7 + m^{-2}$

e $\frac{x-2}{x+2}$

f $\sqrt[3]{4y}$

Solution. The first three are all polynomials. In fact, (a) is a monomial, (b) is a binomial, and (c) is a trinomial. The last three are not polynomials. The variable in (d) has a negative exponent, the variable in (e) occurs in the denominator, and the variable in (f) occurs under a radical. □

In a polynomial containing only one variable, the greatest exponent that appears on the variable is called the **degree** of the polynomial. If there is no variable at all, then the polynomial is called a constant, and the degree of a constant is zero.

Example A.7.2 Give the degree of each polynomial.

a $b^3 - 3b^2 + 3b - 1$

b 10^{10}

c $-4w^3$

d $s^2 - s^6$

Solution.

a This is a polynomial in the variable b, and because the greatest exponent on b is 3, the degree of this polynomial is 3.

b This is a constant polynomial, so its degree is 0. (The exponent on a constant does not affect the degree.)

c This monomial has degree 3.

d This is a binomial of degree 6.

□

We can evaluate a polynomial just as we evaluate any other algebraic expression: We replace the variable with a number and simplify the result.

Example A.7.3 Let $p(x) = -2x^2 + 3x - 1$. Evaluate each of the following.

a $p(2)$

b $p(-1)$

c $p(t)$

d $p(t+3)$

Solution. In each case, we replace x by the given value.

a $p(\mathbf{2}) = -2(\mathbf{2})^2 + 3(\mathbf{2}) - 1 = -8 + 6 - 1 = -3$

b $p(\mathbf{-1}) = -2(\mathbf{-1})^2 + 3(\mathbf{-1}) - 1 = -2 + (-3) - 1 = -6$

c $p(\boldsymbol{t}) = -2(\boldsymbol{t})^2 + 3(\boldsymbol{t}) - 1 = - - 2t^2 + 3t - 1$

d

$$\begin{aligned} p(\boldsymbol{t+3}) &= -2(\boldsymbol{t+3})^2 + 3(\boldsymbol{t+3}) - 1 \\ &= -2(t^2 + 6t + 9) + 3(t+3) - 1 \\ &= -2t^2 - 9t - 10 \end{aligned}$$

□

A.7.2 Products of Polynomials

To multiply polynomials, we use a generalized form of the distributive property:

$$a(b+c+d+\cdots) = ab+ac+ad+\cdots$$

To multiply a polynomial by a monomial, we multiply each term of the polynomial by the monomial.

Example A.7.4

a

$$\begin{aligned} 3x(x+y+z) &= 3x(x)+3x(y)+3x(z) \\ &= 3x^2+3xy+3xz \end{aligned}$$

b

$$\begin{aligned} -2ab^2(3a^2-ab+2b^2) &= -2ab^2(3a^2)-2ab^2(-ab)-2ab^2(2b^2) \\ &= -6a^3b^2+2a^2b^3-4ab^4 \end{aligned}$$

□

A.7.3 Products of Binomials

Products of binomials occur so frequently that it is worthwhile to learn a shortcut for this type of multiplication. We can use the following scheme to perform the multiplication mentally. (See Figure A.7.5, p. 905.)

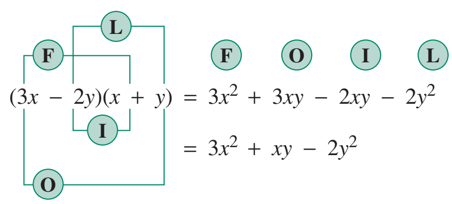


Figure A.7.5

This process is sometimes called the **FOIL** method, where FOIL represents

the product of the ***F***irst terms
the product of the ***O***uter terms
the product of the ***I***nner terms
the product of the ***L***ast terms

Example A.7.6

$$\begin{aligned} (2x-1)(x+3) &= 2x^2+6x-x-3 \\ &= 2x^2+5x-3 \end{aligned}$$

□

A.7.4 Factoring

We sometimes find it useful to write a polynomial as a single *term* composed of two or more *factors*. This process is the reverse of multiplication and is called **factoring**. For example, observe that

$$3x^2+6x = 3x(x+2)$$

We will only consider factorization in which the factors have integer coefficients.

A.7.5 Common Factors

We can factor a common factor from a polynomial by using the distributive property in the form

$$ab + ac = a(b + c)$$

We first identify the common factor. For example, each term of the polynomial

$$6x^3 + 9x^2 - 3x$$

contains the monomial $3x$ as a factor; therefore,

$$6x^3 + 9x^2 - 3x = 3x(\underline{\hspace{3cm}})$$

Next, we insert the proper polynomial factor within the parentheses. This factor can be determined by inspection. We ask ourselves for monomials that, when multiplied by $3x$, yield $6x^3$, $9x^2$, and $-3x$, respectively, and obtain

$$6x^3 + 9x^2 - 3x = 3x(2x^2 + 3x - 1)$$

We can check the result of factoring an expression by multiplying the factors. In the example above,

$$3x(2x^2 + 3x - 1) = 6x^3 + 9x^2 - 3x$$

Example A.7.7

a

$$\begin{aligned} 18x^2y - 24xy^2 &= 6xy(? - ?) \\ &= 6xy(3x - 4y) \end{aligned}$$

because

$$6xy(3x - 4y) = 18x^2y - 24xy^2$$

b

$$\begin{aligned} y(x-2) + z(x-2) &= (x-2)(? - ?) \\ &= (x-2)(y+z) \end{aligned}$$

because

$$(x-2)(y+z) = y(x-2) + z(x-2)$$

□

A.7.6 Opposite of a Binomial

It is often useful to factor -1 from the terms of a binomial.

$$\begin{aligned} a - b &= (-1)(-a + b) \\ &= (-1)(b - a) = -(b - a) \end{aligned}$$

Hence, we have the following important relationship.

Opposite of a Binomial.

$$a - b = -(b - a)$$

That is, $a - b$ and $b - a$ are opposites or negatives of each other.

Example A.7.8

a $3x - y = -(y - 3x)$

b $a - 2b = -(2b - a)$

□

A.7.7 Polynomial Division

We can divide one polynomial by a polynomial of lesser degree. The quotient will be the sum of a polynomial and a simpler algebraic fraction.

If the divisor is a monomial, we can simply divide the monomial into each term of the numerator.

Example A.7.9 Divide $\dfrac{9x^3 - 6x^2 + 4}{3x}$

Solution. Divide $3x$ into each term of the numerator.

$$\begin{aligned}\frac{9x^3 - 6x^2 + 4}{3x} &= \frac{9x^3}{3x} - \frac{6x^2}{3x} + \frac{4}{3x} \\ &= 3x^2 - 2x + \frac{4}{3x}\end{aligned}$$

The quotient is the sum of a polynomial, $3x^2 - 2x$, and an algebraic fraction, $\dfrac{4}{3x}$. □

If the denominator is not a monomial, we can use a method similar to the long division algorithm used in arithmetic.

Example A.7.10 Divide $\dfrac{2x^2 + x - 7}{x + 3}$

Solution. First write

$$x + 3\overline{)\,2x^2 + x - 7}$$

and divide $2x^2$ (the first term of the numerator) by x (the first term of the denominator) to obtain $2x$. (It may be helpful to write down the division: $\dfrac{2x^2}{2x} = x$.) Write $2x$ above the quotient bar as the first term of the quotient, as shown below.

Next, multiply $x + 3$ by $2x$ to obtain $2x^2 + 6x$, and subtract this product from $2x^2 + x - 7$:

$$\begin{array}{r} 2x\phantom{{}+x-7} \\ x + 3\overline{)\,2x^2 + x - 7} \\ \underline{-(2x^2 + 6x)}\phantom{{}-7} \\ -5x - 7 \end{array}$$

Repeating the process, divide $-5x$ by x to obtain -5. Write -5 as the second term of the quotient. Then multiply $x + 3$ by -5 to obtain $-5x - 15$, and subtract:

$$\begin{array}{r} 2x - 5\phantom{{}-7} \\ x + 3\overline{)\,2x^2 + x - 7} \\ \underline{-(2x^2 + 6x)}\phantom{{}-7} \\ -5x - 7 \\ \underline{-\ (-5x - 15)} \\ 8 \end{array}$$

Because the degree of 8 is less than the degree of $x + 3$, the division is finished. The quotient is $2x - 5$, with a remainder of 8. We write the remainder as a fraction to obtain

$$\frac{2x^2 + x - 7}{x + 3} = 2x - 5 + \frac{8}{x + 3}$$

□

When using polynomial division, it helps to write the polynomials in descending powers of the variable. If the numerator is missing any terms, we can insert terms with zero coefficients so that like powers will be aligned. For example, to perform the division

$$\frac{3x - 1 + 4x^3}{2x - 1}$$

we first write the numerator in descending powers as $4x^3 + 3x - 1$. We then insert $0x^2$ between $4x^3$ and $3x$ and set up the quotient as

$$2x - 1\overline{)4x^3 + 0x^2 + 3x - 1}$$

We then proceed as in Example A.7.10, p. 907. You can check that the quotient is

$$2x^2 + x + 2 + \frac{1}{2x - 1}$$

A.7.8 Section Summary

A.7.8.1 Vocabulary

Look up the definitions of new terms in the Glossary.

- Polynomial
- Common factor
- Degree
- Constant
- Trinomial
- Monomial
- Binomial

A.7.8.2 SKILLS

Practice each skill in the exercises listed.

1 Identify polynomials: #1–12

2 Evaluate polynomials: #13–20

3 Multiply polynomials: #21–42

4 Factor out a common factor: #43–68

5 Divide polynomials: #69–80

A.7.9 Exercises A.7

For Problems 1-8, identify the polynomial as a monomial, a binomial, or a trinomial. Give the degree of the polynomial.

1. $2x^3 - x^2$

Answer.
Binomial; 3

2. $x^2 - 2x + 1$

3. $5n^4$

Answer.
Monomial; 4

4. $3n+1$

5. $3r^2-4r+2$

Answer.
Trinomial; 2

6. r^3

7. y^3-2y^2-y

Answer.
Trinomial; 3

8. $3y^2+1$

Which of the expressions in Problems 9-12 are not polynomials?

9.

a $1-0.04t^2$

b $3x^2-4x+\dfrac{2}{x}$

c $2\sqrt{z}-7z^3+2$

d $\sqrt{2}w^3+\dfrac{3}{4}w^2-w$

Answer. b and c

10.

a $\sqrt{3}p^2-7p+2$

b $2h^{4/3}+6h^{1/3}-2$

c $\dfrac{2}{x^2-6x+5}$

d $\dfrac{1}{4}y^{-2}+3y^{-1}+4$

11.

a $\dfrac{1}{m^2+3}$

b v^2-16+2^v

c $\sqrt{x^3-4x}$

d $\dfrac{m^4}{12}$

Answer. a, b, c

12.

a 3^t-5t^3+2

b $\dfrac{q+3}{q-1}$

c $c^{1/2}-c$

d $\sqrt[3]{d+1}$

For Problems 13-20, evaluate the polynomial function for the given values of the variable.

13. $P(x)=x^3-3x^2+x+1$

a $x=2$ b $x=-2$ c $x=2b$

Answer.

a -1 b -21 c $8b^3-12b^2+2b+1$

14. $P(x)=2x^3+x^2-3x+4$

a $x=3$ b $x=-3$ c $x=-a$

15. $Q(t)=t^2+3t+1$

a $t=\dfrac{1}{2}$ b $t=-\dfrac{1}{3}$ c $t=-w$

Answer.

a $\dfrac{11}{4}$ b $\dfrac{1}{9}$ c w^2-3w+1

16. $Q(t)=2t^2-t+1$

a $t=\dfrac{1}{4}$ b $t=-\dfrac{1}{2}$ c $t=3v$

17. $R(z) = 3z^4 - 2z^2 + 3$
a $z = 1.8$ b $z = -2.6$ c $z = k - 1$

Answer.

a 28.0128
b 126.5728
c $3k^4 - 12k^3 + 16k^2 - 8k + 4$

18. $R(z) = z^4 + 4z - 2$
a $z = 2.1$ b $z = -3.1$ c $z = h + 2$

19. $N(a) = a^6 - a^5$
a $a = -1$ b $a = -2$ c $a = \dfrac{m}{3}$

Answer.

a 2
b 96
c $\dfrac{m^6}{729} - \dfrac{m^5}{243}$

20. $N(a) = a^5 - a^4$
a $a = -1$ b $a = -2$ c $a = \dfrac{q}{2}$

For Problems 21-42, write the product as a polynomial and simplify.

21. $4y(x - 2y)$
Answer. $4xy - 8y^2$

22. $3x(2x + y)$

23. $-6x(2x^2 - x + 1)$
Answer. $-12x^3 + 6x^2 - 6x$

24. $-2y(y^2 - 3y + 2)$

25. $a^2b(3a^2 - 2ab - b)$
Answer. $3a^4b - 2a^3b^2 - a^2b^2$

26. $ab^3(-a^2b^2 + 4ab - 3)$

27. $2x^2y^2(4xy^4 - 2xy - 3x^3y^2)$
Answer.
$8x^3y^7 - 4x^3y^4 - 6x^5y^5$

28. $5x^2y^2(3x^4y^2 + 3x^2y - xy^6)$

29. $(n + 2)(n + 8)$
Answer. $n^2 + 10n + 16$

30. $(r - 1)(r - 6)$

31. $(r + 5)(r - 2)$
Answer. $r^2 + 3r - 10$

32. $(z - 3)(z + 5)$

33. $(2z + 1)(z - 3)$
Answer. $2z^2 - 5z - 3$

34. $(3t - 1)(2t + 1)$

35. $(4r + 3s)(2r - s)$
Answer. $8r^2 + 2rs - 3s^2$

36. $(2z - w)(3z + 5w)$

37. $(2x - 3y)(3x - 2y)$
Answer. $6x^2 - 13xy + 6y^2$

38. $(3a + 5b)(3a + 4b)$

39. $(3t - 4s)(3t + 4s)$
Answer. $9t^2 - 16s^2$

40. $(2x - 3z)(2x + 3z)$

41. $(2a^2 + b^2)(a^2 - 3b^2)$
Answer. $2a^4 - 5a^2b^2 - 3b^4$

42. $(s^2 - 5t^2)(3s^2 + 2t^2)$

For Problems 43-60, factor completely. Check your answers by multiplying factors.

43. $4x^2z + 8xz$

Answer. $4xz(x + 2)$

44. $3x^2y + 6xy$

45. $3n^4 - 6n^3 + 12n^2$

Answer. $3n^2(n^2 - 2n + 4)$

46. $2x^4 - 4x^2 + 6x$

47. $15r^2s + 18rs^2 - 3r$

Answer. $3r(5rs + 6s^2 - 1)$

48. $2x^2y^2 - 3xy + 5x^2$

49. $3m^2n^4 - 6m^3n^3 + 14m^3n^2$

Answer.
$m^2n^2(3n^2 - 6mn + 14m)$

50. $6x^3y - 6xy^3 + 12x^2y^2$

51. $15a^4b^3c^4 - 12a^2b^2c^5 + 6a^2b^3c^4$

Answer.
$3a^2b^2c^4(5a^2b - 4c + 2b)$

52. $14xy^4z^3 + 21x^2y^3z^2 - 28x^3y^2z^5$

53. $a(a + 3) + b(a + 3)$

Answer. $(a + b)(a + 3)$

54. $b(a - 2) + a(a - 2)$

55. $y(y - 2) - 3x(y - 2)$

Answer. $(y - 3x)(y - 2)$

56. $2x(x + 3) - y(x + 3)$

57. $4(x - 2)^2 - 8x(x - 2)^3$

Answer.
$4(x - 2)^2(-2x^2 + 4x + 1)$

58. $6(x + 1) - 3x(x + 1)^2$

59. $x(x - 5)^2 - x^2(x - 5)^3$

Answer.
$x(x - 5)^2(-x^2 + 5x + 1)$

60. $x^2(x + 3)^3 - x(x + 3)^2$

For Problems 61-68, supply the missing factors or terms.

61. $3m - n = -(\quad ? \quad)$

Answer. $-(2n - 3m)$

62. $2a - b = -(\quad ? \quad)$

63. $-2x + 2 = -2(\quad ? \quad)$

Answer. $-2(x - 1)$

64. $-6x - 9 = -3(\quad ? \quad)$

65. $-ab - ac = \ ?\ (b + c)$

Answer. $-a(b + c)$

66. $-a^2 + ab = \ ?\ (a - b)$

67. $2x - y + 3z = -(\quad ? \quad)$

Answer. $-(-2x + y - 3z)$

68. $3x + 3y - 2z = -(\quad ? \quad)$

For Problems 69-80, divide.

69. $\dfrac{18r^2s^2 - 15rs + 6}{3rs}$

Answer. $6rs - 5 + \dfrac{2}{rs}$

70. $\dfrac{8a^2x^2 - 4ax^2 + ax}{2ax}$

71. $\dfrac{15s^{10} - 21s^5 + 6}{-3s^2}$

Answer. $-5s^8 + 7s^3 - \dfrac{2}{s^2}$

72. $\dfrac{25m^6 - 15m^4 + 7}{-5m^3}$

73. $\dfrac{4y^2 + 12y + 7}{2y + 1}$

Answer. $2y + 5 + \dfrac{2}{2y + 1}$

74. $\dfrac{4t^2 - 4t - 5}{2t - 1}$

75. $\dfrac{x^3+2x^2+x+1}{x-2}$

Answer. $x^2+4x+9+\dfrac{19}{x-2}$

76. $\dfrac{2x^3-3x^2-2x+4}{x+1}$

77. $\dfrac{4z^2+5z+8z^4+3}{2z+1}$

Answer.
$4z^3-2z^2+3z+1+\dfrac{2}{2z+1}$

78. $\dfrac{7-3t^3-23t^2+10t^4}{2t+3}$

79. $\dfrac{x^4-1}{x-2}$

Answer.
$x^3+2x^2+4x+8+\dfrac{15}{x-2}$

80. $\dfrac{y^5+1}{y-1}$

A.8 Factoring Quadratic Trinomials

Consider the trinomial

$$x^2+10x+16$$

Can we find two binomial factors,

$$(x+a)(x+b)$$

whose product is the given trinomial? The product of the binomials is

$$(x+a)(x+b)=x^2+(a+b)x+ab$$

Thus, we are looking for two numbers, a and b, that satisfy

$$(x+a)(x+b)=x^2+(\boldsymbol{a+b})x+\boldsymbol{ab}=x^2+\mathbf{10}x+\mathbf{16}$$

By comparing the coefficients of the terms in the two trinomials, we see that $\boldsymbol{a+b=10}$ and $\boldsymbol{ab=16}$. That is, the sum of the two numbers is the coefficient of the linear term, 10, and their product is the constant term, 16.

To find the numbers, we list all the possible integer factorizations of 16:

$$1\cdot 16,\quad 2\cdot 8,\quad \text{and}\quad 4\cdot 4$$

We see that only one combination gives the correct linear term: 8 and 2. These are the numbers a and b, so

$$x^2+10x+16=(x+8)(x+2)$$

In Example A.8.1, p. 912 we factor quadratic trinomials in which one or more of the coefficients is negative.

Example A.8.1 Factor.

a $x^2-7x+12$

b x^2-x-12

Solution.

a Find two numbers whose product is 12 and whose sum is -7. Because the product is positive and the sum is negative, the two numbers must both be negative. The possible factors of 12 are -1 and -12, -2 and -6, or -3 and -4. Only -4 and -3 have the correct sum, -7. Hence,

$$x^2 - 7x + 12 = (x-4)(x-3)$$

b Find two numbers whose product is -12 and whose sum is -1. Because the product is negative, the two numbers must be of opposite sign and their sum must be -1. By listing the possible factors of -12, we find that the two numbers are -4 and 3. Hence,

$$x^2 - x - 12 = (x-4)(x+3)$$

□

If the coefficient of the quadratic term is not 1, we must also consider its factors.

Example A.8.2 Factor $8x^2 - 9 - 21x$

Solution.

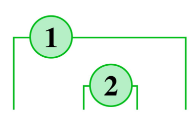
1. Write the trinomial in decreasing powers of x.

$$8x^2 - 21x - 9$$

2. List the possible factors for the quadratic term.

$$\begin{array}{l} (8x \qquad)(x \qquad) \\ (4x \qquad)(2x \qquad) \end{array}$$

3. Consider possible factors for the constant term: 9 may be factored as $9 \cdot 1$ or as $3 \cdot 3$. Form all possible pairs of binomial factor using these factorizations.



$$\begin{array}{l} (8x \quad 9)(x \quad 1) \\ (8x \quad 1)(x \quad 9) \\ (8x \quad 3)(x \quad 3) \\ (4x \quad 9)(2x \quad 1) \\ (4x \quad 1)(2x \quad 9) \\ (4x \quad 3)(2x \quad 3) \end{array}$$

4. Select the combinations of the products "1" and "2" whose sum or difference could be the linear term, $-21x$.

$$(8x \quad 3)(x \quad 3)$$

5. Insert the proper signs:

$$(8x+3)(x-3)$$

□

With practice, you can usually factor trinomials of the form $Ax^2 + Bx + C$ mentally. The following observations may help.

1 If A, B and C are all positive, both signs in the factored form are positive. For example, as a first step in factoring $6x^2 + 11x + 4$, we could write

$$(\quad + \quad)(\quad + \quad)$$

2 If A and C are positive and B is negative, both signs in the factored form are negative. Thus as a first step in factoring $6x^2 - 11x + 4$, we could write

$$(\quad - \quad)(\quad - \quad)$$

3 If C is negative, the signs in the factored form are opposite. Thus as a first step in factoring $6x^2 - 5x - 4$, we could write

$$(\quad + \quad)(\quad - \quad) \text{ or } (\quad - \quad)(\quad + \quad)$$

Example A.8.3

a $$\begin{aligned} 6x^2 + 5x + 1 &= (\quad + \quad)(\quad + \quad) \\ &= (3x + 1)(2x + 1) \end{aligned}$$

b $$\begin{aligned} 6x^2 - 5x + 1 &= (\quad - \quad)(\quad - \quad) \\ &= (3x - 1)(2x - 1) \end{aligned}$$

c $$\begin{aligned} 6x^2 - x - 1 &= (\quad + \quad)(\quad - \quad) \\ &= (3x + 1)(2x - 1) \end{aligned}$$

d $$\begin{aligned} 6x^2 - xy - y^2 &= (\quad + \quad)(\quad - \quad) \\ &= (3x + y)(2x - y) \end{aligned}$$

□

A.8.1 Special Products and Factors

The products below are special cases of the multiplication of binomials. They occur so often that you should learn to recognize them on sight.

Special Products.

I $(a + b)^2 = (a + b)(a + b) = a^2 + 2ab + b^2$

II $(a - b)^2 = (a - b)(a - b) = a^2 - 2ab + b^2$

III $(a + b)(a - b) = a^2 - b^2$

Caution A.8.4 Notice that in (I) $(a + b)^2 \neq a^2 + b^2$, and that in (II) $(a - b)^2 \neq a^2 - b^2$. For example,

$$\begin{aligned} (x + 4)^2 &\neq x^2 + 16, &&\text{instead} && (x + 4)^2 = x^2 + 8x + 16 \\ (t - 5)^2 &\neq t^2 - 16, &&\text{instead} && (t - 5)^2 = t^2 - 10t + 25 \end{aligned}$$

Example A.8.5

a $$\begin{aligned} 3(x + 4)^2 &= 3(x^2 + 2 \cdot 4x + 4^2) \\ &= 3x^2 + 24x + 48 \end{aligned}$$

b $(y+5)(y-5) = y^2 - 5^2$
$\quad = y^2 - 25$

c $(3x-2y)^2 = (3x)^2 - 2(3x)(2y) + (2y)^2$
$\quad = 9x^2 - 12xy + 4y^2$

□

Each of the formulas for special products, when viewed from right to left, also represents a special case of factoring quadratic polynomials.

Special Factorizations.

I $a^2 + 2ab + b^2 = (a+b)^2$

II $a^2 - 2ab + b^2 = (a-b)^2$

III $a^2 - b^2 = (a+b)(a-b)$

IV $a^2 + b^2$ cannot be factored

The trinomials in (I) and (II) are sometimes called **perfect-square trinomials** because they are squares of binomials. Note that the sum of two squares, $a^2 + b^2$, cannot be factored.

Example A.8.6 Factor.

a $x^2 + 8x + 16$

b $y^2 - 10y + 25$

c $4a^2 - 12ab + 9b^2$

d $25m^2n^2 + 20mn + 4$

Solution.

a Because 16 is equal to 4^2 and 8 is equal to $2 \cdot 4$,

$$\begin{aligned} x^2 + 8x + 16 &= x^2 - 2 \cdot 4x + 4^2 \\ &= (x+4)^2 \end{aligned}$$

b Because $25 = 5^2$ and $10 = 2 \cdot 5$,

$$\begin{aligned} y^2 - 10y + 25 &= y^2 - 2 \cdot 5y + 5^2 \\ &= (y-5)^2 \end{aligned}$$

c Because $4a^2 = (2a)^2$, $9b^2 = (3b)^2$, and $2ab = 2(2a)(3b)$,

$$\begin{aligned} 4a^2 - 12ab + 9b^2 &= (2a)^2 - 2(2a)(3b) + (3b)^2 \\ &= (2a-3b)^2 \end{aligned}$$

d Because $25m^2n^2 = (5mn)^2$, $4 = 2^2$, and $20mn = 2(5mn)(2)$,

$$\begin{aligned} 25m^2n^2 + 20mn + 4 &= (5mn)^2 + 2(5mn)(2) + 2^2 \\ &= (5mn+2)^2 \end{aligned}$$

□

Binomials of the form $a^2 - b^2$ are often called the **difference of two squares**.

Example A.8.7 Factor if possible.

a $x^2 - 81$

b $4x^2 - 9y^2$

c $x^2 + 81$

Solution.

a The expression $x^2 - 81$ is the difference of two squares, $x^2 - 9^2$, and thus can be factored according to Special Factorization (III) above.

$$\begin{aligned} x^2 - 81 &= x^2 - 9^2 \\ &= (x + 9)(x - 9) \end{aligned}$$

b Because $4x^2 - 9y^2$ can be written as $(2x)^2 - (3y)^2$,

$$\begin{aligned} 4x^2 - 9y^2 &= (2x^2) - (3y)^2 \\ &= (2x + 3y)(2x - 3y) \end{aligned}$$

c The expression $x^2 + 81$, or $x^2 + 0x + 81$, is *not* factorable, because no two real numbers have a product of 81 and a sum of 0.

□

Caution A.8.8 $x^2 + 81 \neq (x + 9)(x + 9)$, which you can verify by multiplying

$$(x + 9)(x + 9) = x^2 + 18x + 8$$

The factors $x + 9$ and $x - 9$ in Example A.8.7, p. 916a are called **conjugates** of each other. In general, any binomials of the form $a + b$ and $a - b$ are called a **conjugate pair**.

A.8.2 Section Summary

A.8.2.1 Vocabulary

Look up the definitions of new terms in the Glossary.

- Perfect-square trinomial
- Difference of squares
- Conjugate

A.8.2.2 SKILLS

Practice each skill in the exercises listed.

1 Factor quadratic trinomials: #1–36

2 Expand special products: #37–48

3 Factor special quadratic expressions: #49–68

A.8.3 Exercises A.8

For Problems 1-36, factor completely.

1. $x^2 + 5x + 6$

Answer. $(x + 2)(x + 3)$

2. $x^2 + 5x + 4$

3. $y^2 - 7y + 12$

Answer. $(y-3)(y-4)$

4. $y^2 - 7y + 10$

5. $x^2 - 6 - x$

Answer. $(x-3)(x+2)$

6. $x^2 - 15 - 2x$

7. $2x^2 + 3x - 2$

Answer. $(2x-1)(x+2)$

8. $3x^2 - 7x + 2$

9. $7x + 4x^2 - 2$

Answer. $(4x-1)(x+2)$

10. $1 - 5x + 6x^2$

11. $9y^2 - 21y - 8$

Answer. $(3y+1)(3y-8)$

12. $10y^2 - 3y - 18$

13. $10u^2 - 3 - u$

Answer. $(2u+1)(5u-3)$

14. $8u^2 - 3 + 5u$

15. $21x^2 - 43x - 14$

Answer. $(3x-7)(7x+2)$

16. $24x^2 - 29x + 5$

17. $5a + 72a^2 - 25$

Answer. $(9a+4)(8a-3)$

18. $-30a + 72a^2 - 25$

19. $12 - 53x + 30x^2$

Answer. $(2x-3)(15x-4)$

20. $39x + 80x^2 - 20$

21. $-30t - 44 + 54t^2$

Answer. $2(3t+2)(9t-11)$

22. $48t^2 - 122t + 39$

23. $3x^2 - 7ax + 2a^2$

Answer. $(x-2a)(3x-a)$

24. $9x^2 + 9ax - 10a^2$

25. $15x^2 - 4xy - 4y^2$

Answer. $(3x-2y)(5x+2y)$

26. $12x^2 + 7xy - 12y^2$

27. $18u^2 + 20v^2 - 39uv$

Answer. $(3u-4v)(6u-5v)$

28. $24u^2 - 20v^2 + 17uv$

29. $12a^2 - 14b^2 - 13ab$

Answer. $(3a+2b)(4a-7b)$

30. $24a^2 - 15b^2 - 2ab$

31. $10a^2b^2 - 19ab + 6$

Answer. $(5ab-2)(2ab-3)$

32. $12a^2b^2 - ab - 20$

33. $56x^2y^2 - 2xy - 4$

Answer.
$2(4xy+1)(7xy-2)$

34. $54x^2y^2 + 3xy - 2$

35. $22a^2z^2 - 21 - 19az$

Answer. $(2az-3)(11az+7)$

36. $26a^2z^2 - 24 + 23az$

For Problems 37-48, write the expression as a polynomial and simplify.

37. $(x+3)^2$

Answer. $x^2 + 6x + 9$

38. $(y-4)^2$

39. $(2y-5)^2$

Answer. $4y^2 - 20y + 25$

40. $(3x+2)^2$

41. $(x+3)(x-3)$

Answer. $x^2 - 9$

42. $(x-7)(x+7)$

43. $(3t-4s)(3t+4s)$

Answer. $9t^2-16s^2$

44. $(2x+a)(2x-a)$

45. $(5a-2)(5a-2)$

Answer. $25a^2-20ab+4b^2$

46. $(4u+5v)(4u+5v)$

47. $(8xz+3)(8xz+3)$

Answer. $64x^2z^2+48xz+9$

48. $(7yz-2)(7yz-2)$

For Problems 49-68, factor completely.

49. x^2-25

Answer. $(x+5)(x-5)$

50. x^2-36

51. $x^2-24x+144$

Answer. $(x-12)^2$

52. $x^2+26x+169$

53. x^2-4y^2

Answer. $(x+2y)(x-2y)$

54. $9x^2-y^2$

55. $4x^2+12x+9$

Answer. $(2x+3)^2$

56. $4y^2+4y+1$

57. $9u^2-30uv+25v^2$

Answer. $(3u-5v)^2$

58. $16s^2-56st+49t^2$

59. $4a^2-25b^2$

Answer. $(2a+5b)(2a-5b)$

60. $16a^2-9b^2$

61. x^2y^2-81

Answer. $(xy+9)(xy-9)$

62. x^2y^2-64

63. $9x^2y^2+6xy+1$

Answer. $(3xy+1)^2$

64. $4x^2y^2+12xy+9$

65. 16^2y^2-1

Answer. $(4xy-1)(4xy+1)$

66. $64x^2y^2-1$

67. $(x+2)^2-y^2$

Answer.
$(x+2-y)(x+2+y)$

68. $x^2-(y-3)^2$

A.9 Working with Algebraic Fractions

A quotient of two polynomials is called a **rational expression** or an **algebraic fraction**. Operations on algebraic fractions follow the same rules as operations on common fractions.

A.9.1 Reducing Fractions

When we reduce an ordinary fraction such as $\dfrac{24}{36}$, we are using the fundamental principle of fractions.

Fundamental Principle of Fractions.

If we multiply or divide the numerator and denominator of a fraction by the same (nonzero) number, the new fraction is equivalent to the old

one. In symbols,
$$\frac{ac}{bc} = \frac{a}{b}, \qquad (b,\ c \neq 0)$$

Thus, for example,
$$\frac{24}{36} = \frac{2 \cdot 12}{3 \cdot 12} = \frac{2}{3}$$

We use the same procedure to reduce algebraic fractions: We look for common factors in the numerator and denominator and then apply the fundamental principle.

Example A.9.1 Reduce each algebraic fraction.

a $\dfrac{8x^3y}{6x^2y^3}$

b $\dfrac{6x-3}{3}$

Solution. Factor out any common factors from the numerator and denominator. Then divide numerator and denominator by the common factors.

a $\dfrac{8x^3y}{6x^2y^3} = \dfrac{4x \cdot 2x^2y}{3y^2 \cdot 2x^2y} = \dfrac{4x}{3y^2}$

b $\dfrac{6x-3}{3} = \dfrac{\not{3}(2x+1)}{\not{3}} = 2x+1$

□

If the numerator or denominator of the fraction contains more than one term, it is especially important to *factor* before attempting to apply the fundamental principle. We can divide out common *factors* from the numerator and denominator of a fraction, but the fundamental principle does *not* apply to common *terms*.

Caution A.9.2 We can reduce
$$\frac{2xy}{3y} = \frac{2x}{3}$$
because y is a common factor in the numerator and denominator. However,
$$\frac{2x+y}{3+y} \neq \frac{2x}{3}$$
because y is a common term but is *not a common factor* of the numerator and denominator. Furthermore,
$$\frac{5x+3}{5y} \neq \frac{x+3}{y}$$
because 5 is not a factor of the *entire* numerator.

Example A.9.3 Reduce each fraction.

a $\dfrac{4x+2}{4}$

b $\dfrac{9x^2+3}{6x+3}$

Solution. Factor the numerator and denominator. Then divide numerator and denominator by the common factors.

a $\dfrac{4x+2}{4} = \dfrac{\not{2}(2x+1)}{\not{2}(2)} \dfrac{2x+1}{2}$

b $\dfrac{9x^2+3}{6x+3} = \dfrac{\cancel{3}(3x^2+1)}{\cancel{3}(2x+1)} = \dfrac{3x^2+1}{2x+1}$

□

Caution A.9.4 Note that in Example A.9.3, p. 919a above,

$$\frac{4x+2}{4} \neq x+2$$

and in Example A.9.3, p. 919b,

$$\frac{9x^2+3}{6x+3} \neq \frac{9x^2}{6x}$$

We summarize the procedure for reducing algebraic fractions as follows.

To Reduce an Algebraic Fraction:.

1 Factor the numerator and denominator.

2 Divide the numerator and denominator by any common factors.

Example A.9.5 Reduce each fraction.

a $\dfrac{x^2-7x+6}{36-x^2}$

b $\dfrac{27x^3-1}{9x^2-1}$

Solution.

a Factor numerator and denominator to obtain

$$\frac{(x-6)(x-1)}{(6-x)(6+x)}$$

The factor $x-6$ in the numerator is the opposite of the factor $6-x$ in the denominator. That is, $x-6=-1(6-x)$. Thus,

$$\frac{-1\cancel{(6-x)}(x-1)}{\cancel{(6-x)}(6+x)} = \frac{-1(x-1)}{6+x} = \frac{1-x}{6+x}$$

b The numerator of the fraction is a difference of two cubes, and the denominator is a difference of two squares. Factor each to obtain

$$\frac{\cancel{(3x-1)}(9x^2+3x+1)}{\cancel{(3x-1)}(3x+1)} = \frac{9x^2+3x+1}{3x+1}$$

□

A.9.2 Products of Fractions

To multiply two or more common fractions together, we multiply their numerators together and multiply their denominators together. The same is true for a product of algebraic fractions. For example, xy

$$\frac{6x^2}{y}\cdot\frac{xy}{2} = \frac{6x^2\cdot}{y\cdot 2} = \frac{6x^3y}{2y} \qquad \text{Reduce.}$$

$$= \frac{3x^3(2y)}{2y} = 3x^3$$

We can simplify the process by first factoring each numerator and denominator and dividing out any common factors.

$$\frac{6x^2}{y} \cdot \frac{\cancel{2} \cdot 3x^2}{\cancel{y}} \cdot \frac{x\cancel{y}}{\cancel{2}} = 3x^3$$

In general, we have the following procedure for finding the product of algebraic fractions.

To Multiply Algebraic Fractions:.

1 Factor each numerator and denominator.

2 Divide out any factors that appear in both a numerator and a denominator.

3 Multiply together the numerators; multiply together the denominators.

Example A.9.6 Find each product.

a $\dfrac{5}{x^2-1} \cdot \dfrac{x+2}{x}$ b $\dfrac{4y^2-1}{4-y^2} \cdot \dfrac{y^2-2y}{4y+2}$

Solution.

a The denominator of the first fraction factors into $(x+1)(x-1)$. There are no common factors to divide out, so we multiply the numerators together and multiply the denominators together.

$$\frac{5}{x^2-1} \cdot \frac{x+2}{x} = \frac{5(x+2)}{x(x^2-1)} = \frac{5x+10}{x^3-x}$$

b Factor each numerator and each denominator. Look for common factors.

$$\frac{4y^2-1}{4-y^2} \cdot \frac{y^2-2y}{4y+2} = \frac{(2y-1)\cancel{(2y+1)}}{\cancel{(2-y)}(2+y)} \cdot \frac{y\cancel{(y-2)}^{-1}}{2\cancel{(2y+1)}}$$

Divide out common factors.

$$= \frac{-y(2y-1)}{2(y+2)}$$

Note: $\boldsymbol{y-2=-(2-y)}$

□

A.9.3 Quotients of Fractions

To divide two algebraic fractions we multiply the first fraction by the reciprocal of the second fraction. For example,

$$\frac{2x^3}{3y} \div \frac{4x}{5y^2} = \frac{2x^3}{3y} \cdot \frac{5y^2}{4x}$$
$$= \frac{\cancel{2x} \cdot x^2}{3\cancel{y}} \cdot \frac{5y \cdot \cancel{y}}{2 \cdot \cancel{2x}} = \frac{5x^2y}{6}$$

If the fractions involve polynomials of more than one term, we may need to factor each numerator and denominator in order to recognize any common factors. This suggests the following procedure for dividing algebraic fractions.

To Divide Algebraic Fractions:.

1 Multiply the first fraction by the reciprocal of the second fraction.

2 Factor each numerator and denominator.

3 Divide out any factors that appear in both a numerator and a denominator.

4 Multiply together the numerators; multiply together the denominators.

Example A.9.7 Find each quotient.

a $\dfrac{x^2-1}{x+3} \div \dfrac{x^2-x-2}{x^2+5x+6}$

b $\dfrac{6ab}{2a+b} \div (4a^2b)$

Solution.

a Multiply the first fraction by the reciprocal of the second fraction.

$$\begin{aligned}\frac{x^2-1}{x+3} \div \frac{x^2-x-2}{x^2+5x+6} &= \frac{x^2-1}{x+3} \cdot \frac{x^2+5x+6}{x^2-x-2} \qquad \text{Factor.}\\ &= \frac{(x-1)\cancel{(x+1)}}{\cancel{x+3}} \cdot \frac{\cancel{(x+3)}(x+2)}{\cancel{(x+1)}(x-2)}\\ &= \frac{(x-1)(x+2)}{x-2}\end{aligned}$$

b Multiply the first fraction by the reciprocal of the second fraction.

$$\begin{aligned}\frac{6ab}{2a+b} \div (4a^2b) &= \frac{\overset{3}{\cancel{6}}\cancel{ab}}{2a+b} \cdot \frac{1}{\underset{2}{\cancel{4}}a \cdot \cancel{ab}} \qquad \text{Divide out common factors.}\\ &= \frac{3}{2a(2a+b)} = \frac{3}{4a^2+2ab}\end{aligned}$$

□

A.9.4 Sums and Differences of Like Fractions

Algebraic fractions with the same denominator are called **like fractions**. To add or subtract like fractions, we combine their numerators and keep the same denominator for the sum or difference. This method is an application of the distributive law.

Example A.9.8 Find each sum or difference.

a $\dfrac{2x}{9z^2} + \dfrac{5x}{9z^2}$

b $\dfrac{2x-1}{x+3} - \dfrac{5x-3}{x+3}$

Solution.

a Because these are like fractions, we add their numerators and keep the same denominator.

$$\frac{2x}{9z^2} + \frac{5x}{9z^2} = \frac{2x+5x}{9z^2} = \frac{7x}{9z^2}$$

b Be careful to subtract the *entire* numerator of the second fraction: Use parentheses to show that the subtraction applies to both terms of $5x - 3$.

$$\frac{2x-1}{x+3} - \frac{5x-3}{x+3} = \frac{2x-1\mathbf{-(5x-3)}}{x+3}$$
$$= \frac{2x-1\mathbf{-5x+3}}{x+3} = \frac{-3x+2}{x+3}$$

□

A.9.5 Lowest Common Denominator

To add or subtract fractions with different denominators, we must first find a **common denominator**.

For arithmetic fractions, we use the smallest natural number that is exactly divisible by each of the given denominators. For example, to add the fractions $\frac{1}{6}$ and $\frac{3}{8}$, we use 24 as the common denominator because 24 is the smallest natural number that both 6 and 8 divide into evenly.

We define the **lowest common denominator** (**LCD**) of two or more algebraic fractions as the polynomial of least degree that is exactly divisible by each of the given denominators.

Example A.9.9 Find the LCD for the fractions $\dfrac{3x}{x+2}$ and $\dfrac{2x}{x-3}$

Solution. The LCD is a polynomial that has as factors both $x + 2$ and $x - 3$. The simplest such polynomial is $(x + 2)(x - 3)$, or $x^2 - x - 6$. For our purposes, it will be more convenient to leave the LCD in factored form, so the LCD is $(x + 2)(x - 3)$. □

The LCD in Example A.9.9, p. 923 was easy to find because each original denominator consisted of a single factor; that is, neither denominator could be factored. In that case, the LCD is just the product of the original denominators.

We can always find a common denominator by multiplying together all the denominators in the given fractions, but this may not give us the *simplest* or *lowest* common denominator. Using anything other than the simplest possible common denominator will complicate our work needlessly.

If any of the denominators in the given fractions can be factored, we factor them before looking for the LCD.

To Find the LCD of Algebraic Fractions:.

1 Factor each denominator completely.

2 Include each different factor in the LCD as many times as it occurs in any *one* of the given denominators.

Example A.9.10 Find the LCD for the fractions $\dfrac{2x}{x^2-1}$ and $\dfrac{x+3}{x^2+x}$.

Solution. Factor the denominators of each of the given fractions.

$$x^2 - 1 = (x-1)(x+1) \quad \text{and} \quad x^2 + x = x(x+1)$$

The factor $(x - 1)$ occurs once in the first denominator, the factor x occurs once in the second denominator, and the factor $(x + 1)$ occurs once in each denominator. Therefore, we include in our LCD one copy of each of these factors. The LCD is $x(x + 1)(x - 1)$. □

Caution A.9.11 In Example A.9.10, p. 923, we do not include two factors of $(x+1)$ in the LCD. We need only one factor of $(x+1)$ because $(x+1)$ occurs only once in either denominator. You should check that each original denominator divides evenly into our LCD, $x(x+1)(x-1)$.

A.9.6 Building Fractions

After finding the LCD, we **build** each fraction to an equivalent one with the LCD as its denominator. The new fractions will be like fractions, and we can combine them as explained above.

Building a fraction is the opposite of reducing a fraction, in the sense that we multiply, rather than divide, the numerator and denominator by an appropriate factor. To find the **building factor**, we compare the factors of the original denominator with those of the desired common denominator.

Example A.9.12 Build each of the fractions $\dfrac{3x}{x+2}$ and $\dfrac{2x}{x-3}$ to equivalent fractions with the LCD $(x+2)(x-3)$ as denominator.

Solution. Compare the denominator of the given fraction to the LCD. We see that the fraction $\dfrac{3x}{x+2}$ needs a factor of $(x-3)$ in its denominator, so $(x-3)$ is the building factor for the first fraction. We multiply the numerator and denominator of the first fraction by $(x-3)$ to obtain an equivalent fraction:

$$\frac{3x}{x+2} = \frac{3x\boldsymbol{(x-3)}}{(x+2)\boldsymbol{(x-3)}} = \frac{3x^2-9x}{x^2-x-6}$$

The fraction $\dfrac{2x}{x-3}$ needs a factor of $(x+2)$ in the denominator, so we multiply numerator and denominator by $(x+2)$:

$$\frac{2x}{x-3} = \frac{2x\boldsymbol{(x+2)}}{(x-3)\boldsymbol{(x+2)}} = \frac{2x^2+4x}{x^2-x-6}$$

□

The two new fractions we obtained in Example A.9.12, p. 924 are like fractions; they have the same denominator.

A.9.7 Sums and Differences of Unlike Fractions

We are now ready to add or subtract algebraic fractions with unlike denominators. We will do this in four steps.

To Add or Subtract Fractions with Unlike Denominators:.

1 Find the LCD for the given fractions.

2 Build each fraction to an equivalent fraction with the LCD as its denominator.

3 Add or subtract the numerators of the resulting like fractions. Use the LCD as the denominator of the sum or difference.

4 Reduce the sum or difference, if possible.

Example A.9.13 Subtract $\dfrac{3x}{x+2} - \dfrac{2x}{x-3}$.

Solution.

1. The LCD for these fractions is $(x+2)(x-3)$.

2. We build each fraction to an equivalent one with the LCD, as we did in Example A.9.12, p. 924.

$$\frac{3x}{x+2} = \frac{3x^2-9x}{x^2-x-6} \quad \text{and} \quad \frac{2x}{x-3} = \frac{2x^2+4x}{x^2-x-6}$$

3. Combine the numerators over the same denominator.

$$\begin{aligned}\frac{3x}{x+2} - \frac{2x}{x-3} &= \frac{3x^2-9x}{x^2-x-6} - \frac{2x^2+4x}{x^2-x-6} && \text{Subtract the numerators.}\\ &= \frac{(3x^2-9x)-(2x^2+4x)}{x^2-x-6}\\ &= \frac{x^2-13x}{x^2-x-6}\end{aligned}$$

4. Reduce the result, if possible. If we factor both numerator and denominator, we find

$$\frac{x(x-13)}{(x-3)(x+2)}$$

The fraction cannot be reduced.

□

Example A.9.14 Write as a single fraction: $1 + \frac{2}{a} - \frac{a^2+2}{a^2+a}$.

Solution.

1. To find the LCD, factor each denominator:

$$\begin{aligned} a &= a\\ a^2+a &= a(a+1)\end{aligned}$$

The LCD is $a(a+1)$.

2. Build each term to an equivalent fraction with the LCD as denominator. (The building factors for each fraction are shown in color.) The third fraction already has the LCD for its denominator.

$$\begin{aligned} 1 = \frac{1\cdot \boldsymbol{a(a+1)}}{1\cdot \boldsymbol{a(a+1)}} &= \frac{a^2+a}{a(a+1)}\\ \frac{2}{a} = \frac{2\cdot \boldsymbol{(a+1)}}{a\cdot \boldsymbol{(a+1)}} &= \frac{2a+2}{a(a+1)}\\ \frac{a^2+2}{a^2+a} &= \frac{a^2+2}{a(a+1)}\end{aligned}$$

3. Combine the numerators over the LCD.

$$\begin{aligned}1 + \frac{2}{a} - \frac{a^2+2}{a^2+a} &= \frac{a^2+a}{a(a+1)} + \frac{2a+2}{a(a+1)} - \frac{a^2+2}{a(a+1)}\\ &= \frac{a^2+a+(2a+2)-(a^2+2)}{a(a+1)}\\ &= \frac{3a}{a(a+1)}\end{aligned}$$

4. Reduce the fraction to find

$$\frac{3\cancel{a}}{\cancel{a}(a+1)} = \frac{3}{a+1}$$

□

A.9.8 Complex Fractions

A fraction that contains one or more fractions in either its numerator or its denominator or both is called a complex fraction. For example,

$$\frac{\frac{2}{3}}{\frac{5}{6}} \quad \text{and} \quad \frac{x+\frac{3}{4}}{x-\frac{1}{2}}$$

are complex fractions. Like simple fractions, complex fractions represent quotients. For the examples above,

$$\frac{\frac{2}{3}}{\frac{5}{6}} = \frac{2}{3} \div \frac{5}{6} \qquad \text{and} \qquad \frac{x+\frac{3}{4}}{x-\frac{1}{2}} = \left(x+\frac{3}{4}\right) \div \left(x-\frac{1}{2}\right)$$

We can always simplify a complex fraction into a standard algebraic fraction. If the denominator of the complex fraction is a single term, we can treat the fraction as a division problem and multiply the numerator by the reciprocal of the denominator. Thus,

$$\frac{\frac{2}{3}}{\frac{5}{6}} = \frac{2}{3} \div \frac{5}{6} = \frac{2}{3} \cdot \frac{6}{5} = \frac{4}{5}$$

If the numerator or denominator of the complex fraction contains more than one term, it is easier to use the fundamental principle of fractions to simplify the expression.

Example A.9.15 Simplify $\dfrac{x+\frac{3}{4}}{x-\frac{1}{2}}$

Solution. Consider all of the simple fractions that appear in the complex fraction; in this example $\frac{1}{2}$ and $\frac{3}{4}$. The LCD of these fractions is 4. If we multiply the numerator and denominator of the complex fraction by 4, we will eliminate the fractions within the fraction.

Be sure to multiply *each* term of the numerator and *each* term of the denominator by 4.

$$\frac{\mathbf{4}\left(x+\frac{3}{4}\right)}{\mathbf{4}\left(x-\frac{1}{2}\right)} = \frac{\mathbf{4}(x)+\mathbf{4}\left(\frac{3}{4}\right)}{\mathbf{4}(x)-\mathbf{4}\left(\frac{1}{2}\right)} = \frac{4x+3}{4x-2}$$

Thus, the original complex fraction is equivalent to the simple fraction $\dfrac{4x+3}{4x-2}$.

□

We summarize the method for simplifying complex fractions as follows.

To Simplify a Complex Fraction:.

1 Find the LCD of all the fractions contained in the complex fraction.

2 Multiply each term in the numerator and the denominator of the complex fraction by the LCD.

3 Reduce the resulting simple fraction, if possible.

A.9.9 Negative Exponents

Algebraic fractions are sometimes written using negative exponents. (You can review negative exponents in Section 3.1, p. 291.

Example A.9.16 Write each expression as a single algebraic fraction.

a $x^{-1} - y^{-1}$

b $\left(x^{-2} + y^{-2}\right)^{-1}$

Solution.

a $x^{-1} - y^{-1} = \frac{1}{x} - \frac{1}{y}$ or $\frac{y-x}{xy}$

b $\left(x^{-2} + y^{-2}\right)^{-1} = \left(\frac{1}{x^2} + \frac{1}{y^2}\right)^{-1} = \left(\frac{y^2+x^2}{x^2y^2}\right)^{-1} = \frac{x^2y^2}{y^2+x^2}$

□

When working with fractions and exponents, it is important to avoid some tempting but *incorrect* algebraic operations.

Caution A.9.17

1 In Example A.9.16, p. 927a, note that

$$\frac{1}{x} - \frac{1}{y} \neq \frac{1}{x-y}$$

For example, you can check that for $x = 2$ and $y = 3$,

$$\frac{1}{2} - \frac{1}{3} \neq \frac{1}{2-3} = -1$$

2 In Example A.9.16, p. 927b, note that

$$\left(x^{-2} + y^{-2}\right)^{-1} \neq x^2 + y^2$$

In general, the fourth law of exponents does *not* apply to sums and differences; that is,

$$(a+b)^n \neq a^n + b^n$$

A.9.10 Section Summary

A.9.10.1 Vocabulary

Look up the definitions of new terms in the Glossary.

- Rational expression
- Building factor
- Like fraction
- Reciprocal
- Common factor
- Algebraic fraction
- Complex fraction
- Common denominator
- Numerator
- Common term
- Reduce
- Polynomial division
- Denominator
- Opposite

A.9.10.2 SKILLS

Practice each skill in the exercises listed.

1 Reduce fractions: #1–24

2 Multiply fractions: #25–36

3 Divide fractions: #37–48

4 Add like fractions: #49–56

5 Find the LCD: #57–62

6 Add unlike fractions: #63–82

7 Simplify complex fractions: #83–106

A.9.11 Exercises A.9

For Problems 1-20, reduce the algebraic fraction.

1. $\dfrac{14c^2d}{-7c^2d^3}$

Answer. $\dfrac{-2}{d^2}$

2. $\dfrac{-12r^2st}{-6rst^2}$

3. $\dfrac{4x+6}{6}$

Answer. $\dfrac{2x+3}{3}$

4. $\dfrac{2y-8}{8}$

5. $\dfrac{6a^3-4a^2}{4a}$

Answer. $\dfrac{3a^2-2a}{2}$

6. $\dfrac{3x^3-6x^2}{6x^2}$

7. $\dfrac{6-6t^2}{(t-1)^2}$

Answer. $\dfrac{6(1+t)}{1-t}$

8. $\dfrac{4-4x^2}{(x+1)^2}$

9. $\dfrac{2y^2-8}{2y+4}$

Answer. $y-2$

10. $\dfrac{5y^2-20}{2y-4}$

11. $\dfrac{6-2v}{v^3-27}$

Answer. $\dfrac{-2}{v^2+3v+9}$

12. $\dfrac{4-2u}{u^3-8}$

13. $\dfrac{4x^3-36x}{6x^2+18x}$

Answer. $\dfrac{2(x-3)}{3}$

14. $\dfrac{5x^2+10x}{5x^3+20x}$

15. $\dfrac{y^2-9x^2}{(3x-y)^2}$

Answer. $\dfrac{y+3x)}{y-3x}$

16. $\dfrac{(2x-y)^2}{y^2-4x^2}$

17. $\dfrac{2x^2+x-6}{x^2+x-2}$

Answer. $\dfrac{2x-3}{x-1}$

18. $\dfrac{6x^2-x-1}{2x^2+9x-5}$

19. $\dfrac{8z^3-27}{4z^2-9}$

Answer. $\dfrac{4z^2+6z+9}{2z+3}$

20. $\dfrac{8z^3-1}{4z^2-1}$

21. Which of the following fractions are equivalent to $2a$ (on their common domain)?

a $\dfrac{2a+4}{4}$ b $\dfrac{4a^2-2a}{2a-1}$ c $\dfrac{4a^2-2a}{2a}$ d $\dfrac{a+3}{2a^2+6a}$

Answer. (b)

22. Which of the following fractions are equivalent to $3b$ (on their common domain)?

a $\dfrac{9b^2-3b}{3b}$ b $\dfrac{b+2}{3b^2+6b}$ c $\dfrac{3b-9}{9}$ d $\dfrac{9b^2-3b}{3b-1}$

23. Which of the following fractions are equivalent to -1 (where they are defined)?

a $\dfrac{2a+b}{2a-b}$ b $\dfrac{-(a+b)}{b-a}$ c $\dfrac{2a^2-1}{2a^2}$ d $\dfrac{-a^2+3}{a^2+3}$

Answer. None

24. Which of the following fractions are equivalent to -1 (where they are defined)?

a $\dfrac{2a-b}{b-2a}$ b $\dfrac{-b^2-2}{b^2+2}$ c $\dfrac{3b^2-1}{3b^2+1}$ d $\dfrac{b-1}{b}$

For Problems 25-36, write the product as a single fraction in lowest terms.

25. $\dfrac{-4}{3np}\cdot\dfrac{6n^2p^3}{16}$

Answer. $\dfrac{-np^2}{2}$

26. $\dfrac{14a^3b}{3b}\cdot\dfrac{-6}{7a^2}$

27. $5a^2b^2\cdot\dfrac{1}{a^3b^3}$

Answer. $\dfrac{5}{ab}$

28. $15x^2y\cdot\dfrac{3}{35xy^2}$

29. $\dfrac{5x+25}{2x}\cdot\dfrac{4x}{2x+10}$

Answer. 5

30. $\dfrac{3y}{4xy-6y^2}\cdot\dfrac{2x-3y}{12x}$

31. $\dfrac{4a^2-1}{a^2-16}\cdot\dfrac{a^2-4a}{2a+1}$

Answer. $\dfrac{a(2a-1)}{a+4}$

32. $\dfrac{9x^2-25}{2x-2}\cdot\dfrac{x^2-1}{6x-10}$

33. $\dfrac{2x^2-x-6}{3x^2+4x+1}\cdot\dfrac{3x^2+7x+2}{2x^2+7x+6}$

Answer. $\dfrac{x-2}{x+1}$

34. $\dfrac{3x^2-7x-6}{2x^2-x-1}\cdot\dfrac{2x^2-9x-5}{3x^2-13x-10}$

35. $\dfrac{3x^4-48}{x^4-4x^2-32}\cdot\dfrac{4x^4-8x^3+4x^2}{2x^4+16x}$

Answer. $\dfrac{6x(x-2)(x-1)^2}{(x^2-8)(x^2-2x+4)}$

36. $\dfrac{x^4-3x^3}{x^4+6x^2-27}\cdot\dfrac{x^4-81}{3x^4-81x}$

For Problems 37-48, write the quotient as a single fraction in lowest terms.

37. $\dfrac{4x-8}{3y}\div\dfrac{6x-12}{y}$

Answer. $\dfrac{2}{9}$

38. $\dfrac{6y-27}{5x}\div\dfrac{4y-18}{x}$

39. $\dfrac{a^2-a-6}{a^2+2a-15}\div\dfrac{a^2-4}{a^2+6a+5}$

Answer. $\dfrac{a+1}{a-2}$

40. $\dfrac{a^2+2a-15}{a^2+3a-10}\div\dfrac{a^2-9}{a^2-9a+14}$

41. $\dfrac{x^3+y^3}{x}\div\dfrac{x+y}{3x}$

Answer. $3(x^2-xy+y^2)$

42. $\dfrac{8x^3-y^3}{x+y}\div\dfrac{2x-y}{x^2-y^2}$

43. $1\div\dfrac{x^2-1}{x+2}$

Answer. $\dfrac{x+2}{x^2-1}$

44. $1\div\dfrac{x^2+3x+1}{x-2}$

45. $(x^2-5x+4)\div\dfrac{x^2-1}{x^2}$

Answer. $\dfrac{x^2(x-4)}{x+1}$

46. $(x^2-9)\div\dfrac{x^2-6x+9}{3x}$

47. $\dfrac{x^2+3x}{2y}\div(3x)$

Answer. $\dfrac{x+3}{6y}$

48. $\dfrac{2y^2+y}{3x}\div(2y)$

For Problems 49-56, write the sum or difference as a single fraction in lowest terms.

49. $\dfrac{x}{2}-\dfrac{3}{2}$

Answer. $\dfrac{x-3}{2}$

50. $\dfrac{y}{7}-\dfrac{5}{7}$

51. $\dfrac{1}{6}a+\dfrac{1}{6}b-\dfrac{5}{6}c$

Answer. $\dfrac{a+b-5c}{6}$

52. $\dfrac{1}{3}x-\dfrac{2}{3}y+\dfrac{1}{3}z$

53. $\dfrac{x-1}{2y}+\dfrac{x}{2y}$

Answer. $\dfrac{2x-1}{2y}$

54. $\dfrac{y+1}{b}-\dfrac{y-1}{b}$

55. $\dfrac{3}{x+2y}-\dfrac{x-3}{x+2y}-\dfrac{x-1}{x+2y}$

Answer. $\dfrac{-2x+7}{x+2y}$

56. $\dfrac{2}{a-3b}-\dfrac{b-2}{a-3b}+\dfrac{b}{a-3b}$

For Problems 57-62, find the LCD for the pair of fractions.

57. $\dfrac{5}{6(x+y)^2}$, $\dfrac{3}{4xy^2}$

Answer. $12xy^2(x+y)^2$

58. $\dfrac{1}{8(a-b)^2}$, $\dfrac{5}{12a^2b^2}$

59. $\dfrac{2a}{a^2+5a+4}$, $\dfrac{2}{(a+1)^2}$

Answer. $(a+4)(a+1)^2$

60. $\dfrac{3x}{x^2-3x+2}$, $\dfrac{3}{(x-1)^2}$

61. $\dfrac{x+2}{x^2-x}$, $\dfrac{x+1}{(x-1)^3}$

Answer. $x(x-1)^3$

62. $\dfrac{y-1}{y^2+2y}$, $\dfrac{y-3}{(y+2)^2}$

For Problems 63-82, write the sum or difference as a single fraction in lowest terms.

63. $\dfrac{x}{2}+\dfrac{2x}{3}$

Answer. $\dfrac{7x}{6}$

64. $\dfrac{3y}{4}+\dfrac{y}{3}$

65. $\dfrac{5}{6}y-\dfrac{3}{4}y$

Answer. $\dfrac{y}{12}$

66. $\dfrac{3}{4}x-\dfrac{1}{6}x$

67. $\dfrac{x+1}{2x}+\dfrac{2x-1}{3x}$

Answer. $\dfrac{7x+1}{6}$

68. $\dfrac{y-2}{4y}+\dfrac{2y-3}{3y}$

69. $\dfrac{5}{x}+\dfrac{3}{x-1}$

Answer. $\dfrac{8x-5}{x(x-1)}$

70. $\dfrac{2}{y+2}+\dfrac{3}{y}$

71. $\dfrac{y}{2y-1}-\dfrac{2y}{y+1}$

Answer. $\dfrac{3y-3y^2}{(y+1)(2y-1)}$

72. $\dfrac{2x}{3x+1}-\dfrac{x}{x-2}$

73. $\dfrac{y-1}{y+1}-\dfrac{y-2}{2y-3}$

Answer. $\dfrac{y^2-4y+5}{(y+1)(2y-3)}$

74. $\dfrac{x-2}{2x+1}-\dfrac{x+1}{x-1}$

75. $\dfrac{7}{5x-10}-\dfrac{5}{3x-6}$

Answer. $\dfrac{-4}{15(x-2)}$

76. $\dfrac{2}{3y+6}-\dfrac{3}{2y+4}$

77. $\dfrac{y-1}{y^2-3y}-\dfrac{y+1}{y^2+2y}$

Answer. $\dfrac{3y+1}{y(y-3)(y+2)}$

78. $\dfrac{x+1}{x^2+2x}-\dfrac{x-1}{x^2-3x}$

79. $x-\dfrac{1}{x}$

Answer. $\dfrac{x^2-1}{x}$

80. $1+\dfrac{1}{y}$

81. $x+\dfrac{1}{x-1}-\dfrac{1}{(x-1)^2}$

Answer. $\dfrac{x^3-2x^2+2x-2}{(x-1)^2}$

82. $y-\dfrac{2}{y^2-1}+\dfrac{3}{y+1}$

For Problems 83-94, write the complex fraction as a simple fraction in lowest terms.

83. $\dfrac{\dfrac{2}{a}+\dfrac{3}{2a}}{5+\dfrac{1}{a}}$

Answer. $\dfrac{7}{10a+2}$

84. $\dfrac{\dfrac{2}{y}+\dfrac{1}{2y}}{y+\dfrac{y}{2}}$

85. $\dfrac{1+\dfrac{2}{a}}{1-\dfrac{4}{a^2}}$

Answer. $\dfrac{a}{a-2}$

86. $\dfrac{9-\dfrac{1}{x^2}}{3-\dfrac{1}{x}}$

87. $\dfrac{h+\dfrac{h}{m}}{1+\dfrac{1}{m}}$

Answer. h

88. $\dfrac{1+\dfrac{1}{p}}{1-\dfrac{1}{p}}$

89. $\dfrac{1}{1-\dfrac{1}{q}}$

Answer. $\dfrac{q}{q-1}$

90. $\dfrac{4}{\dfrac{2}{v}+2}$

91. $\dfrac{L+C}{\dfrac{1}{L}+\dfrac{1}{C}}$

Answer. LC

92. $\dfrac{H-T}{\dfrac{H}{T}-\dfrac{T}{H}}$

93. $\dfrac{\dfrac{4}{x^2}-\dfrac{4}{z^2}}{\dfrac{2}{z}-\dfrac{2}{x}}$

Answer. $\dfrac{-2(x+z)}{xz}$

94. $\dfrac{\dfrac{6}{b}-\dfrac{6}{a}}{\dfrac{3}{a^2}-\dfrac{3}{b^2}}$

For Problems 95-106, write the expression as a single algebraic fraction.

95. $x^{-2}+y^{-2}$

Answer. $\dfrac{x^2+y^2}{x^2y^2}$

96. $x^{-2}-y^{-2}$

97. $2w^{-1}-(2w)^{-2}$

Answer. $\dfrac{8w-1}{4w^2}$

98. $3w^{-3}+(3w)^{-1}$

99. $a^{-1}b-ab^{-1}$

Answer. $\dfrac{b^2-a^2}{ab}$

100. $a-b^{-1}a-b^{-1}$

101. $(x^{-1}+y^{-1})^{-1}$

Answer. $\dfrac{xy}{x+y}$

102. $(1-xy^{-1})^{-1}$

103. $\dfrac{x+x^{-2}}{x}$

Answer. $\dfrac{x^3+1}{x^3}$

104. $\dfrac{x^{-1}-y}{x^{-1}}$

105. $\dfrac{a^{-1}+b^{-1}}{(ab)^{-1}}$

Answer. $b+a$

106. $\dfrac{x}{x^{-2}-y^{-2}}$

A.10 Working with Radicals

In some situations, radical notation is more convenient to use than exponents. In these cases, we usually simplify radical expressions algebraically as much as possible before using a calculator to obtain decimal approximations.

A.10.1 Properties of Radicals

$\sqrt[n]{a} = a^{1/n}$, we can use the laws of exponents to derive two important properties that are useful in simplifying radicals.

Properties of Radicals.

1 $\sqrt[n]{ab} = \sqrt[n]{a}\,\sqrt[n]{b}$, for $a, b \geq 0$

2 $\sqrt[n]{\dfrac{a}{b}} = \dfrac{\sqrt[n]{a}}{\sqrt[n]{b}}$, for $a \geq 0, \quad b > 0$

As examples, you can verify that

$$\sqrt{36} = \sqrt{4}\sqrt{9} \quad \text{and} \quad \sqrt[3]{\frac{1}{8}} = \frac{\sqrt[3]{1}}{\sqrt[3]{8}}$$

Example A.10.1 Which of the following are true?

a Is $\sqrt{36+64} = \sqrt{36} + \sqrt{64}$?

b Is $\sqrt[3]{8(64)} = \sqrt[3]{8}\sqrt[3]{64}$?

c Is $\sqrt{x^2+4} = x+2$?

d Is $\sqrt[3]{8x^3} = 2x$?

Solution. The statements in (b) and (d) are true, and both are examples of the first property of radicals.

Statements (a) and (c) are false. In general, $\sqrt[n]{a+b}$ is not equal to $\sqrt[n]{a} + \sqrt[n]{b}$, and $\sqrt[n]{a-b}$ is not equal to $\sqrt[n]{a} - \sqrt[n]{b}$. □

A.10.2 Simplifying Radicals

We use Property (1) to simplify radical expressions by factoring the radicand. For example, to simplify $\sqrt[3]{108}$, we look for perfect cubes that divide evenly into 108. The easiest way to do this is to try the perfect cubes in order:

$$1,\ 8,\ 27,\ 64,\ 125, \ldots$$

and so on, until we find one that is a factor. For this example, we find that $108 = 27 \cdot 4$. Using Property (1), we write

$$\sqrt[3]{108} = \sqrt[3]{27}\sqrt[3]{4}$$

Simplify the first factor to find

$$\sqrt[3]{108} = 3\sqrt[3]{4}$$

This expression is considered simpler than the original radical because the new radicand, 4, is smaller than the original, 108.

We can also simplify radicals containing variables. If the exponent on the variable is a multiple of the index, we can extract the variable from the radical. For instance,

$$\sqrt[3]{12} = x^{12/3} = x^4$$

(You can verify this by noting that $(x^4)^3 = x^{12}$.) If the exponent on the variable is not a multiple of the index, we factor out the highest power that is a multiple. For example,

$$\begin{aligned} \sqrt[3]{x^{11}} &= \sqrt[3]{x^9 \cdot x^2} && \text{Apply Property (1).} \\ &= \sqrt[3]{x^9} \cdot \sqrt[3]{2} && \text{Simplify } \boldsymbol{\sqrt[3]{x^9} = x^{9/3}}. \\ &= x^3 \sqrt[3]{x^2} \end{aligned}$$

Example A.10.2 Simplify each radical.

a $\sqrt{18x^5}$

b $\sqrt[3]{24x^6y^8}$

Solution.

a The index of the radical is 2, so we look for perfect square factors of $18x^5$. The factor 9 is a perfect square, and x^4 has an exponent divisible by 2. Thus,

$$\begin{aligned} \sqrt{18x^5} &= \sqrt{9x^4 \cdot 2x} && \text{Apply Property (1).} \\ &= \sqrt{9x^4}\sqrt{2x} && \text{Take square roots.} \\ &= 3x^2\sqrt{2x} \end{aligned}$$

b The index of the radical is 3, so we look for perfect cube factors of $24x^6y^8$. The factor 8 is a perfect cube, and x^6 and y^6 have exponents divisible by 3. Thus,

$$\begin{aligned} \sqrt[3]{24x^6y^8} &= \sqrt[3]{8x^6y^6 \cdot 3y^2} && \text{Apply Property (1).} \\ &= \sqrt[3]{8x^6y^6}\,\sqrt[3]{3y^2} && \text{Take cube roots.} \\ &= 2x^2y^2\,\sqrt[3]{3y^2} \end{aligned}$$

□

Caution A.10.3 Property (1) applies only to products under the radical, not to sums or differences. Thus, for example,

$$\sqrt{4 \cdot 9} = \sqrt{4}\sqrt{9} = 2 \cdot 3, \quad \text{but} \quad \sqrt{4+9} \neq \sqrt{4} + \sqrt{9}$$

and

$$\sqrt[3]{x^3y^6} = \sqrt[3]{x^3}\,\sqrt[3]{y^6} = xy^2, \quad \text{but} \quad \sqrt[3]{x^3 - y^6} \neq \sqrt[3]{x^3} - \sqrt[3]{y^6}$$

To simplify roots of fractions, we use Property (2), which allows us to write the expression as a quotient of two radicals.

Example A.10.4

a $\sqrt{\dfrac{3}{4}} = \dfrac{\sqrt{3}}{\sqrt{4}} = \dfrac{\sqrt{3}}{2}$

b $\sqrt[3]{\dfrac{5}{8}} = \dfrac{\sqrt[3]{5}}{\sqrt[3]{8}} = \dfrac{\sqrt[3]{5}}{2}$

□

We can also use Properties (1) and (2) to simplify products and quotients of radicals.

Example A.10.5 Simplify.

a $\sqrt[4]{6x^2}\sqrt[4]{8x^3}$

b $\dfrac{\sqrt[3]{16y^5}}{\sqrt[3]{y^2}}$

Solution.

a First apply Property (1) to write the product as a single radical, then simplify.

$$\begin{aligned}\sqrt[4]{6x^2}\sqrt[4]{8x^3} &= \sqrt[4]{48x^5} && \text{Factor out perfect fourth powers.}\\ &= \sqrt[4]{16x^4}\sqrt[4]{3x} && \text{Simplify.}\\ &= 2x\sqrt[4]{3x}\end{aligned}$$

b Apply Property (2) to write the quotient as a single radical.

$$\begin{aligned}\frac{\sqrt[3]{16y^5}}{\sqrt[3]{y^2}} &= \sqrt[3]{\frac{16y^5}{y^2}} && \text{Reduce.}\\ &= \sqrt[3]{16y^3} && \text{Simplify: factor out perfect cubes.}\\ &= \sqrt[3]{8y^3}\sqrt[3]{2}\\ &= 2y\sqrt[3]{2}\end{aligned}$$

□

A.10.3 Sums and Differences of Radicals

You know that sums or differences of like terms can be combined by adding or subtracting their coefficients:

$$3xy + 5xy = (3 + 5)xy = 8xy$$

Like radicals, that is, radicals of the same index and radicand, can be combined in the same way.

Example A.10.6

a $3\sqrt{3} + 4\sqrt{3} = (3 + 4)\sqrt{3}$
$= 7\sqrt{3}$

b $4\sqrt[3]{y} - 6\sqrt[3]{y} = (4 - 6)\sqrt[3]{y}$
$= -2\sqrt[3]{y}$

□

Caution A.10.7

1 In Example A.10.6, p. 935a, $3\sqrt{3} + 4\sqrt{3} \neq 7\sqrt{6}$. Only the coefficients are added; the radicand does not change.

2 Sums of radicals with different radicands or different indices cannot be combined. Thus,

$$\sqrt{11} + \sqrt{5} \neq \sqrt{16}$$
$$\sqrt[3]{10x} - \sqrt[3]{2x} \neq \sqrt[3]{8x}$$

and

$$\sqrt[3]{7} + \sqrt{7} \neq \sqrt[5]{7}$$

None of the expressions above can be simplified.

A.10.4 Products of Radicals

According to Property (1), radicals of the same index can be multiplied together.

Product of Radicals.

$$\sqrt[n]{a}\,\sqrt[n]{b} = \sqrt[n]{ab} \qquad (a, b \geq 0)$$

Thus, for example,

$$\sqrt{2}\sqrt{18}\sqrt{36} = 6 \quad \text{and} \quad \sqrt[3]{2x}\,\sqrt[3]{4x^2} = \sqrt[3]{8x^3} = 2x$$

For products involving binomials, we can apply the distributive law.

Example A.10.8

a $\sqrt{3}\left(\sqrt{2x} + \sqrt{6}\right) = \sqrt{3 \cdot 2x} + \sqrt{3 \cdot 6}$
$= \sqrt{6x} + \sqrt{18} = \sqrt{6x} + 3\sqrt{2}$

b $\left(\sqrt{x} - \sqrt{y}\right)\left(\sqrt{x} + \sqrt{y}\right) = \sqrt{x^2} + \sqrt{xy} - \sqrt{xy} - \sqrt{y^2}$
$= x - y$

□

A.10.5 Rationalizing the Denominator

It is easier to work with radicals if there are no roots in the denominators of fractions. We can use the fundamental principle of fractions to remove radicals from the denominator. This process is called **rationalizing the denominator**. For square roots, we multiply the numerator and denominator of the fraction by the radical in the denominator.

Example A.10.9 Rationalize the denominator of each fraction.

a $\sqrt{\dfrac{1}{3}}$

b $\dfrac{\sqrt{2}}{\sqrt{50x}}$

Solution.

a Apply Property (2) to write the radical as a quotient.

$$\sqrt{\frac{1}{3}} = \frac{\sqrt{1}}{\sqrt{3}}$$
$$= \frac{1}{\sqrt{3}} \qquad \text{Multiply numerator and denominator by } \boldsymbol{\sqrt{3}}.$$
$$= \frac{1 \cdot \boldsymbol{\sqrt{3}}}{\sqrt{3} \cdot \boldsymbol{\sqrt{3}}}$$
$$= \frac{\sqrt{3}}{3}$$

b It is always best to simplify the denominator before rationalizing.

$$\begin{aligned}\frac{\sqrt{2}}{\sqrt{50x}} &= \frac{\sqrt{2}}{5\sqrt{2x}} && \text{Multiply numerator and denominator by } \boldsymbol{\sqrt{2x}}. \\ &= \frac{\sqrt{2}\cdot\boldsymbol{\sqrt{2x}}}{5\sqrt{2x}\cdot\boldsymbol{\sqrt{2x}}} && \text{Simplify.} \\ &= \frac{\sqrt{4x}}{5(2x)} \\ &= \frac{2\sqrt{x}}{10x} = \frac{\sqrt{x}}{5x}\end{aligned}$$

□

If the denominator of a fraction is a *binomial* in which one or both terms is a radical, we can use a special building factor to rationalize it. First, recall that

$$(p-q)(p+q) = p^2 - q^2$$

where the product consists of perfect squares only. Each of the two factors $p-q$ and $p+q$ is said to be the **conjugate** of the other.

Now consider a fraction of the form

$$\frac{a}{b+\sqrt{c}}$$

If we multiply the numerator and denominator of this fraction by the conjugate of the denominator, we get

$$\frac{a\boldsymbol{(b-\sqrt{c})}}{(b+\sqrt{c})\boldsymbol{(b-\sqrt{c})}} = \frac{ab-a\sqrt{c}}{b^2-(\sqrt{c})^2} = \frac{ab-a\sqrt{c}}{b^2-c}$$

The denominator of the fraction no longer contains any radicals -- it has been rationalized.

Multiplying numerator and denominator by the conjugate of the denominator also works on fractions of the form

$$\frac{a}{\sqrt{b}+c} \quad \text{and} \quad \frac{a}{\sqrt{b}+\sqrt{c}}$$

We leave the verification of these cases as exercises.

Example A.10.10 Rationalize the denominator: $\dfrac{x}{\sqrt{2}+\sqrt{x}}$.

Solution. Multiply numerator and denominator by the conjugate of the denominator, $\sqrt{2}-\sqrt{x}$.

$$\frac{x(\boldsymbol{\sqrt{2}-\sqrt{x}})}{(\sqrt{2}+\sqrt{x})(\boldsymbol{\sqrt{2}-\sqrt{x}})} = \frac{x(\sqrt{2}-\sqrt{x})}{2-x}$$

□

A.10.6 Simplifying $\sqrt[n]{x^n}$

Raising to a power is the inverse operation for extracting roots; that is,

$$\left(\sqrt[n]{a}\right)^n = a$$

as long as $\sqrt[n]{a}$ is a real number. For example,

$$\left(\sqrt[4]{16}\right)^4 = 2^4 = 16, \quad \text{and} \quad \left(\sqrt[3]{-125}\right)^3 = (-5)^3 = -125$$

Now consider the power and root operations in the opposite order; is it true that $\sqrt[n]{a^n} = a$? If the index n is an odd number, then the statement is always true. For example,

$$\sqrt[3]{2^3} = \sqrt[3]{8} = 2 \quad \text{and} \quad \sqrt[3]{(-2)^3} = \sqrt[3]{-8} = -2$$

However, if n is even, we must be careful. Recall that the principal root $\sqrt[n]{x}$ is always positive, so if a is a negative number, it cannot be true that $\sqrt[n]{a^n} = a$. For example, if $a = -3$, then

$$\sqrt{(-3)^2} = \sqrt{9} = 3$$

Instead, we see that, for even roots, $\sqrt[n]{a^n} = |a|$.

We summarize our results in below.

Roots of Powers.

1 If n is odd, $\sqrt[n]{a^n} = a$

2 If n is even, $\sqrt[n]{a^n} = |a|$

In particular, $\sqrt{a^2} = |a|$

Example A.10.11

a $\sqrt{16x^2} = 4\,|x|$

b $\sqrt{(x-1)^2} = |x-1|$

□

A.10.7 Extraneous Solutions to Radical Equations

It is important to check the solution to a radical equation, because it is possible to introduce false, or **extraneous**, solutions when we square both sides of the equation. For example, the equation

$$\sqrt{x} = -5$$

has no solution, because $\sqrt{x}$ is never a negative number. However, if we try to solve the equation by squaring both sides, we find

$$\begin{aligned}(\sqrt{x})^{\mathbf{2}} &= (-5)^{\mathbf{2}}\\ x &= 25\end{aligned}$$

You can check that 25 is *not* a solution to the original equation, $\sqrt{x} = -5$, because $\sqrt{25}$ does not equal -5.

If each side of an equation is raised to an odd power, extraneous solutions will not be introduced. However, if we raise both sides to an even power, we should check each solution in the original equation.

Example A.10.12 Solve the equation $\sqrt{x+2} + 4 = x$.

Solution. First, isolate the radical expression on one side of the equation. (This will make it easier to square both sides.)

$$\begin{aligned}\sqrt{x+2} &= x - 4 && \text{Square both sides of the equation.}\\ \left(\sqrt{x+2}\right)^{\mathbf{2}} &= (x-4)^{\mathbf{2}}\\ x + 2 &= x^2 - 8x + 16 && \text{Subtract } \boldsymbol{x+2} \text{ from both sides.}\end{aligned}$$

$x^2 - 9x + 14 = 0$ Factor the left side.

$x = 2 \quad \text{or} \quad x = 7$

Check

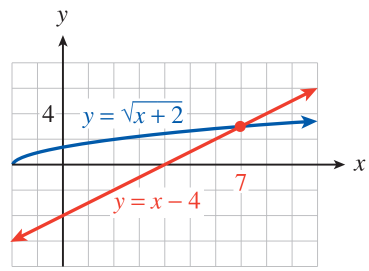
Does $\sqrt{2+2}+4 = 2$? No; **2** is not a solution.

Does $\sqrt{7+2}+4 = 7$? Yes; **7** is a solution.

The apparent solution 2 is extraneous. The only solution to the original equation is 7. We can verify the solution by graphing the equations

$$y_1 = \sqrt{x+2} \quad \text{and} \quad y_2 = x - 4$$

as shown at right. The graphs intersect in only one point, $(7, 3)$, so there is only one solution, $x = 7$.



□

Caution A.10.13 When we square both sides of an equation, it is *not* correct to square each term of the equation separately. Thus, in Example A.10.12, p. 938, the original equation is not equivalent to

$$(\sqrt{x+2})^2 + 4^2 = x^2$$

This is because $(a+b)^2 \neq a^2 + b^2$. Instead, we must square the *entire* left side of the equation as a binomial, like this,

$$(\sqrt{x+2}+4)^2 = x^2$$

or we may proceed as shown in Example A.10.12, p. 938.

A.10.8 Equations with More than One Radical

Sometimes it is necessary to square both sides of an equation more than once in order to eliminate all the radicals.

Example A.10.14 Solve $\sqrt{x-7} + \sqrt{x} = 7$.

Solution. First, isolate the more complicated radical on one side of the equation. (This will make it easier to square both sides.) We will subtract $\sqrt{x}$ from both sides.

$$\sqrt{x-7} = 7 - \sqrt{x}$$

Now square each side to remove one radical. Be careful when squaring the binomial $7 - \sqrt{x}$.

$$(\sqrt{x-7})^{\mathbf{2}} = (7 - \sqrt{x})^{\mathbf{2}}$$
$$x - 7 = 49 - 14\sqrt{x} + x$$

Collect like terms, and isolate the radical on one side of the equation.

$-56 = -14\sqrt{x}$ Divide both sides by **−14.**

$4 = \sqrt{x}$

Now square again to obtain

$$(4)^{\mathbf{2}} = (\sqrt{x})^{\mathbf{2}}$$

$$16 = x$$

Check

Does $\sqrt{\mathbf{16} - 7} + \sqrt{\mathbf{16}} = 7$? Yes. The solution is **16.**

□

Caution A.10.15 Recall that we cannot solve a radical equation by squaring each term separately. In other words, it is *incorrect* to begin Example 14, p. 939 by writing

$$(\sqrt{x-7})^2 + (\sqrt{x})^2 = 7^2$$

We must square the *entire expression* on each side of the equal sign as one piece.

A.10.9 Section Summary

A.10.9.1 Vocabulary

Look up the definitions of new terms in the Glossary.

- Radical
- Extraneous
- Rationalize
- Conjugate
- Like radicals
- Radicand
- Index

A.10.9.2 SKILLS

Practice each skill in the exercises listed.

1 Simplify radicals: #1–6

2 Simplify products and quotients of radcials: #7–10

3 Combine like radicals: #11–18

4 Multiply radical expressions: #19–36

5 Rationalize the denominator: #37–50

6 Simplify $\sqrt[n]{a^n}$: #51–54

7 Solve radical equations: #55–80

A.10.10 Exercises A.10

For Problems 1-6, simplify. Assume that all variables represent positive numbers.

1.

a $\sqrt{18}$ b $\sqrt[3]{24}$ c $-\sqrt[4]{64}$

Answer.

a $3\sqrt{2}$ b $2\sqrt[3]{3}$ c $-2\sqrt[4]{4} = -2\sqrt{2}$

2.

a $\sqrt{50}$ b $\sqrt[3]{54}$ c $-\sqrt[4]{162}$

3.

a $\sqrt{60,000}$ b $\sqrt[3]{900,000}$ c $\sqrt[3]{\dfrac{-40}{27}}$

Answer.

a $100\sqrt{6}$ b $10\sqrt[3]{900}$ c $\dfrac{-2}{3}\sqrt[3]{5}$

4.

a $\sqrt{800,000}$ b $\sqrt[3]{24,000}$ c $\sqrt[4]{\dfrac{80}{625}}$

5.

a $\sqrt[3]{x^{10}}$ b $\sqrt{27z^3}$ c $\sqrt[4]{48a^9}$

Answer.

a $x^3\sqrt[3]{x}$ b $3z\sqrt{3z}$ c $2a^2\sqrt[4]{3a}$

6.

a $\sqrt[3]{y^{16}}$ b $\sqrt{12t^5}$ c $\sqrt[3]{81b^8}$

For Problems 7-10, simplify.

7.

a $-\sqrt{18s}\sqrt{2s^3}$ b $\sqrt[3]{7h^2}\sqrt[3]{-49h}$ c $\sqrt{16-4x^2}$

Answer.

a $-6s^2$ b $-7h$ c $2\sqrt{4-x^2}$

8.

a $\sqrt{3w^3}\sqrt{27w^3}$ b $-\sqrt[4]{2m^3}\sqrt[4]{8m}$ c $\sqrt{9Y^2+18}$

9.

a $\sqrt[3]{8A^3+A^6}$ b $\dfrac{\sqrt{45x^3y^3}}{\sqrt{5y}}$ c $\dfrac{\sqrt[3]{8b^7}}{\sqrt[3]{a^6b^2}}$

Answer.

a $A\sqrt[3]{8+A^3}$ b $3xy\sqrt{x}$ c $\dfrac{2b\sqrt[3]{b^2}}{a^2}$

10.

a $\sqrt[3]{b^9-27b^3}$ b $\dfrac{\sqrt{98x^2y^3}}{\sqrt{xy}}$ c $\dfrac{\sqrt[3]{16r^4}}{\sqrt[3]{4t^3}}$

For Problems 11-18, simplify and combine like terms.

11. $3\sqrt{7}+2\sqrt{7}$

Answer. $5\sqrt{7}$

12. $5\sqrt{2}-3\sqrt{2}$

13. $4\sqrt{3}-\sqrt{27}$

Answer. $\sqrt{3}$

14. $\sqrt{75}+2\sqrt{3}$

15. $\sqrt{50x}+\sqrt{32x}$

Answer. $9\sqrt{2x}$

16. $\sqrt{8y}-\sqrt{18y}$

17. $3\sqrt[3]{16}-\sqrt[3]{2}-2\sqrt[3]{54}$

Answer. $-\sqrt[3]{2}$

18. $\sqrt[3]{81}+2\sqrt[3]{24}-3\sqrt[3]{3}$

For Problems 19-32, multiply.

19. $2(3-\sqrt{5})$

Answer. $6-2\sqrt{5}$

20. $5(2-\sqrt{7})$

21. $\sqrt{2}(\sqrt{6}+\sqrt{10})$

Answer. $2\sqrt{3}+2\sqrt{5}$

22. $\sqrt{3}(\sqrt{12}-\sqrt{15})$

23. $\sqrt[3]{2}(\sqrt[3]{20}-2\sqrt[3]{12})$

Answer. $2\sqrt[3]{5}-4\sqrt[3]{3}$

24. $\sqrt[3]{3}(2\sqrt[3]{18}+\sqrt[3]{36})$

25. $(\sqrt{x}-3)(\sqrt{x}+3)$

Answer. $x-9$

26. $(2+\sqrt{x})(2-\sqrt{x})$

27. $(\sqrt{2}-\sqrt{3})(\sqrt{2}+2\sqrt{3})$

Answer. $-4+\sqrt{6}$

28. $(\sqrt{3}-\sqrt{5})(2\sqrt{3}+\sqrt{5})$

29. $(\sqrt{5}-\sqrt{2})^2$

Answer. $7-2\sqrt{10}$

30. $(\sqrt{2}-2\sqrt{3})^2$

31. $(\sqrt{a}-2\sqrt{b})^2$

Answer. $a-4\sqrt{ab}+4b$

32. $(\sqrt{2a}-2\sqrt{b})(\sqrt{2a}+2\sqrt{b})$

For Problems 33-36, verify by substitution that the number is a solution of the quadratic equation.

33. $x^2-2x-2=0,\ \ 1+\sqrt{3}$

Answer.
$(1+\sqrt{3})^2-2(1+\sqrt{3})-2=0$

34. $x^2+4x-1=0,\ \ -2+\sqrt{5}$

35. $x^2+6x-9=0,\ \ -3+3\sqrt{2}$

Answer.
$(-3+3\sqrt{2})^2+6(-3+3\sqrt{2})-9=0$

36. $4x^2-20x+22=0,\ \ \dfrac{5-\sqrt{3}}{2}$

For Problems 37-50, rationalize the denominator.

37. $\dfrac{6}{\sqrt{3}}$

Answer. $2\sqrt{3}$

38. $\dfrac{10}{\sqrt{5}}$

39. $\sqrt{\dfrac{7x}{18}}$

Answer. $\dfrac{\sqrt{14x}}{6}$

40. $\sqrt{\dfrac{27x}{20}}$

41. $\sqrt{\dfrac{2a}{b}}$

Answer. $\dfrac{\sqrt{2ab}}{b}$

42. $\sqrt{\dfrac{5p}{q}}$

43. $\dfrac{2\sqrt{3}}{\sqrt{2k}}$

Answer. $\dfrac{\sqrt{6k}}{k}$

44. $\dfrac{6\sqrt{2}}{\sqrt{3v}}$

45. $\dfrac{4}{1+\sqrt{3}}$

Answer. $-2(1-\sqrt{3})$

46. $\dfrac{3}{7-\sqrt{2}}$

47. $\dfrac{x}{x-\sqrt{3}}$

Answer. $\dfrac{x(x+\sqrt{3})}{x^2-3}$

48. $\dfrac{y}{\sqrt{5}-y}$

49. $\dfrac{\sqrt{6}-3}{2-\sqrt{6}}$

Answer. $\dfrac{\sqrt{6}}{2}$

50. $\dfrac{\sqrt{x}+\sqrt{y}}{\sqrt{x}-\sqrt{y}}$

51. Use your calculator to graph each function, and explain the result.

a $y = \sqrt{x^2}$ b $y = \sqrt[3]{x^3}$

Answer.

a $y = \sqrt{x^2} = |x|$ b $y = \sqrt[3]{x^3} = x$

52. Use your calculator to graph each function, and explain the result.

a $y = (x^4)^{1/4}$ b $y = (x^5)^{1/5}$

For Problems 53-54, do not assume that variables represent positive numbers. Use absolute value bars as necessary to simplify the radicals.

53.

a $\sqrt{4x^2}$ b $\sqrt{(x-5)^2}$ c $\sqrt{x^2 - 6x + 9}$

Answer.

a $2|x|$ b $|x-5|$ c $|x-3|$

54.

a $\sqrt{9x^2y^4}$ b $\sqrt{(2x-1)^2}$ c $\sqrt{9x^2 - 6x + 1}$

For Problems 55-78, solve

55. $\sqrt{x} - 5 = 3$

Answer. 64

56. $\sqrt{x} - 4 = 1$

57. $\sqrt{y+6} = 2$

Answer. -2

58. $\sqrt{y-3} = 5$

59. $4\sqrt{z} - 8 = -2$

Answer. $\dfrac{9}{4}$

60. $-3\sqrt{z} + 14 = 8$

61. $5 + 2\sqrt{6-2w} = 13$

Answer. -5

62. $8 - 3\sqrt{9+2w} = -7$

63. $3z + 4 = \sqrt{3z+10}$

Answer. $\dfrac{-1}{3}$

64. $2x - 3 = \sqrt{7x-3}$

65. $2x + 1 = \sqrt{10x+5}$

Answer. $\dfrac{-1}{2}$, 2

66. $4x + 5 = \sqrt{3x+4}$

67. $\sqrt{y+4} = y - 8$

Answer. 12

68. $4\sqrt{x-4} = x$

69. $\sqrt{2y-1} = \sqrt{3y-6}$

Answer. 5

70. $\sqrt{4y+1} = \sqrt{6y-3}$

71. $\sqrt{x-3}\sqrt{x} = 2$

Answer. 4

72. $\sqrt{x}\sqrt{x-5} = 6$

73. $\sqrt{y+4} = \sqrt{y+20} - 2$

Answer. 5

74. $4\sqrt{y} + \sqrt{1+16y} = 5$

75. $\sqrt{x} + \sqrt{2} = \sqrt{x+2}$

Answer. 0

76. $\sqrt{4x+17} = 4 - \sqrt{x+1}$

77. $\sqrt{5+x} + \sqrt{x} = 5$

Answer. 4

78. $\sqrt{y+7} + \sqrt{y+4} = 3$

79. Explain why the following first step for solving the radical equation is incorrect:

$$\sqrt{x-5}+\sqrt{2x-1}=8$$
$$(x-5)+(2x-1)=64$$

Answer. We cannot square each term separately; we must square each side of the equation.

80. Explain why the following first step for solving the radical equation is incorrect:

$$\sqrt{x+2}+1=\sqrt{2x-3}$$
$$(x+2)+1=2x-3$$

For Problems 81-84, write the complex fraction as a simple fraction in lowest terms, and rationalize the denominator.

81. $\dfrac{\dfrac{2}{\sqrt{7}}}{1-\dfrac{\sqrt{3}}{\sqrt{7}}}$

Answer. $\dfrac{\sqrt{7}+\sqrt{3}}{2}$

82. $\dfrac{\dfrac{1}{4}}{\dfrac{\sqrt{5}}{2\sqrt{2}}-\dfrac{\sqrt{3}}{2\sqrt{2}}}$

83. $\dfrac{\dfrac{\sqrt{3}}{2}+\dfrac{1}{\sqrt{2}}}{1-\dfrac{\sqrt{3}}{2}\cdot\dfrac{1}{\sqrt{2}}}$

Answer. $\dfrac{6\sqrt{3}+7\sqrt{2}}{5}$

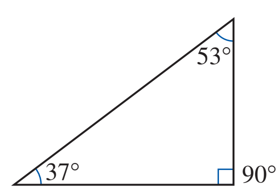
84. $\dfrac{\dfrac{1}{\sqrt{3}}-\dfrac{\sqrt{5}}{3}}{1+\dfrac{1}{\sqrt{3}}\cdot\dfrac{\sqrt{5}}{3}}$

A.11 Facts from Geometry

In this section, we review some information you will need from geometry. You are already familiar with the formulas for the area and perimeter of common geometric figures; you can find these formulas in the reference section Geometry formulas, p. 1027.

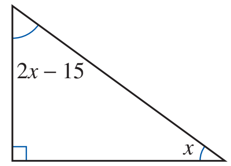
A.11.1 Right Triangles and the Pythagorean Theorem

A **right triangle** is a triangle in which one of the angles is a right angle, or 90°. Because the sum of the three angles in any triangle is 180°, this means that the other two angles in a right triangle must have a sum of 180° − 90°, or 90°. For instance, if we know that one of the angles in a right triangle is 37°, then the remaining angle must be 90° − 37°, or 53°, as shown at right.



Example A.11.1

In the right triangle shown at right, the medium-sized angle is 15° less than twice the smallest angle. Find the sizes of the three angles.



Solution.

Step 1 Let x stand for the size of the smallest angle. Then the medium-sized angle must be $2x - 15$.

Step 2 Because the right angle is the largest angle, the sum of the smallest and medium-sized angles must be the remaining 90°. Thus,

$$x + (2x - 15) = 90$$

Step 3 Solve the equation. Begin by simplifying the left side.

$$3x - 15 = 90 \qquad \text{Add 15 to both sides.}$$
$$3x = 105 \qquad \text{Divide both sides by 3.}$$
$$x = 35$$

Step 4 The smallest angle is 35°, and the medium-sized angle is $2(35°) - 15°$, or 55°.

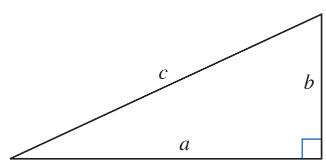
□

In a right triangle, the longest side is opposite the right angle and is called the **hypotenuse**. Ordinarily, even if we know the lengths of two sides of a triangle, it is not easy to find the length of the third side (to solve this problem we need trigonometry), but for the special case of a right triangle, there is an equation that relates the lengths of the three sides. This property of right triangles was known to many ancient cultures, and we know it today by the name of a Greek mathematician, Pythagoras, who provided a proof of the result.

Pythagorean Theorem.

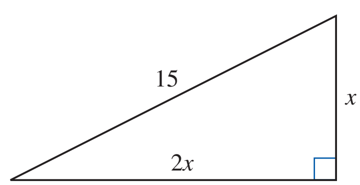
In a right triangle, if c stands for the length of the hypotenuse and a and b stand for the lengths of the two sides, then

$$a^2 + b^2 = c^2$$



Example A.11.2

The hypotenuse of a right triangle is 15 feet long. The third side is twice the length of the shortest side. Find the lengths of the other 2 sides.



Solution.

Step 1 Let x represent the length of the shortest side, so that the third side has length $2x$.

Step 2 Substituting these expressions into the Pythagorean theorem, we find

$$x^2 + (2x)^2 = 15^2$$

Step 3 This is a quadratic equation with no linear term, so we simplify and then isolate x^2.

$$\begin{aligned} x^2 + 4x^2 &= 225 && \text{Combine like terms.} \\ 5x^2 &= 225 && \text{Divide both sides by 5.} \\ x^2 &= 45 \end{aligned}$$

Taking square roots of both sides yields

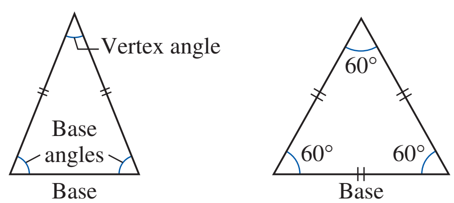
$$x = \pm\sqrt{45} \approx \pm 6.708203932$$

Step 4 Because a length must be a positive number, the shortest side has length approximately 6.71 feet, and the third side has length 2(6.71), or approximately 13.42 feet.

□

A.11.2 Isosceles and Equilateral Triangles

Recall also that an **isosceles** triangle is one that has at least two sides of equal length. In an isosceles triangle, the angles opposite the equal sides, called the **base** angles, are equal in measure. In an equilateral triangle, all three sides have equal length, and all three angles have equal measure.

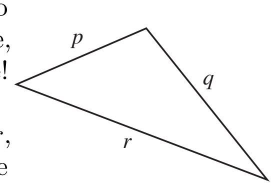


(a) Isosceles triangle **(b)** Equilateral triangle

A.11.3 The Triangle Inequality

The longest side in a triangle is always opposite the largest angle, and the shortest side is opposite the smallest angle.

It is also true that the sum of the lengths of any two sides of a triangle must be greater than the third side, or else the two sides will not meet to form a triangle! This fact is called the **triangle inequality**. In the triangle at right, we must have that $p + q > r$, where p, q, and r are the lengths of the sides of the triangle.



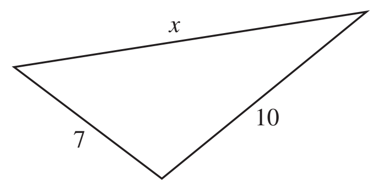
Now we can use the triangle inequality to discover information about the sides of a triangle.

Example A.11.3 Two sides of a triangle have lengths 7 inches and 10 inches. What can you say about the length of the third side?

Solution.

Let x represent the length of the third side of the triangle. By the triangle inequality, we must have that

$$x < 7 + 10, \quad \text{or} \quad x < 17$$



Looking at another pair of sides, we must also have that

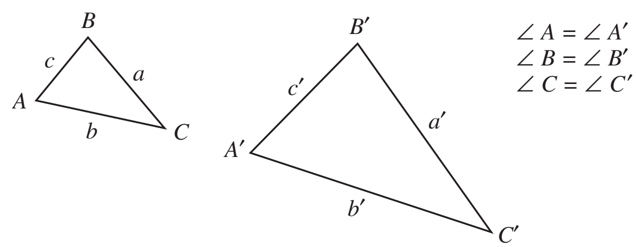
$$10 < x + 7, \quad \text{or} \quad x > 3$$

Thus the third side must be greater than 3 inches but less than 17 inches long. □

A.11.4 Similar Triangles

Two triangles are said to be **similar** if their corresponding angles are equal. This means that the two triangles will have the same shape but not necessarily the same size. One of the triangles will be an enlargement or a reduction of the other; so their corresponding sides are proportional.

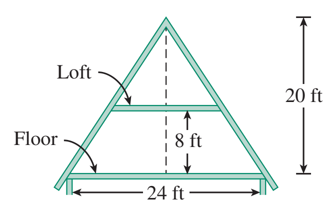
In other words, *for similar triangles, the ratios of the corresponding sides are equal.*



If any two pairs of corresponding angles of two triangles are equal, then the third pair must also be equal, because in both triangles the sum of the angles is 180°. Thus, to show that two triangles are similar, we need only show that two pairs of angles are equal.

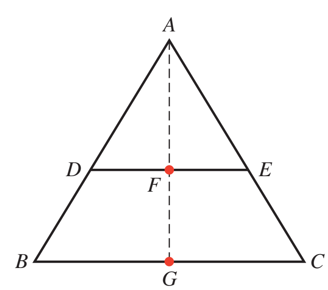
Example A.11.4

The roof of an A-frame ski chalet forms an isosceles triangle with the floor. The floor of the chalet is 24 feet wide, and the ceiling is 20 feet tall at the center. If a loft is built at a height of 8 feet from the floor, how wide will the loft be?



Solution.

Look at the diagram of the chalet at right. We can show that ΔABC is similar to ΔADE. Both triangles include $\angle A$, and because $\overline{DE}$ is parallel to $\overline{BC}$, $\angle ADE$ is equal to $\angle ABC$. Thus, the triangles have two pairs of equal angles and are therefore similar triangles.



1 *Step 1.*

Let w stand for the width of the loft.

2 *Step 2.*

First note that if $FG = 8$, then $AF = 12$. Because ΔABC is similar to ΔADE, the ratios of their corresponding sides (or corresponding altitudes) are equal. In particular,

$$\frac{w}{24} = \frac{12}{20}$$

3 *Step 3.*

Solve the proportion for w. Begin by cross-multiplying.

$$20w = (12)(24) \qquad \text{Apply the fundamental principle.}$$
$$w = \frac{288}{20} = 14.4 \qquad \text{Divide by 20.}$$

4 *Step 4.*

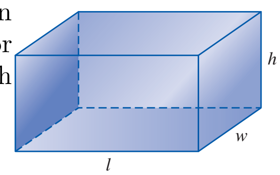
The floor of the loft will be 14.4 feet wide.

□

A.11.5 Volume and Surface Area

The **volume** of a three-dimensional object measures its capacity, or how much space it encloses. Volume is measured in cubic units, such as cubic inches or cubic meters.

The volume of a rectangular prism, or box, is given by the product of its length, width, and height. For example, the volume of the box of length 4 inches, width 3 inches, and height 2 inches shown at right is



$$V = lwh = 4(3)(2) = 24 \text{ cubic inches}$$

Formulas for the volumes of other common objects can be found inside the front cover of the book.

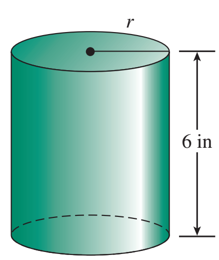
Example A.11.5 A cylindrical can must have a height of 6 inches, but it can have any reasonable radius.

a Write an algebraic expression for the volume of the can in terms of its radius.

b If the volume of the can should be approximately 170 cubic inches, what should its radius be?

Solution.

a



The formula for the volume of a right circular cylinder is $V = \pi r^2 h$. If the height of the cylinder is 6 inches, then $V = \pi r^2(6)$, or $V = 6\pi r^2$.

b Substitute 170 for V and solve for r.

$$170 = 6\pi r^2 \qquad \text{Divide both sides by } \boldsymbol{6\pi}.$$
$$r^2 = \frac{170}{6\pi} \qquad \text{Take square roots.}$$

$$r = \sqrt{\frac{170}{6\pi}} \approx 3.00312$$

Thus, the radius of the can should be approximately 3 inches. A calculator keying sequence for the expression above is

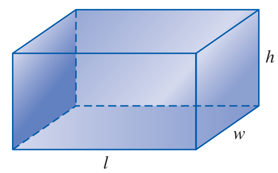
$$\boxed{\sqrt{\ }}\ \boxed{(}\ 170\ \boxed{\div}\ \boxed{(}\ 6\ \boxed{\pi}\ \boxed{)}\ \boxed{)}\ \boxed{\text{ENTER}}$$

□

The **surface area** of a solid object is the sum of the areas of all the exterior faces of the object. It measures the amount of paper that would be needed to cover the object entirely. Since it is an area, it is measured in square units.

Example A.11.6

Write a formula for the surface area of a closed box in terms of its length, width, and height.



Solution. The box has six sides; we must find the area of each side and add them.

- The top and bottom of the box each have area lw, so together they contribute $2lw$ to the surface area.
- The back and front of the box each have area lh, so they contribute $2lh$ to the surface area.
- Finally, the left and right sides of the box each have area wh, so they add $2wh$ to the surface area.

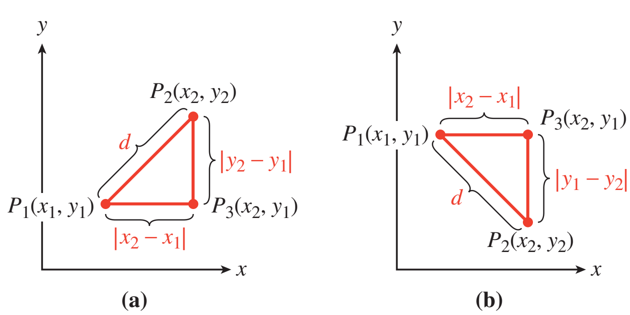
Thus, the total surface area is

$$S = 2lw + 2lh + 2wh$$

□

A.11.6 The Distance Formula

By using the Pythagorean theorem, we can derive a formula for the distance between two points, P_1 and P_2, in terms of their coordinates. We first label a right triangle, as we did in the example above. Draw a horizontal line through P_1 and a vertical line through P_2. These lines meet at a point P_3, as shown below. The x-coordinate of P_3 is the same as the x-coordinate of P_2, and the y-coordinate of P_3 is the same as the y-coordinate of P_1. Thus, the coordinates of P_3 are (x_2, y_1).



The distance between P_1 and P_3 is $|x_2 - x_1|$, and the distance between P_2 and P_3 is $|y_2 - y_1|$. (See Section 2.5, p. 234 to review distance and absolute value.)

These two numbers are the lengths of the legs of the right triangle. The length of the hypotenuse is the distance between P_1 and P_2, which we will call d. By the Pythagorean theorem,

$$d^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$$

Taking the (positive) square root of each side of this equation gives us the **distance formula**.

> **Distance Formula.**
>
> The **distance** d between points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ is
>
> $$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Example A.11.7 Find the distance between $(2, -1)$ and $(4, 3)$

Solution. Substitute $(2, -1)$ for (x_1, y_1) and substitute $(4, 3)$ for (x_2, y_2) in the distance formula to obtain

$$\begin{aligned} d &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\ &= \sqrt{(4-2)^2 + \left[3 - (-1)\right]^2} \\ &= \sqrt{4 + 16} = \sqrt{20} \approx 4.47 \end{aligned}$$

□

Note A.11.8 In Example A.11.7, p. 950, we obtain the same answer if we use $(4, 3)$ for P_1 and use $(2, -1)$ for P_2:

$$\begin{aligned} d &= \sqrt{(2-4)^2 + \left[(-1) - 3\right]^2} \\ &= \sqrt{4 + 16} = \sqrt{20} \end{aligned}$$

A.11.7 The Midpoint Formula

If we know the coordinates of two points, we can calculate the coordinates of the point halfway between them using the **midpoint** formula. Each coordinate of the midpoint is the average of the corresponding coordinates of the two points.

> **Midpoint Formula.**
>
> The **midpoint** of the line segment joining the points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ is the point $M(\overline{x}, \overline{y})$, where
>
> $$\overline{x} = \frac{x_1 + x_2}{2} \quad \text{and} \quad \overline{y} = \frac{y_1 + y_2}{2}$$

Example A.11.9 Find the midpoint of the line segment joining the points $(-2, 1)$ and $(4, 3)$.

Solution. Substitute $(-2, 1)$ for (x_1, y_1) and $(4, 3)$ for (x_2, y_2) in the midpoint formula to obtain

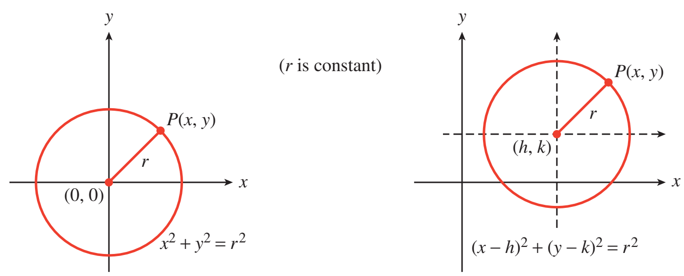
$$\overline{x} = \frac{x_1 + x_2}{2} = \frac{-2+4}{2} = 1 \qquad \text{and}$$
$$\overline{y} = \frac{y_1 + y_2}{2} = \frac{1+3}{2} = 2$$

The midpoint of the segment is the point $(\overline{x}, \overline{y}) = (1, 2)$. □

A.11.8 Circles

A **circle** is the set of all points in a plane that lie at a given distance, called the **radius**, from a fixed point called the **center**.

We can use the distance formula to find an equation for a circle. First consider the circle (a) below, whose center is the origin, $(0, 0)$.



(a) (b)

The distance from the origin to any point $P(x, y)$ on the circle is r. Therefore,

$$\sqrt{(x-0)^2 + (y-0)^2} = r$$

Or, squaring both sides,

$$(x-0)^2 + (y-0)^2 = r^2$$

Thus, the equation for a circle of radius r centered at the origin is

$$x^2 + y^2 = r^2$$

Now consider the circle (b) above, whose center is the point (h, k). Every point $P(x, y)$ on the circle lies a distance r from (h, k), so the equation of the circle is given by the following formula.

Standard Form for a Circle.

The equation for a **circle** of **radius** r centered at the point (h, k) is

$$(x-h)^2 + (y-k)^2 = r^2$$

This equation is the **standard form** for a circle of radius r with center at (h, k). It is easy to graph a circle if its equation is given in standard form.

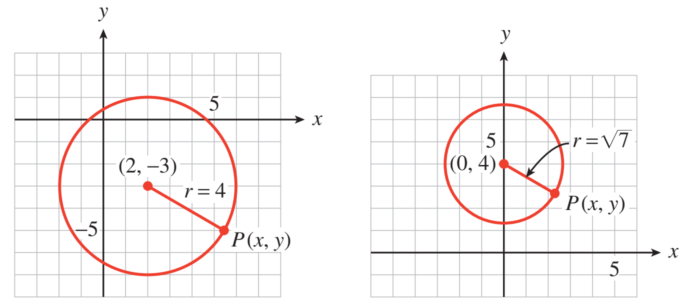
Example A.11.10 Graph the circles.

a $(x-2)^2 + (y+3)^2 = 16$ b $x^2 + (y-4)^2 = 7$

Solution.

a The graph of $(x-2)^2 + (y+3)^2 = 16$ is a circle with radius 4 and center at $(2, -3)$. To sketch the graph, first locate the center of the circle. (The center is not part of the graph of the circle.)

From the center, move a distance of 4 units (the radius of the circle) in each of four directions: up, down, left, and right. This locates four points that lie on the circle: $(2, 1)$, $(2, -7)$, $(-2, -3)$, and $(6, -3)$. Sketch the circle through these four points.



(a) **(b)**

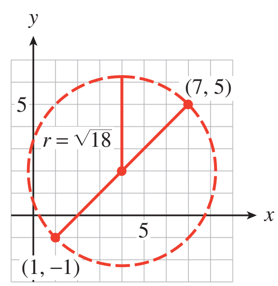
b The graph of $x^2 + (y - 4)^2 = 7$ is a circle with radius $\sqrt{7}$ and center at $(0, 4)$. From the center, move $\sqrt{7}$, or approximately 2.6, units in each of the four coordinate directions to obtain the points $(0, 6.6)$, $(0, 1.4)$, $(-2.6, 4)$, and $(2.6, 4)$. Sketch the circle through these four points.

□

We can write an equation for any circle if we can find its center and radius.

Example A.11.11 Find an equation for the circle whose diameter has endpoints $(7, 5)$ and $(1, -1)$.

Solution. The center of the circle is the midpoint of its diameter. Use the midpoint formula to find the center:



$$h = \overline{x} = \frac{7+1}{2} = 4$$
$$k = \overline{y} = \frac{5-1}{2} = 2$$

Thus, the center is the point $(h, k) = (4, 2)$. The radius is the distance from the center to either of the endpoints of the diameter, say the point $(7, 5)$. Use the distance formula with the points $(7, 5)$ and $(4, 2)$ to find the radius.

$$r = \sqrt{(7-4)^2 + (5-2)^2}$$
$$= \sqrt{3^2 + 3^2} = \sqrt{18}$$

Finally, substitute 4 for h and 2 for k (the coordinates of the center) and $\sqrt{18}$ for 4 (the radius) into the standard form to obtain

$$(x - h)^2 + (y - k)^2 = r^2$$
$$(x - 4)^2 + (y - 2)^2 = 18$$

□

A.11.9 Section Summary

A.11.9.1 Vocabulary

Look up the definitions of new terms in the Glossary.

- Right triangle
- Isosceles
- Volume
- Circle
- Hypotenuse
- Triangle inequality
- Surface area
- Center
- Equilateral

A.11.9.2 SKILLS

Practice each skill in the exercises listed.

1 Use properties of triangles: #1–10

2 Use similar triangles to solve problems: #11–16

3 Calculate volumes and surface areas: #17–20

4 Use the distance and midpoint formulas: #21–32

5 Sketch a circle: #33–40

6 Find the equation for a circle: #41–46

A.11.10 Exercises A.11

For Problems 1-10, use properties of triangles to answer the questions.

1. One angle of a triangle is 10° larger than another, and the third angle is 29° larger than the smallest. How large is each angle?

Answer. $47°,\ 57°,\ 76°$

2. One angle of a triangle is twice as large as the second angle, and the third angle is 10° less than the larger of the other two. How large is each angle?

3. One acute angle of a right triangle is twice the other acute angle. How large is each acute angle?

Answer. $30°,\ 60°$

4. One acute angle of a right triangle is 10° less than three times the other acute angle. How large is each acute angle?

5. The vertex angle of an isosceles triangle is 20° less than the sum of the equal angles. How large is each angle?

Answer. $50°,\ 50°,\ 80°$

6. The vertex angle of an isosceles triangle is 30° less than one of the equal angles. How large is each angle?

7. The perimeter of an isosceles triangle is 42 centimeters and its base is 12 centimeters long. How long are the equal sides?

Answer. $15\ \ \text{cm}$

8. The altitude of an equilateral triangle is $\dfrac{\sqrt{3}}{2}$ times its base. If the perimeter of an equilateral triangle is 18 inches, what is its area?

9. If two sides of a triangle are 6 feet and 10 feet long, what can you say about the length of the third side?

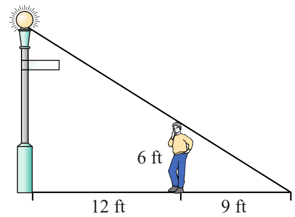
Answer. It is more than 4 and less than 16 feet long.

10. If one of the equal sides of an isosceles triangle is 8 millimeters long, what can you say about the length of the base?

For Problems 11-16, use properties of similar triangles to answer the questions.

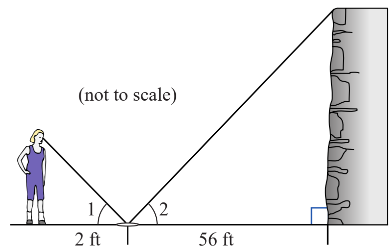
11.

A 6-foot man stands 12 feet from a lamppost. His shadow is 9 feet long. How tall is the lamppost?



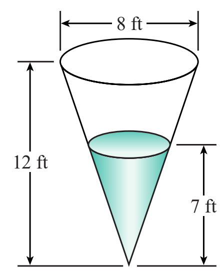
Answer. 14 ft

12. A rock climber estimates the height of a cliff she plans to scale as follows: She places a mirror on the ground so that she can just see the top of the cliff in the mirror while she stands straight. (The angles 1 and 2 formed by the light rays are equal.) She then measures the distance to the mirror (2 feet) and the distance from the mirror to the base of the cliff. If she is 5 feet 6 inches tall, how high is the cliff?



13.

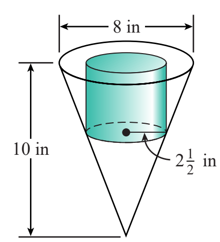
A conical tank is 12 feet deep and the diameter of the top is 8 feet. If the tank is filled with water to a depth of 7 feet, what is the area of the exposed surface of the water?



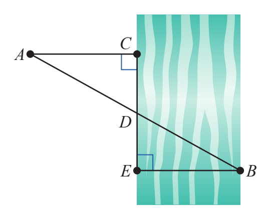
Answer. 17.1 sq ft

14.

A florist fits a cylindrical piece of foam into a conical vase that is 10 inches high and measures 8 inches across the top. If the radius of the foam cylinder is 212 inches, how tall should it be just to reach the top of the vase?

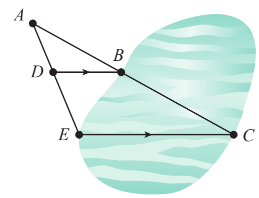


15. To measure the distance across a river, stand at point A and sight across the river to a convenient landmark at B. Then measure the distances AC, CD, and DE. If $AC = 20$ feet, $CD = 13$ feet, and $DE = 58$ feet, how wide is the river?



Answer. 89.23 ft

16. To measure the distance EC across a lake, stand at point A and sight point C across the lake, then mark point B. Then sight to point E and mark point D so that DB is parallel to CE. If $AD = 25$ yards, $AE = 60$ yards, and $BD = 30$ yards, how wide is the lake?



For Problems 17-20, use formulas to find volumes and surface areas.

17.

a How much helium (in cubic meters) is needed to inflate a spherical balloon to a radius of 1.2 meters?

b How much gelatin (in square centimeters) is needed to coat a spherical pill whose radius is 0.7 centimeter?

Answer.

a 7.24 cu m   b 6.16 sq cm

18.

a How much storage space is there in a rectangular box whose length is 12.3 inches, whose width is 4 inches, and whose height is 7.3 inches?

b How much marine sealer will be needed to paint a rectangular wooden storage locker with length 6.2 feet, width 5.8 feet, and height 2.6 feet?

19.

a How much grain can be stored in a cylindrical silo whose radius is 6 meters and whose height is 23.2 meters?

b How much paint is needed to cover a cylindrical storage drum whose radius is 15.3 inches and whose height is 4.5 inches?

Answer.

a 2623.86 cu m   b 1903.43 sq in

20.

a A conical pile of sand is 8.1 feet high and has a radius of 4.6 feet. How much sand is in the pile?

b How much plastic is needed to line a conical funnel with a radius of 16 centimeters and a slant height of 42 centimeters?

For Problems 21-26, find the distance between the given pairs of points, and find the midpoint of the segment joining them.

21. $(1,1),\ (4,5)$

Answer. 5; $\left(\dfrac{5}{2},3\right)$

22. $(-1,1),\ (5,9)$

23. $(2,-3),\ (-2,-1)$

Answer. $\sqrt{20}$; $(0,-2)$

24. $(5,-4),\ (-1,1)$

25. $(3,5),\ (-2,5)$

Answer. 5; $\left(\dfrac{1}{2},5\right)$

26. $(-2,-5),\ (-2,3)$

27. Leanne is sailing 3 miles west and 5 miles south of the harbor. She heads directly toward an island that is 8 miles west and 7 miles north of the harbor.

a How far is Leanne from the island?

b How far will Leanne be from the harbor when she is halfway to the island?

Answer.

a 13 miles

b $\dfrac{\sqrt{125}}{2} \approx 5.6$ miles

28. Dominic is 100 meters east and 250 meters north of Kristy. He is walking directly toward a tree that is 220 meters east and 90 meters north of Kristy.

a How far is Dominic from the tree?

b How far will Dominic be from the Kristy when he is halfway to the tree?

For Problems 29-32, sketch a diagram on graph paper, then solve the problem.

29. Find the perimeter of the triangle with vertices $(10,1),\ (3,1),\ (5,9)$.

Answer. $7+\sqrt{89}+\sqrt{67} \approx 24.7$

30. Find the perimeter of the triangle with vertices $(-1,5),\ (8,-7),\ (4,1)$.

31. Show that the point $C(\sqrt{5}, 2+\sqrt{5})$ is the same distance from $A(2,0)$ and $B(-2,4)$.

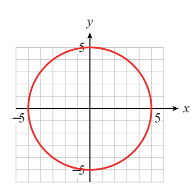
Answer. $AC = BC = \sqrt{18}$

32. Show that the points $(-2,1),\ (0,-1)$, and $(\sqrt{3}-1, \sqrt{3})$ are the vertices of an equilateral triangle.

For Problems 33-40, graph the equation.

33. $x^2+y^2=25$

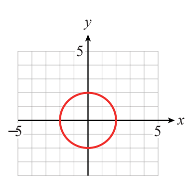
Answer.



34. $x^2+y^2=16$

35. $4x^2+4y^2=16$

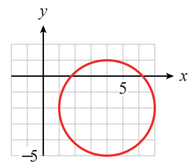
Answer.



36. $2x^2+2y^2=18$

37. $(x-4)^2+(y+2)^2=9$

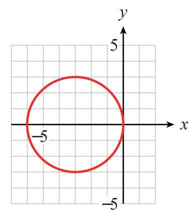
Answer.



38. $(x-1)^2+(y-3)^2=16$

39. $(x+3)^2+y^2=10$

Answer.



40. $x^2+(y+4)^2=12$

For Problems 41-46, write an equation for the circle with the given properties.

41. Center at $(-2,5)$, radius $2\sqrt{3}$.

Answer. $(x+2)^2+(y-5)^2=12$

42. Center at $(4,-3)$, radius $2\sqrt{6}$.

43. Center at $\left(\frac{3}{2},-4\right)$, one point on the cirlce $(4,-3)$.

Answer. $\left(x-\frac{3}{2}\right)^2+(y+4)^2=\frac{29}{4}$

44. Center at $\left(\frac{-3}{2},\frac{-1}{2}\right)$, one point on the cirlce $(-4,-2)$.

45. Endpoints of a diameter at $(1,5)$ and $(3,-1)$.

Answer. $(x-2)^2+(y-2)^2=10$

46. Endpoints of a diameter at $(3,6)$ and $(-5,2)$.

A.12 Properties of Lines

A.12.1 Horizontal and Vertical Lines

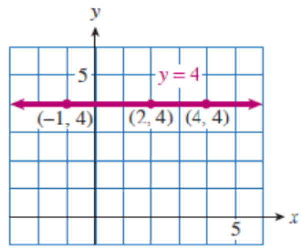
Two special cases of linear equations are worth noting. First, an equation such as $y=4$ can be thought of as an equation in two variables,

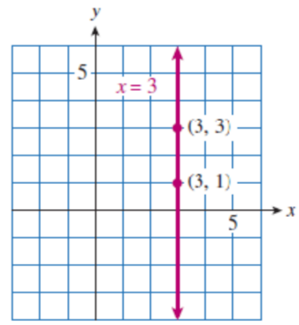
$$0x+y=4$$

For each value of x, this equation assigns the value 4 to y. Thus, any ordered pair of the form $(x,4)$ is a solution of the equation. For example,

$$(-1,4),\quad (2,4)\quad \text{and}\quad (4,4)$$

are all solutions of the equation. If we draw a straight line through these points, we obtain the **horizontal** line shown at left below.





The other special case of a linear equation is of the type $x = 3$, or

$$x + 0y = 3$$

Here, only one value is permissible for x, namely **3**, while any value may be assigned to y. Any ordered pair of the form $(3, y)$ is a solution of this equation. If we choose two solutions, say $(\mathbf{3}, 1)$ and $(\mathbf{3}, 3)$, and draw a straight line through these two points, we have the **vertical** line shown at right above. In general, we have the following results.

> The graph of $x = k$ (k a constant) is a vertical line.
> The graph of $y = k$ (k a constant) is a horizontal line.

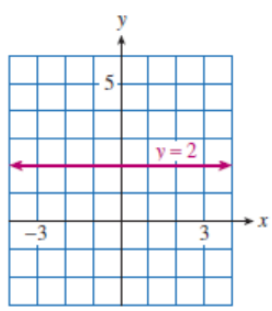
Example A.12.1

a Graph $y = 2$.

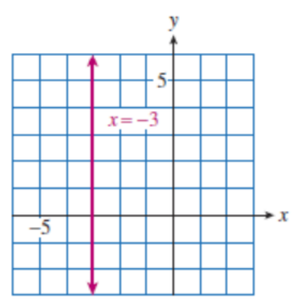
b Graph $x = -3$.

Solution.

a



b



□

Now let's compute the slopes of the two lines in the previous example. Choose two points on the graph of $y = 2$, say $(-5, 2)$ and $(4, 2)$. Use these points to compute the slope.

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{2 - 2}{4 - (-5)} = \frac{0}{9} = 0$$

The slope of the horizontal line $y = 2$ is zero. In fact, the slope of any horizontal line is zero, because the y -coordinates of all the points on the line are equal. Thus

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{0}{x_2 - x_1} = 0$$

On a vertical line, the x-coordinates of all the points are equal. For example, two points on the line $x = -3$ are $(-3, 1)$ and $(-3, 6)$. Using these points to compute the slope, we find

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{6 - 1}{-3 - (-3)} = \frac{5}{0}$$

which is undefined. The slope of any vertical line is undefined because the expression $x_2 - x_1$ equals zero.

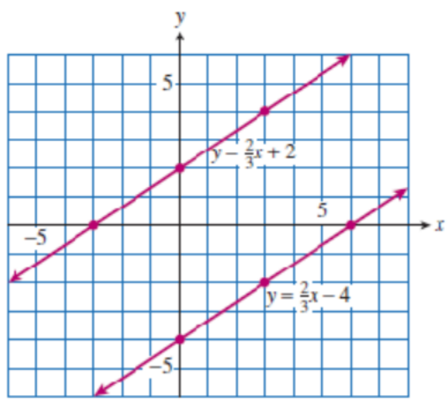
> The slope of a horizontal line is zero.
> The slope of a vertical line is undefined.

A.12.2 Parallel and Perpendicular Lines

Consider the graphs of the equations

$$y = \frac{2}{3}x - 4$$
$$y = \frac{2}{3}x + 2$$

shown below.



The lines have the same slope, $\frac{2}{3}$, but different y-intercepts. Because slope measures the steepness or inclination of a line, lines with the same slope are **parallel**.

> Two lines with slopes m_1 and m_2 are **parallel** if and only if $m_1 = m_2$.

Example A.12.2 Are the graphs of the equations $3x+6y = 6$ and $y = -\frac{1}{2}x+5$ parallel?

Solution. The lines are parallel if their slopes are equal. We can find the slope of the first line by putting its equation into slope-intercept form. We solve for y:

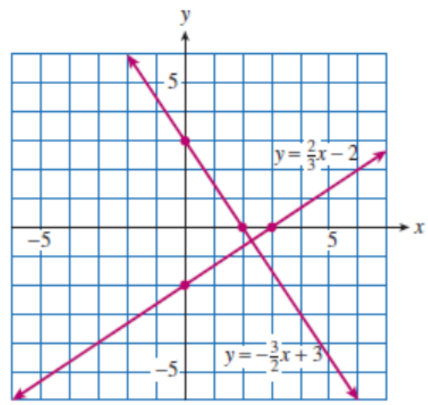
$$
\begin{aligned}
3x + 6y &= 6 && \text{Subtract } \mathbf{3x} \text{ from both sides.}\\
6y &= -3x + 6 && \text{Divide both sides by } \mathbf{6.}\\
y &= \frac{-3x}{6} + \frac{6}{6} && \text{Simplify.}\\
y &= = -\frac{1}{2}x + 1
\end{aligned}
$$

The slope of the first line is $m_1 = -\dfrac{1}{2}$. The equation of the second line is already in slope-intercept form, and its slope is $m_2 = -\dfrac{1}{2}$. Thus, $m_1 = m_2$, so the lines are parallel. □

Now consider the graphs of the equations

$$y = \frac{2}{3}x - 2$$
$$y = -\frac{3}{2}x + 3$$

shown below.



The lines appear to be **perpendicular**. The relationship between the slopes of perpendicular lines is not as easy to see as the relationship for parallel lines. However, for this example, $m_1 = \dfrac{2}{3}$ and $m_2 = -\dfrac{3}{2}$. Note that

$$m_2 = -\frac{3}{2} = \frac{-1}{\frac{2}{3}} = \frac{-1}{m_1}$$

This relationship holds for any two perpendicular lines with slopes m_1 and m_2, as long as $m_1 \neq 0$ and $m_2 \neq 0$.

> Two lines with slopes m_1 and m_2 are **perpendicular** if
>
> $$m_2 = \frac{-1}{m_1}.$$

We say that m_2 is the **negative reciprocal** of m_1.

Example A.12.3 Are the graphs of $3x-5y = 5$ and $2y = \dfrac{10}{3}x+3$ perpendicular?

Solution. We find the slope of each line by putting the equations into slope-intercept form. For the first line,

$$5y = 3x - 5 \quad \text{Divide both sides by } \mathbf{5}.$$
$$y = \frac{3}{5}x - 1$$

so $m_1 = \dfrac{3}{5}$. For the second line, $y = \dfrac{5}{3}x + \dfrac{3}{2}$, so $m_2 = \dfrac{5}{3}$. Now, the negative reciprocal of m_1 is

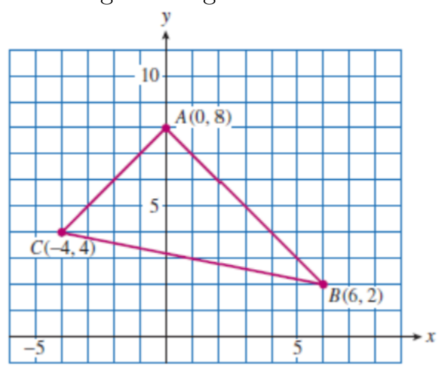
$$\frac{-1}{m_1} = \frac{-1}{\frac{3}{5}} = \frac{-5}{3}$$

but $m_2 = \frac{5}{3}$. Thus, $m_2 \neq \frac{-1}{m_1}$, so the lines are not perpendicular. □

A.12.3 Applications to Geometry

These relationships for the slopes of parallel and perpendicular lines can help us solve numerous geometric problems.

Example A.12.4 Show that the triangle with vertices $A(0,8)$, $B(6,2)$, and $C(-4,4)$ shown below is a right triangle.



Solution. We will show that two of the sides of the triangle are perpendicular. The line segment $\overline{AB}$ has slope

$$m_1 = \frac{2-8}{6-0} = \frac{-6}{6} = -1$$

and the line segment $\overline{AC}$ has slope

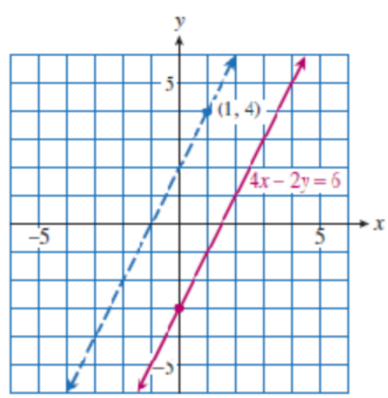
$$m_2 = \frac{4-8}{-4-0} = \frac{-4}{-4} = 1$$

Because

$$\frac{-1}{m_1} = \frac{-1}{-1} = 1 = m_2,$$

the sides $\overline{AB}$ and $\overline{AC}$ are perpendicular, and the triangle is a right triangle. □

Consider the graph of $4x - 2y = 6$ shown below.



Can we find the equation of the line that is parallel to this line, but passes through the point $(1, 4)$? If we can find the slope of the desired line, we can use the slope-intercept formula to find its equation.

Now because the line we want is parallel to the given line, they must have the same slope. To find the slope of the given line, we write its equation in slope-intercept form:

$$\begin{aligned} 4x - 2y &= 6 && \text{Subtract } \mathbf{4x} \text{ from both sides.} \\ -2y &= -4x + 6 && \text{Divide both sides by } \mathbf{-2.} \\ y &= 2x - 3 \end{aligned}$$

The slope of the given line is 2. Because the unknown line is parallel to this line, its slope is also 2. Now we know the slope of the desired line, $m = 2$, and one point on the line, $(1, 4)$. Substituting these values into the point-slope formula will give us the equation.

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - \mathbf{4} &= \mathbf{2}(x - \mathbf{1}) && \text{Apply the distributive law.} \\ y - 4 &= 2x - 2 && \text{Add 4 to both sides.} \\ y &= 2x + 2 \end{aligned}$$

Example A.12.5 Find an equation for the line that passes through the point $(1, 4)$ and is perpendicular to the line $4x - 2y = 6$.

Solution. We follow the same strategy as in the discussion above: First find the slope of the desired line, then use the point-slope formula to write its equation.

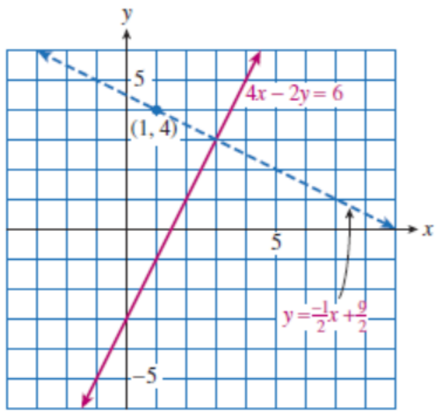
The line we want is perpendicular to the given line, so its slope is the negative reciprocal of $m_1 = 2$, the slope of the given line. Thus

$$m_2 = \frac{-1}{m_1} = \frac{-1}{2}$$

Now we use the point-slope formula with $m_2 = \frac{-1}{2}$ and $(x_1, y_1) = (1, 4)$.

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - \mathbf{4} &= \frac{\mathbf{-1}}{\mathbf{2}}(x - \mathbf{1}) && \text{Apply the distributive law.} \\ y - 4 &= \frac{-1}{2}x + \frac{1}{2} && \text{Add 4 to both sides.} \\ y &= \frac{-1}{2}x + \frac{9}{2} && \frac{\mathbf{1}}{\mathbf{2}} + \mathbf{4} = \frac{\mathbf{1}}{\mathbf{2}} + \frac{\mathbf{8}}{\mathbf{2}} = \frac{\mathbf{9}}{\mathbf{2}} \end{aligned}$$

The given line and the perpendicular line are shown below.



□

A.12.4 Section Summary

A.12.4.1 Vocabulary

Look up the definitions of new terms in the Glossary.

- Horizontal
- Vertical
- Parallel
- Perpendicular

A.12.4.2 SKILLS

Practice each skill in the exercises listed.

1 Sketch horizontal and vertical lines: #1–6

2 Find an equation for a horizontal or vertical line: #7–12

3 Identify parallel or perpendicular lines: #15–24

4 Find equations for parallel or perpendicular lines: #25–36

A.12.4.3 Reading Questions

1 Give an example of an equation for a vertical line, and for a horizontal line.

2 Why is the slope of a vertical line undefined?

3 What is the best way to determine whether two lines are parallel?

4 Suppose you know the equation of a certain line. Explain how to find the slope of a second line perpendicular to the first line.

A.12.5 Exercises A.12

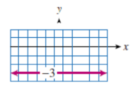
For Problems 1–6,

a Sketch a rough graph of each equation, and label its intercept.

b State the slope of each line.

1. $y = -3$

Answer.

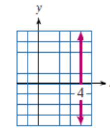
a 

b $m = 0$

2. $x = -2$

3. $2x = 8$

Answer.

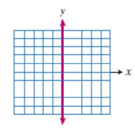
a 

b m is undefined

4. $3y = 15$

5. $x = 0$

Answer.

a 

b m is undefined

6. $y = 0$

For Problems 7–12, find the equation of the line described.

7. A vertical line through the point $(-5, 8)$

Answer. $x = -5$

8. A horizontal line through the point $(2, -4)$

9. The x-axis

Answer. $y = 0$

10. The y-axis

11. Perpendicular to $x = 3$ and intersecting it at $(3, 9)$

Answer. $y = 9$

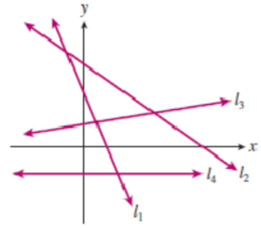
12. Parallel to the y-axis and including the point $(-1, -2)$

For Problems 13 and 14,

a Determine whether the slope of each line is positive, negative, zero, or undefined.

b List the lines in order of increasing slope.

13.

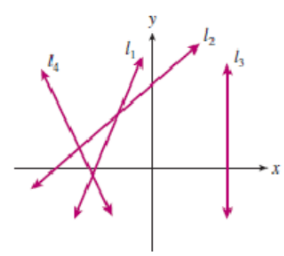


Answer.

a l_1 negative, l_2 negative, l_3 positive, l_4 zero

b l_1, l_2, l_4, l_3

14.



15.

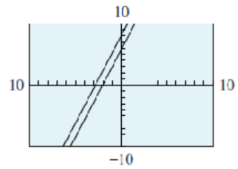
a Use your calculator to graph the equations $y = 3x+8$ and $y = 3.1x+6$ together in the standard window. Do you think the lines are parallel?

b Find the slope of each line in part (a). Are the lines parallel?

c Find the y-value for each equation when $x = 20$. What do your answers tell you about the two lines?

Answer.

a



b $m = 3,\ m = 3.1,$ No

c $y = 68$ for both lines. The lines intersect at $(20, 68)$.

16.

a Use your calculator to graph the equation $y = 0.001x + 4$ in the standard window. Do you think the line is horizontal?

b Find the slope and the x-intercept of the line in part (a). Is the line horizontal?

c Graph the equation in part (a) in the window

Xmin $= -5000$; Xmax $= 5000$

Ymin $= -10$; Ymax $= 10$

Find the coordinates of two convenient points on the line, and compute its slope using the slope formula.

17. The slopes of several lines are given below. Which of the lines are parallel to the graph of $y = 0.75x + 2$, and which are perpendicular to it?

a $m = \dfrac{3}{4}$

b $m = \dfrac{8}{6}$

c $m = \dfrac{-20}{15}$

d $m = \dfrac{-39}{52}$

e $m = \dfrac{4}{3}$

f $m = \dfrac{-16}{12}$

g $m = \dfrac{36}{48}$

h $m = \dfrac{9}{12}$

Answer. parallel: a, g, h; perpendicular: c, f

18. The slopes of several lines are given below. Which of the lines are parallel to the graph of $y = 2.5x - 3$, and which are perpendicular to it?

a $m = \dfrac{2}{5}$

b $m = \dfrac{25}{10}$

c $m = \dfrac{-8}{20}$

d $m = \dfrac{-45}{18}$

e $m = \dfrac{40}{16}$

f $m = 25$

g $m = \dfrac{-1}{25}$

h $m = \dfrac{-5}{10}$

19. In each part, determine whether the two lines are parallel, perpendicular, or neither.

a $y = \dfrac{3}{5}x - 7,\quad 3x - 5y = 2$

b $y = 4x + 3,\quad y = \dfrac{1}{4}x - 3$

c $6x + 2y = 1,\quad x = 1 - 3y$

d $2y = 5,\quad 5y = -2$

Answer.

a parallel

b neither

c neither

d parallel

20. In each part, determine whether the two lines are parallel, perpendicular, or neither.

a $2x - 7y = 14, \quad 7x - 2y = 14$

b $x + y = 6, \quad x - y = 6$

c $x = -3, \quad 3y = 5$

d $\frac{1}{4}x - \frac{3}{4}y = \frac{2}{3}, \quad \frac{1}{6}x + \frac{1}{2}y = \frac{1}{3}$

21.

a Sketch the triangle with vertices $A(2, 5)$, $B(5, 2)$, and $C(10, 7)$.

b Show that the triangle is a right triangle. (Hint: What should be true about the slopes of the two sides that form the right angle?)

Answer. Slope $\overline{AB} = -1$, slope $\overline{BC} = 1$, slope $\overline{AC} = \frac{1}{4}$. Hence $\overline{AB} \perp \overline{AB}$, so the triangle is a right triangle.

22.

a Sketch the triangle with vertices $P(-1, 3)$, $Q(-3, 8)$, and $R(4, 5)$.

b Show that the triangle is a right triangle. (See the hint for Problem 21.)

23.

a Sketch the quadrilateral with vertices $P(2, 4)$, $Q(3, 8)$, $R(5, 1)$, and $S(4, -3)$.

b Show that the quadrilateral is a parallelogram. (Hint: What should be true about the slopes of the opposite sides of the parallelogram?)

Answer. Slope $\overline{PQ} = -4$, slope $\overline{QR} = \frac{-7}{2}$, slope $\overline{RS} = 4$, slope $\overline{SP} = \frac{-7}{2}$. Hence $\overline{PQ} \parallel \overline{RS}$ and $\overline{QR} \parallel \overline{SP}$, so the points are the vertices of a parallelogram.

24.

a Sketch the quadrilateral with vertices $A(-5, 4)$, $B(7, -11)$, $C(12, 25)$, and $D(0, 40)$.

b Show that the quadrilateral is a parallelogram. (See the hint for Problem 23.)

25. Show that the line passing through the points $A(0, -3)$ and $B(3, \frac{1}{2})$ also passes through the point $C(-6, -10)$.

Answer. Slope $\overline{AB} = \frac{7}{6}$ = slope $\overline{BC}$, so A, B, and C lie on the same line.

26. Do the points $P(-5, -3\frac{1}{2})$, $Q(4, -2)$ and $R(9\frac{1}{2}, -1)$ lie on the same line? Why or why not?

Use graph paper for Problems 27-30.

27.

a Put the equation $x - 2y = 5$ into slope-intercept form, and graph the equation.

b What is the slope of any line that is parallel to $x - 2y = 5$?

c On your graph for part (a), sketch by hand a line that is parallel to $x - 2y = 5$ and passes through the point $(2, -1)$.

d Use the point-slope formula to write an equation for the line that is parallel to the graph of $x - 2y = 5$ and passes through the point $(2, -1)$.

Answer.

a $y = \frac{1}{2}x - \frac{5}{2}$

b $\frac{1}{2}$

c (sketch)

d $y = \frac{1}{2}x - 2$

28.

a Put the equation $2y - 3x = 5$ into slope-intercept form, and graph the equation.

b What is the slope of any line that is parallel to $2y - 3x = 5$?

c On your graph for part (a), sketch by hand a line that is parallel to $2y - 3x = 5$ and passes through the point $(-3, 2)$.

d Use the point-slope formula to write an equation for the line that is parallel to the graph of $2y - 3x = 5$ and passes through the point $(-3, 2)$.

29.

a Put the equation $2y - 3x = 5$ into slope-intercept form, and graph the equation.

b What is the slope of any line that is perpendicular to $2y - 3x = 5$?

c On your graph for part (a), sketch by hand a line that is perpendicular to $2y - 3x = 5$ and passes through the point $(1, 4)$.

d Use the point-slope formula to write an equation for the line that is perpendicular to the graph of $2y - 3x = 5$ and passes through the point $(1, 4)$.

Answer.

a $y = \frac{3}{2}x + \frac{5}{2}$

b $\frac{-2}{3}$

c (sketch)

d $y = \frac{-2}{3}x + \frac{14}{3}$

30.

a Put the equation $x - 2y = 5$ into slope-intercept form, and graph the equation.

b What is the slope of any line that is perpendicular to $x - 2y = 5$?

c On your graph for part (a), sketch by hand a line that is perpendicular to $x - 2y = 5$ and passes through the point $(4, -3)$.

d Use the point-slope formula to write an equation for the line that is perpendicular to the graph of $x - 2y = 5$ and passes through the point $(4, -3)$.

31. Two of the vertices of rectangle $ABCD$ are $A(-5, 2)$ and $B(-2, 4)$.

a Find an equation for the line that includes side $\overline{AB}$.

b Find an equation for the line that includes side $\overline{BC}$.

Answer.

a $y = -2x - 8$

b $y = \dfrac{1}{2}x - 3$

32. Two of the vertices of rectangle $PQRS$ are $P(-2, -6)$ and $Q(4, -4)$.

a Find an equation for the line that includes side $\overline{PQ}$.

b Find an equation for the line that includes side $\overline{QR}$.

For Problems 33 and 33, recall from geometry that the altitude from one vertex of a triangle to the opposite side is perpendicular to that side.

33.

a Sketch the triangle with vertices $A(-6, -3)$, $B(-6, 3)$ and $C(4, 5)$.

b Find the slope of the side $\overline{AC}$.

c Find the slope of the altitude from point B to side $\overline{BC}$.

d Find an equation for the line that includes the altitude from point B to side $\overline{BC}$.

Answer.

a (sketch)

b $m = \dfrac{4}{5}$

c $m = \dfrac{-5}{4}$

d $y = \dfrac{-5}{4}x - \dfrac{9}{2}$

34.

a Sketch the triangle with vertices $A(-5, 12)$, $B(4, -2)$ and $C(1, -6)$.

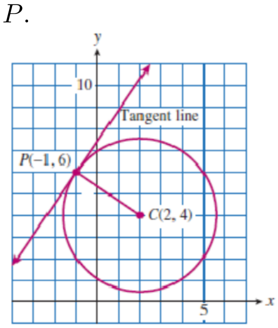
b Find the slope of the side $\overline{AC}$.

c Find the slope of the altitude from point B to side $\overline{BC}$.

d Find an equation for the line that includes the altitude from point B to side $\overline{BC}$.

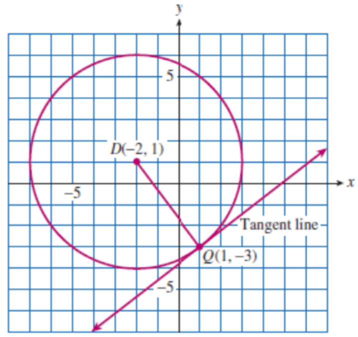
For Problems 35 and 36, recall from geometry that the tangent line to a circle is perpendicular to the radius to the point of tangency.

35. The center of a circle is the point $C(2, 4)$, and $P(-1, 6)$ is a point on the circle, as shown below. Find the equation of the line tangent to the circle at the point P.

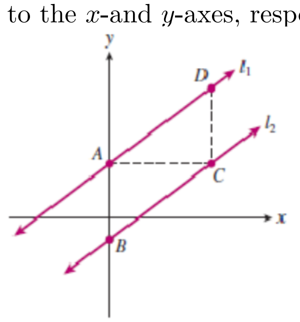


Answer. $y = \dfrac{3}{2}x + \dfrac{15}{2}$

36. The center of a circle is the point $D(-2, 1)$, and $Q(1, -3)$ is a point on the circle, as shown below. Find the equation of the line tangent to the circle at the point Q.



37. In this exercise we will show that parallel lines have the same slope. In the figure below, l_1 and l_2 are two parallel lines that are neither horizontal nor vertical. Their y-intercepts are A and B. The segments $\overline{AC}$ and $\overline{CD}$ are constructed parallel to the x-and y-axes, respectively.



Explain why each of the following statements is true.

a Angle ACD equals angle CAB.

b Angle DAC equals angle ACB.

c Triangle ACD is similar to triangle CAB.

d $m_1 = \dfrac{CD}{AC}$; $m_2 = \dfrac{AB}{AC}$

e $m_1 = m_2$

Answer.

a Right angles are equal.

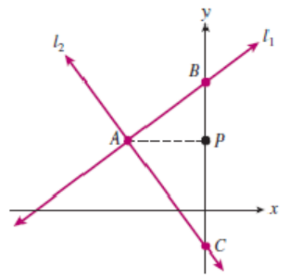
b Alternate interior angles are equal.

c Two angles of one triangle equal two angles of the other.

d Definition of slope.

e Corresponding sides of similar triangles are proportional.

38. In this exercise we will show that if two lines with slopes m_1 and m_2 (where neither line is vertical) are perpendicular, then m_2 is the negative reciprocal of m_1. In the figure below, lines l_1 and l_2 are perpendicular. Their y-intercepts are B and C. The segment $\overline{AP}$ is constructed through the point of intersection of l_1 and l_2 parallel to the x-axis.



Explain why each of the following statements is true.

a Angles ABC and ACB are complementary.

b Angles ABC and BAP are complementary.

c Angle BAP equals angle ACB.

d Angles CAP and ACB are complementary.

e Angle CAP equals angle ABC.

f Triangle ABP is similar to triangle CAP.

g $m_1 = \frac{BP}{AP}$; $m_2 = -\frac{CP}{AP}$

h $m_2 = \frac{-1}{m_1}$

A.13 The Real Number System

A.13.1 Subsets of the Real Numbers

The numbers associated with points on a number line are called the **real numbers**. The set of real numbers is denoted by $\mathbb{R}$. You are already familiar with several types, or subsets, of real numbers:

- The set $\mathbb{N}$ of **natural**, or **counting numbers**, as its name suggests, consists of the numbers $1, 2, 3, 4, \ldots$, where "$\ldots$" indicates that the list continues without end.
- The set $\mathbb{W}$ of **whole numbers** consists of the natural numbers and zero: $0, 1, 2, 3 \ldots$.
- The set $\mathbb{Z}$ of **integers** consists of the natural numbers, their negatives, and zero: $\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots$.

All of these numbers are subsets of the rational numbers.

A.13.2 Rational Numbers

A number that can be expressed as the quotient of two integers $\frac{a}{b}$ where $b \neq 0$, is called a **rational number**. The integers are rational numbers, and so are common fractions. Some examples of rational numbers are $5, -2, 0, \frac{2}{9}, \sqrt{16}$, and $\frac{-4}{17}$. The set of rational numbers is denoted by $\mathbb{Q}$.

Every rational number has a decimal form that either terminates or repeats a pattern of digits. For example,

$$\frac{3}{4} = 3 \div 4 = 0.75,\ \text{a } \textbf{terminating decimal}$$

and

$$\frac{2}{37} = 9 \div 37 = 0.243243243\ldots$$

where the pattern of digits 243 is repeated endlessly. We use the **repeater bar** notation to write a repeating decimal fraction:

$$\frac{9}{37} = 0.\overline{243}$$

A.13.3 Irrational Numbers

Some real numbers *cannot* be written in the form $\frac{a}{b}$, where a and b are integers. For example, the number $\sqrt{2}$ is not equal to any common fraction. Such numbers are called **irrational numbers**. Examples of irrational numbers are $\sqrt{15}$, π, and $-\sqrt[3]{7}$.

The decimal form of an irrational number never terminates, and its digits do not follow a repeating pattern, so it is impossible to write down an exact decimal equivalent for an irrational number. However, we can obtain decimal *approximations* correct to any desired degree of accuracy by rounding off. A graphing calculator gives the decimal representation of π as 3.141592654. This is not the *exact* value of π, but for most calculations it is quite adequate.

Some nth roots are rational numbers and some are irrational numbers. For example,

$$\sqrt{49}, \quad \sqrt[3]{\frac{27}{8}}, \quad \text{and} \quad 81^{1/4}$$

are rational numbers because they are equal to 7, $\frac{3}{2}$, and 3, respectively. On the other hand,

$$\sqrt{5}, \quad \sqrt[3]{54}, \quad \text{and} \quad 7^{1/5}$$

are irrational numbers. We can use a calculator to obtain decimal approximations for each of these numbers:

$$\sqrt{5} \approx 2.236, \quad \sqrt[3]{54} \approx 3.826, \quad \text{and} \quad 7^{1/5} \approx 1.476$$

The subsets of the real numbers are related as shown in Figure A.13.1, p. 972. Every natural number is also a whole number, every whole number is an integer, every integer is a rational number, and every rational number is real. Also, every real number is either rational or irrational.

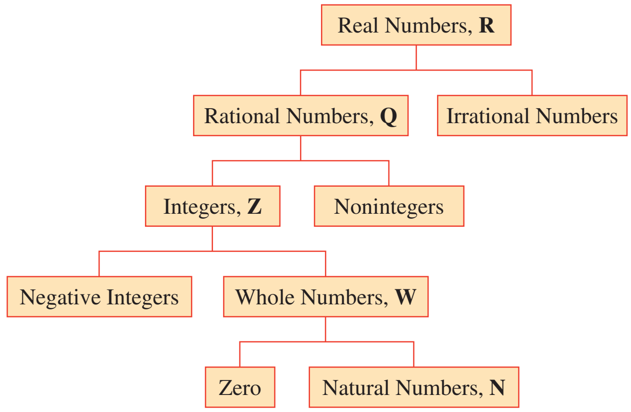


Figure A.13.1

Example A.13.2

a 2 is a natural number, a whole number, an integer, a rational number, and a real number.

b $\sqrt{15}$ is an irrational number and a real number.

c The number π, whose decimal representation begins $3.14159\ldots$ is irrational and real.

d 3.14159 is a rational and real number (which is close but not exactly equal to π).

□

A.13.4 Properties of the Real Numbers

The real numbers have several useful properties governing the operations of addition and multiplication. If a, b, and c represent real numbers, then each of the following equations is true:

- $a + b = b + a$ Commutative properties
  $ab = ba$
- $(a + b) + c = a + (b + c)$ Associative properties
  $(ab) = a(bc)$
- $a(b + c) = ab + ac$ Distributive property
- $a + 0 = a$ Identity properties
  $a \cdot 1 = a$

These properties do not mention subtraction or division. But we can define *subtraction* and *division* in terms of addition and multiplication. For example, we can define the difference $a - b$ as follows:

$$a - b = a + (-b)$$

where $-b$, the **additive inverse** (or **opposite**) of b, is the number that satisfies

$$b + (-b) = 0$$

Similarly, we can define the quotient $\frac{a}{b}$:

$$\frac{a}{b} = a\left(\frac{1}{b}\right) \qquad (b \neq 0)$$

where $\frac{1}{b}$, the **multiplicative inverse** (or **reciprocal**) of b, is the number that satisfies

$$b \cdot \frac{1}{b} = 1 \qquad (b \neq 0)$$

Division by zero is not defined.

Example A.13.3 Use the commutative and associative laws to simplify the computations.

a $24 + 18 + 6$ b $4 \cdot 27 \cdot 25$

Solution.

a Apply the commutative law of addition.

$$\begin{aligned} 24 + 18 + 6 &= (24 + 6) + 18 \\ &= 30 + 18 = 48 \end{aligned}$$

b Apply the commutative law of multiplication.

$$\begin{aligned} 4 \cdot 27 \cdot 25 &= (4 \cdot 25) \cdot 27 \\ &= 100 \cdot 27 = 2700 \end{aligned}$$

□

A.13.5 Order Properties of the Real Numbers

Real numbers obey properties about order, that is, properties about inequalities. The familiar inequality symbols, $<$ and $>$, have the following properties:

- If a and b are any real numbers, then one of three things is true:

$$a < b, \quad \text{or} \quad a > b, \quad \text{or} \quad a = b$$

- (Transitive property) For real numbers a, b, and c,

$$\text{if } a < b \text{ and } b < c, \text{ then } a < c$$

We also have three properties that are useful for solving inequalities:

- If $a < b$, then $a + c < b + c$.
- If $a < b$ and $c > 0$, then $ac < bc$.
- If $a < b$ and $c < 0$, then $ac > bc$.

Example A.13.4

a If $x < y$ and $y < -2$, then $x < -2$

b $\pi < 3.1416$, so $10\pi < 31.416$.

c $\frac{1}{3} > 0.33$, so $-\frac{1}{3} < -0.33$.

□

A.13.6 Section Summary

A.13.6.1 Vocabulary

Look up the definitions of new terms in the Glossary.

- Real number
- Multiplicative inverse
- Additive inverse
- Distributive property
- Whole number
- Natural number
- Reciprocal
- Opposite
- Irrational number
- Integers
- Counting number
- Transitive property
- Commutative property
- Terminating decimal
- Rational number
- Identity property
- Associative property
- Repeater bar

A.13.6.2 SKILLS

Practice each skill in the exercises listed.

1 Identify types of numbers: #1–12

2 Write the decimal form of a fraction: #13–20

3 Use the properties governing arithmetic operations: #21–40

4 Use the properties of order: #41–46

A.13.7 Exercises A.13

For Problems 1-12, name the subsets of the real numbers to which the number belongs

1. $-\frac{5}{8}$
Answer. Rationals

2. 137

3. $\sqrt{8}$
Answer. Irrationals

4. $2.71828\ldots$

5. -36
Answer. Integers

6. $\sqrt{49}$

7. 0
Answer. Whole numbers

8. $0.0\overline{357}$

9. $13\overline{289}$
Answer. Whole numbers

10. $\sqrt{\frac{4}{9}}$

11. 2π
Answer. Irrationals

12. $\frac{13}{7}$

For Problems 13-20, write the rational number in decimal form. Does the decimal terminate or does it repeat a pattern?

13. $\frac{3}{8}$

Answer. 0.375, terminates

14. $\frac{5}{6}$

15. $\frac{2}{7}$

Answer. $0.\overline{285714}$, repeats a pattern

16. $\frac{43}{11}$

17. $\frac{7}{16}$

Answer. 0.4375, terminates

18. $\frac{5}{12}$

19. $\frac{11}{13}$

Answer. $0.\overline{846153}$, repeats a pattern

20. $\frac{25}{6}$

For Problems 21-30, fill in the blank according to the indicated property.

21. Commutative property
$7 + 10 = 10 + \_\_\_\_\_$

Answer. 7

22. Associative property
$(6 \cdot 4) \cdot 3 = 6 \cdot (4 \cdot \_\_\_\_\_)$

23. Associative property
$(3 + 6) + 9 = \_\_\_\_\_ + (6 + 9)$

Answer. 3

24. Commutative property
$(8 \cdot 12) = \_\_\_\_\_ \cdot 8$

25. Commutative property
$36 \cdot 147 = \_\_\_\_\_ \cdot 36$

Answer. 147

26. Commutative property
$13 + 87 = 87 + \_\_\_\_\_$

27. Associative property
$(17 \cdot 2) \cdot 5 = 17 \cdot (\_\_\_\_\_ \cdot \_\_\_\_\_)$

Answer. $2 \cdot 5$

28. Associative property
$(44 + 12) + 8 = 44 + (\_\_\_ + \_\_\_)$

29. Commutative property
$(5 + 9) + 4 = (9 + \_\_\_\_\_) + 4$

Answer. 5

30. Commutative property
$(8 \cdot 9) \cdot 3 = (9 \cdot \_\_\_\_\_) \cdot 3$

For Problems 31-40, use the commutative and associative properties to compute mentally.

31. $47 + 28 + 3$

Answer. 78

32. $12 + 147 + 8$

33. $26 + 37 + 3 + 4$

Answer. 70

34. $55 + 32 + 5 + 8$

35. $2 \cdot 7 \cdot 5$

Answer. 70

36. $15 \cdot 6 \cdot 2$

37. $50 \cdot 13 \cdot 2$

Answer. 1300

38. $4 \cdot 26 \cdot 25$

39. $4 \cdot 6 \cdot 5 \cdot 5$

Answer. 600

40. $8 \cdot 8 \cdot 5 \cdot 5$

For Problems 41-46, fill in the blank with the correct symbol: $<$, $>$, or $=$.

41. $-0.667 \_\_\_\_\_ -\frac{2}{3}$

Answer. $<$

42. $\sqrt{2} \_\_\_\_\_ 1.4$

43. If $x > 8$, then $x - 7 \_\_\_ 1$.

Answer. $>$

44. If $x < -6$, then $x - 6 \_\_\_ -12$.

45. If $x > -2$, then $-9x \_\_\_ 18$.

Answer. $<$

46. If $x < -4$, then $3x \_\_\_ -12$.

Appendix B

Using a Graphing Calculator

This appendix provides instructions for TI-84 or TI-83 calculators from Texas Instruments, but most other calculators work similarly.We describe only the basic operations and features of the graphing calculator used in your textbook.

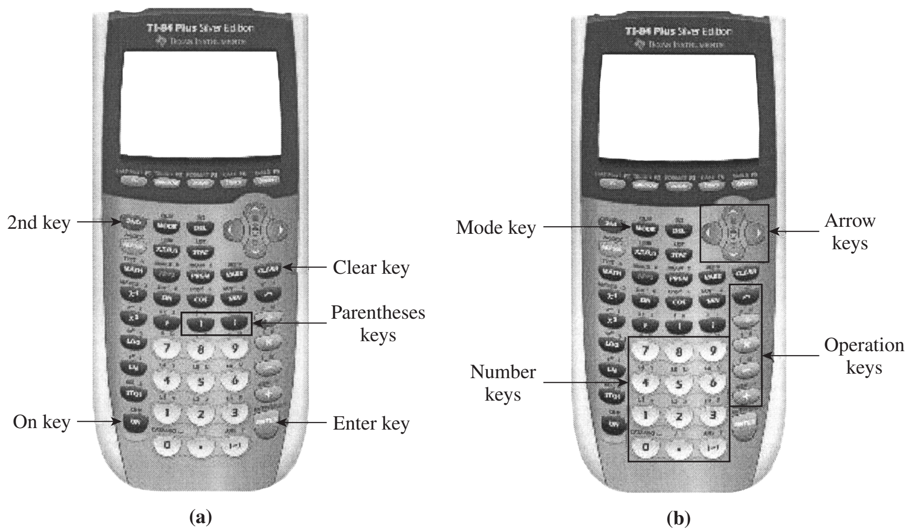


Figure B.0.1

B.1 Getting Started

B.1.1 On and Off

Press `ON` to turn *on* the calculator (see Figure B.0.1, p. 977a). You will see a cursor blinking in the upper left corner of the Home screen. Press `2ndON` to turn *off* the calculator.

B.1.2 Numbers and Operations

The parentheses keys, the Clear key, and the Enter key are shown in Figure B.0.1, p. 977a. Locate the number keys, operation keys, and arrow keys on your calculator, as shown in Figure B.0.1, p. 977b.

We use the `-` key for subtraction, but we use the `(-)` key (located next to `ENTER`) for negative numbers.

Example B.1.1 Compute $5 - 8$. Press
`5 - 8ENTER` Ans. -3

□

Example B.1.2 Compute $-5 + 8$. Press
`(-)5 + 8ENTER` Ans. 3

□

We press `ENTER` to tell the calculator to compute.
The calculator has a key for the value of π.

Example B.1.3 Compute 2π. Press
`2X2nd ^ ENTER` or `22nd ^ ENTER` Ans. 6.283185307

□

B.1.3 Clear and Delete

Press `DEL` to delete the character under the cursor.
Press `CLEAR` to clear the contents of the current input line.
In the Home screen, press `CLEAR CLEAR` to clear the entire screen.

Troubleshooting.

1 If your screen is too light, press `2nd` [↑] several times to make it darker. If it is too dark, press `2nd ↓`.

2 For the features we use in this book, the MODE and FORMAT should be in their default settings. Press `MODE` to see the menu in Figure B.2a, and `2nd ZOOM` to see the format menu in Figure B.1.4, p. 978b. Use the Arrow keys and `ENTER` to alter the menus to the default settings if necessary. *Note*: The Set Clock function does not appear on the T1-83.

(a) **(b)**

Figure B.1.4

B.2 Entering Expressions

B.2.1 Parentheses

Order of Operations: The calculator follows the standard order of operations.

Example B.2.1 Compute $2 + 3 \cdot 4$. Press
`2 + 3 X 4 ENTER` Ans. 14

□

Example B.2.2 Compute $(2 + 3) \cdot 4$. Press
`( 2 + 3 ) X 4 ENTER` Ans. 20

□

Example B.2.3 Compute $\frac{1}{2 \cdot 3}$. Press
`1` [÷] `( 2 X 3 ) ENTER` Ans. 0.1666666667

□

Example B.2.4 Compute $\dfrac{1+3}{2}$. Press

(1 + 3) $\boxed{\div}$ 2 ENTER Ans. 2

□

B.2.2 Exponents and Powers

Exponents: We use the caret key, ^ , to enter exponents or powers.

Example B.2.5 Evaluate 2^{10}.

2 ^ 10 ENTER Ans. 1024

□

Squaring: There is a short-cut key for squaring, $\boxed{x^2}$.

Example B.2.6 Evaluate 57^2.

57 $\boxed{x^2}$ ENTER Ans. 3249

□

Fractional Exponents: Fractional exponents must be enclosed in parentheses!

Example B.2.7 Evaluate $8^{2/3}$.

8 ^ (2 $\boxed{\div}$ 3) ENTER Ans. 4

□

B.2.3 Roots

Square Roots: We access the square root by pressing 2nd $\boxed{x^2}$, and the display shows $\sqrt{}($. The calculator automatically gives an open parenthesis for the square root, but not a close parenthesis.

Example B.2.8 Evaluate $\sqrt{2}$.

2nd $\boxed{x^2}$ 2) ENTER Ans. 1.414213562

□

Example B.2.9 Evaluate $\sqrt{9+16}$.

2nd $\boxed{x^2}$ $9+16$) ENTER Ans. 5

□

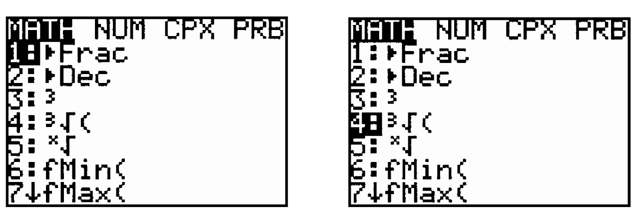
In the next example, note that we must enter) at the end of the radicand to tell the calculator where the radical ends.

Example B.2.10 Evaluate $\sqrt{9}+16$.

2nd $\boxed{x^2}$ 9) $+16$ ENTER Ans. 19

□

Cube Roots: For cube roots, we press MATH to open the Math menu and press 4 (see Figure B.2.11, p. 979).



(a) (b)

Figure B.2.11

Example B.2.12 Compute $\sqrt[3]{1728}$.
`MATH 4 1728 ) ENTER` Ans. 12

□

For evaluating cube roots and square roots, `)` can be omitted if there are no operations following the radical.

Other Roots: For nth roots, we press `MATH` to open the Math menu and press 5 (see Figure B.2.11, p. 979a). The calculator symbol for nth roots, $\sqrt[x]{}$, does not include an open parenthesis, `(`. If the radicand includes an operation, we must enclose it in parentheses.

Example B.2.13 Compute $\sqrt[10]{2 \cdot 512}$.
`10 MATH 5 ( 2 x 512 ) ENTER` Ans. 2

□

Notice that we enter the index 10 *before* the radical symbol.

B.2.4 Absolute Value

TI calculators use $abs(x)$ instead of $|x|$ to denote the absolute value of x. The absolute value function is the first entry in the MATH NUM menu (see Figure B.2.14, p. 980). The calculator gives `(` for the absolute value function, but not `)`.

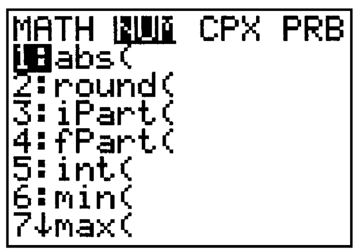


Figure B.2.14

Example B.2.15 Evaluate $\dfrac{|21 \cdot 54 - 81|}{-9}$.
`MATH` [→] `ENTER 21 X 54 - 81 )` [÷] `(-) 9 ENTER` Ans. −117

□

B.2.5 Scientific Notation

The TI calculators display numbers in scientific notation when the numbers use too many digits to display.

Example B.2.16 Compute $123,456,789^2$. Enter
`123456789` [x^2] `ENTER` Ans. 1.524157875 E 16

□

This is how the calculator displays the number $1.524157875 \times 10^{16}$. Notice that the power 10^{16} is displayed as E 16.

To enter a number in scientific form, we use the key labeled **EE**, or `2nd ,`.

Example B.2.17 To enter 3.26×10^{18}, use the keying sequence
`3.26 2nd , (-) 18 ENTER` Ans. 3.26 E −18

□

Troubleshooting.

Figure B.2.18

If your calculator gives you an error message like this, you may have made one of the following common mistakes:

1 Using the negative key, `(-)`, when you wanted the subtraction key, `-`, or vice versa.

2 Omitting a `(` or `)`. Each `(` should have a matching `)`.

Press 2 to **Go to** the error, and see Editing Expressions B.3, p. 981 below.

B.3 Editing Expressions

B.3.1 Overwriting

We can edit an expression without starting again. If we place the cursor over a symbol and press a new key, the new symbol replaces (overwrites) the old one.

Example B.3.1 Correct the error in the following keystrokes for $120 - 36$:

$$120 \boxed{(-)} 36 \boxed{\text{ENTER}}$$

The calculator gives an error message (See Figure B.2.18, p. 981.) Select "2:Goto", and a blinking cursor appears over the error. Press `-` to replace the negative symbol by the subtraction symbol. □

B.3.2 Recalling an Entry

We can recall a previous entry by pressing `2nd ENTER`.

Example B.3.2 Evaluate $\sqrt{3}+\sqrt{2}$ and $\sqrt{3}-\sqrt{2}$. First evaluate $\sqrt{3}+\sqrt{2}$:
`2nd` $\boxed{x^2}$ `3 ) + 2nd` $\boxed{x^2}$ `2 ENTER` Ans. 3.14626437

Now press `2nd ENTER` to recall the last entry, and use the left arrow key $\boxed{\leftarrow}$ to position the cursor over +. Press `-` to change to $-$, then press `ENTER`. Your screen should look like Figure B.3.3, p. 981.

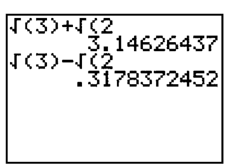


Figure B.3.3

□

B.3.3 Inserting a Character

To insert a new character *before* a symbol, position the cursor over that symbol and press `2nd DEL` to get the **INS** (insert) command.

Example B.3.4 Evaluate $\sqrt{3} - 5\sqrt{2}$ by editing the example above.

Press 2nd ENTER to recall the last entry, and use the left arrow key [←] to position the cursor over the second $\sqrt{}$ from left to right. Press 2nd DEL 5 to insert 5 *before* the $\sqrt{}$ symbol, then press ENTER. Your screen should look like Figure B.3.5, p. 982.

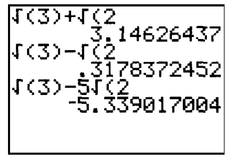


Figure B.3.5

□

B.3.4 Recalling an Answer

We often want to use the result from a previous calculation in a new calculation, without having to type in the number. We use the **ANS** key, 2nd (-), to recall the answer to the last calculation.

Example B.3.6

a Evaluate $\sqrt{5} - 1$

2nd [x^2] 5) - 1 ENTER   Ans. 1.236067977

b Evaluate $x^2 + 2x + 1$ for $x = \sqrt{5} - 1$. Because x is the last answer the calculator computed, we enter

2nd (-) [x^2] + 2 2nd (-) + 1 ENTER   Ans. 5

Your screen should look like Figure B.3.7, p. 982.

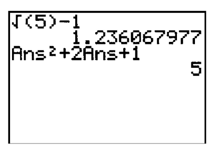


Figure B.3.7

□

B.4 Graphing an Equation

We can graph equations written in the form $y =$ (expression in x). The graphing keys are located on the top row of the keypad. There are two steps to graphing an equation:

1 Entering the equation

2 Setting the graphing window

B.4.1 Standard Window

The standard window displays values from -10 to 10 on both axes.

Example B.4.1

1 Press Y= and enter $2X - 5$ after $Y_1 =$ by keying in

2 $\boxed{X, T, \theta, n}$ $\boxed{-}$ 5 Use the $\boxed{X, T, \theta, n}$ to enter X.

2 Press ZOOM 6 to set the standard window, and the graph will appear (see Figure B.4.2, p. 983).

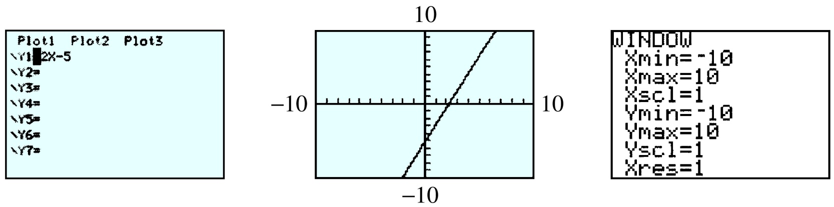


Figure B.4.2

You can press 2nd WINDOW to see the settings for the standard window. $X\mathit{scl} = 1$ means that the tick marks on the x-axis are spaced 1 unit apart.

Press 2nd MODE to Quit the graph and return to the Home screen, where we enter computations. From the Home screen, press GRAPH to return to the graph. □

B.4.2 Tracing

The calculator can display the coordinates of selected points on the graph. Press the TRACE key to see a "bug" blinking on the graph. The coordinates of the bug are displayed at the bottom of the screen. Use the left and right arrow keys to move the bug along the graph, as shown in Figure B.4.3, p. 983. Note that the **Trace** feature does not show every point on the graph!

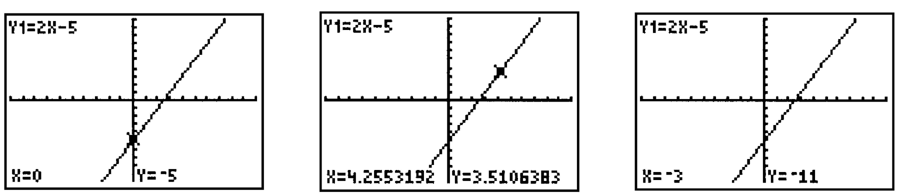


Figure B.4.3

Example B.4.4 Use the **Trace** to find the point on the graph with $x = -3$. Press

$\boxed{\text{TRACE}}$ $\boxed{(-)}$ 3 $\boxed{\text{ENTER}}$

The bug is off the bottom of the screen, but the coordinates are still shown. □

B.4.3 Multiple Graphs

You can enter more than one graph at a time. Press ↓ to enter a second equation at $Y_2 =$, at $Y_3 =$, and so on. When Tracing, press the ↓ and $\boxed{\uparrow}$ keys to move from one graph to another.

To turn off a graph without deleting its equation, press and move the cursor over the = sign in the equation. Press ENTER to deactivate that equation. (When you move the cursor away, the = sign is no longer highlighted.) To reactivate the equation, move the cursor back over the = sign and press ENTER again.

B.4.4 Setting the Window

Of course, the standard window is not suitable for every graph.

Example B.4.5 Graph $y = 0.01x^2 - 50$ in the window

$$\text{Xmin} = -100 \qquad \text{Xmax} = 100$$
$$\text{Ymin} = -60 \qquad \text{Ymax} = 50$$

1 Press `Y=` and enter $0.01X^2 - 50$ by keying in

0.01 $\boxed{X, T, \theta, n}$ $\boxed{x^2}$ $\boxed{-}$ 50 Use the $\boxed{X, T, \theta, n}$ key to enter X.

2 Press `WINDOW` and enter the settings as shown in Figure B.4.6, p. 984. Use the up and down arrow keys to move from line to line. Then press `GRAPH`.

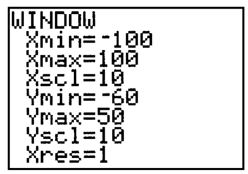


Figure B.4.6

□

B.4.5 Intersect Feature

We can use the calculator to find the intersection point of two graphs:

1 Enter the equations for the two graphs in the `Y=` menu.

2 Choose window settings so that the intersection point is visible in the window.

3 Press `2nd` `TRACE` 5 to activate the intersect feacture.

4 Use the left and right arrow keys to position the bug near the intersection point.

5 Respond to each of the calculator's questions, First curve?, Second curve?, and Guess? by pressing `ENTER`. The coordinates of the intersection point are then displayed at the bottom of the screen.

Figure B.4.7, p. 984 shows one of the intersection points of $y = 0.01x^2 - 50$ and $y = -0.5x$.

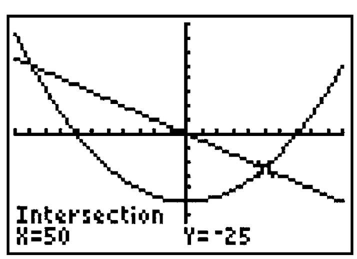


Figure B.4.7

B.4.6 Other Windows

1 The **ZDecimal** (Zoom Decimal) window, accessed by pressing `ZOOM` 4, shows x-values from -4.7 to 4.7 only, but the **Trace** feature shows "nice" x-values in increments of 0.1.

2 The **ZInteger** (Zoom Integer) window shows nice x-values in increments of 1 unit. Access the **ZInteger** window as follows: Press `ZOOM` 8, move the bug with the arrow keys to the center of your new window, and press `ENTER`.

3 The **ZSquare** window, accessed by pressing `ZOOM` 5, makes the tick marks on both axes have the same size. In this window, squares look like squares, circles look like circles, and all angles appear true.

4 **"Friendly" Windows**: If the difference between Xmin and Xmax is a multiple of 94, the **Trace** feature gives nice values for x. A useful example of a friendly window is $Xmin = -9.4$, $Xmax = 9.4$.

Troubleshooting.

1 If the graph is not visible, you may need to adjust your window. Or, the equation may not be activated. Press `Y=` and check to see if the =sign is highlighted.

2 If you get a range error, ERR: WINDOW RANGE, quit the message and press `WINDOW`. Alter the window settings so that Xmin is smaller than Xmax and so that Ymin is smaller than Ymax.

3 If you press and get an unfamiliar window, or if the axes are not visible in the ZStandard window, you may need to return the **Mode** or **Format** menus to their default settings. See Troubleshooting , p. 978 in Section B.1, p. 977.

4 If you get a dimension error, **ERR: INVALID DIM**, you may have a StatPlot turned on. Press `2nd` `Y=` 4 `ENTER` to turn off the StatPlots.

5 If the bug does not move along the curve, `TRACE` may not be activated. Press `TRACE` and then the left or right arrow key.

6 If you get the error, **ERR: INVALID**, you have probably entered a value of x that is outside the window. Adjust the window settings accordingly.

7 If the x-axis or y-axis is too thick, the tick marks are too close together. Press `WINDOW` and make Xscl or Yscl larger. Set $Xscl = 0$ or $Yscl = 0$ to remove the tick marks.

8 If you get **ERR: NO SIGN CHNG** when using the intersect feature, the calculator did not find any intersection point within the current window. Alter the window settings so that the two curves meet within the window. If the two curves are tangent, the calculator may simply fail to find the point of intersection.

B.5 Making a Table

The table feature gives us a convenient tool for evaluating expressions. There are two steps to making a table:

1 Entering the equation

2 Setting the Table features

B.5.1 Using the Auto Option

If we want a table with evenly spaced x-values, we use the Automatic setting.

1 Press `Y=`, clear any previous entries, and enter the expression or expressions you want to evaluate.

2 Press `2nd WINDOW` (**TBLSET**) to access the Table Setup menu. Enter the first x-value after **TblStart**=, and the x-increment after ΔTbl =. Highlight **Auto** for both the Independent and Dependent variables.

3 Finally, press `2nd GRAPH` (**TABLE**) to see the table. You can use the arrows to scroll up and down the table. Figure B.5.1, p. 986 shows the Table Setup and the resulting table for $y = 5 - x^3$.

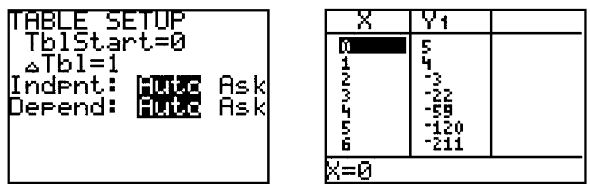


Figure B.5.1

B.5.2 Using the Ask Option

If we want to choose specific x-values for the table, we proceed as above but choose **Ask** instead of **Auto** for the Independent variable. Press `2nd GRAPH` to see the table, then enter the x-values, pressing `ENTER` after each one.

Troubleshooting.

1 If the table shows **ERROR** for a y-value, the expression you entered is undefined at the corresponding x-value.

2 If the table shows only x-values, no equation is activated. Press `Y=` to enter or activate an equation.

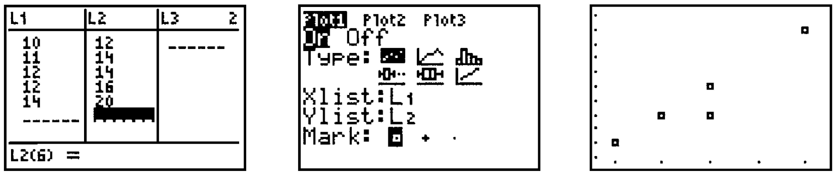
B.6 Regression

We use the statistics features to plot data and calculate regression equations. Press `STAT` to see the Statistics menus. We will use only the first two, **EDIT** and **CALC**.

B.6.1 Making a Scatterplot

1 Press `STAT ENTER` to access the list editor.

2 Enter the x-coordinates of the data points under L1 and the y-coordinates in L2. An example is shown in Figure B.6.1, p. 987a. (If there is a previous list under L1 or L2, move the cursor up to L1 or L2 and press `CLEAR ENTER`.)



(a) (b) (c)

Figure B.6.1

3 Access the STAT PLOT menu by pressing `2nd Y=`, select and turn on Plot1 by pressing `ENTER ENTER`, and set the menu options as shown in Figure B.6.1, p. 987b.

4 Clear out any old equations from the `Y=` menu, then press `ZOOM` 9 (ZoomStat) to see the scatterplot.

Note: You can use any of the lists L1–L6 to store the data. Change Xlist and Ylist to reflect the appropriate lists.

Caution B.6.2 When you are through with the scatterplot, press `Y=` [↑] `ENTER` to turn off Plot1 (or press `2nd Y=` 4 to turn off all the StatPlots). If you neglect to do this, you will continue to see the scatterplot even after you graph a new equation.

B.6.2 Finding a Regression Equation

1 Enter the data as in steps 1 and 2 of Making a Scatterplot.

2 Press `STAT` [→] to open the Calculate menu, and select the type of regression equation you want.

3 Press `ENTER` and the calculator will display the parameters of the regression equation. See the example in Figure B.6.3, p. 987. (You may also see information about r and r^2.)

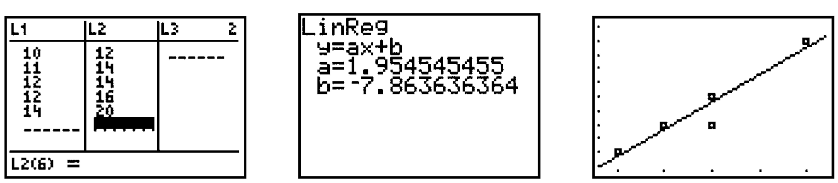


Figure B.6.3

Note: If you do not use L1 and L2 to store the data, enter the appropriate lists, separated by a comma, after the regression command.

B.6.3 Graphing the Regression Equation

If you would like to graph the regression equation on top of the scatterplot, first follow the steps in Making a Scatterplot B.6.1, p. 986 and Finding a Regression Equation B.6.2, p. 987 above.

1 Press `Y=` and clear out any old equations.

2 Position the cursor after $Y_1 =$.

3 Press `VARS` 5 $\boxed{\rightarrow}$ $\boxed{\rightarrow}$ `ENTER` to copy the regression equation.

4 Press `GRAPH` to see graph of the regression equation and the scatterplot.

B.7 Function Notation and Transformation of Graphs

B.7.1 Function Notation

The calculator uses $Y_1(X)$, $Y_2(X)$, and so on, instead of $f(x)$, $g(x)$, and so on, for function notation.

Example B.7.1 Evaluate $f(x) = x^2 + 6x + 9$ for $x = 3$.

1 Set $Y_1 = X^2 + 6X + 9$, and quit (`2nd MODE`) to the Home screen.

2 To evaluate this function for $X = 3$, press

`VARS` $\boxed{\rightarrow}$ `ENTER ENTER ( 3 ) ENTER`

See Figure B.7.2, p. 988.

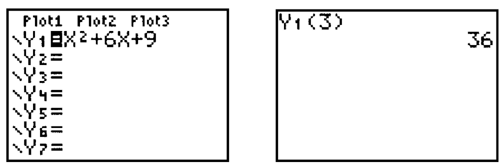


Figure B.7.2

□

B.7.2 Transformation of Graphs

We can use function notation to facilitate graphing transformations. In the examples below, we use $f(x) = x^2$.

B.7.2.1 Translations

Example B.7.3 Compare the graphs of $y = f(x) - 8$ and $y = f(x - 8)$ with that of $y = f(x)$.

Define $Y_1 = X^2$ and $Y_2 = Y_1(X) - 8$. Press `ZOOM` 6 to see the graphs (Figure B.7.4, p. 988).

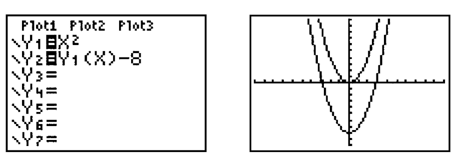


Figure B.7.4

Define $Y_1 = X^2$ and $Y_2 = Y_1(X - 8)$. Press `ZOOM` 6 to see the graphs (Figure B.7.5, p. 989).

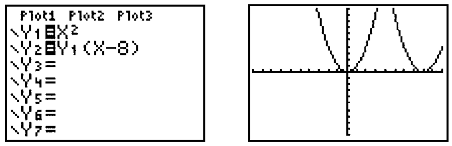


Figure B.7.5

□

B.7.2.2 Vertical Scalings and Reflections

Compare the graph of $y = \frac{-1}{2} f(x)$ with that of $y = f(x)$.

Define $Y_1 = X^2$ and $Y_2 = -1/2 * Y_1(X)$. Press to see the graphs (Figure B.7.6, p. 989).

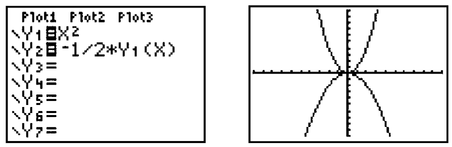


Figure B.7.6

Appendix C

Glossary

Abbreviations used in this glossary: **n** (noun), **v** (verb), **adj** (adjective)

A.

absolute value n, the distance on the number line from a number to 0. For example, the absolute value of -7 is 7. This fact is expressed by the equation $|-7| = 7$.

absolute value equation n, an equation in which the variable occurs between the absolute value bars.

absolute value inequality n, an inequality in which the variable occurs between the absolute value bars.

algebraic expression n, a meaningful combination of numbers, variables, and operation symbols. Also called an **expression**.

algebraic fraction n, a fraction whose numerator and denominator are polynomials. Also called a **rational expression**.

algebraic solution n, a method for solving equations (or inequalities) by manipulating the equations (or inequalities). Compare with **graphical solution** and **numerical solution**.

allometric equation n, an equation showing the (approximate) relationship between a living organism's body mass and another of the organism's properties or processes, usually given in the form $y = k(\text{mass})^p$.

altitude n, (i) the distance above the ground or above sea level; (ii) the vertical distance between the base and the opposite vertex of a triangle, pyramid, or cone; (iii) the distance between parallel sides of a parallelogram, trapezoid, or rectangle. Also called **height**.

amortization n, the payment of a debt through regular installments over a period of time.

amount (in an interest-bearing account), **n**, the sum of the principal that was invested and all the interest earned.

amplitude n, the vertical distance between the midline and the maximum value of a sinusoidal function.

annuity n, sequence of equal payments or deposits made at equal time intervals.

approximation n, an inexact result.

area n, a measure of the two-dimensional space enclosed by a polygon or curve, typically expressed in terms of square units, such as square meters or square feet, etc.

ascending powers n, an ordering of the terms of a polynomial so that the exponents on the variable are increasing, such as in the polynomial $1 + x + x^2$.

associative law of addition n, the property that when adding three or more terms, the grouping of terms does not affect the sum. We express this formally by saying that if a, b, and c are any numbers, then $(a + b) + c = a + (b + c)$.

associative law of multiplication n, the property that when multiplying three or more factors, the grouping of factors does not affect the product. We express this formally by saying that if a, b, and c are any numbers, then $(a \cdot b) \cdot c = a \cdot (b \cdot c)$.

asymptote n, a reference line (or curve) towards which the graph of an equation tends as the value of x and/or y grows or diminishes without bound.

augmented matrix (for a linear systterm with n variables in standard form), n, the matrix obtained by making each row of the matrix correspond to an equation of the systterm, with the coefficients of the variables filling the first n columns, and the last (that is, the $n + 1$) column having the constants.

axis n, (**plural axes**), a line used as a reference for position and/or orientation.

axis of symmetry n, a line that cuts a plane figure into two parts, each a mirror image of the other.

B.

back substitution n, a technique for solving a triangular systterm of linear equations.

bar graph n, a picture of numerical information in which the lengths or heights of bars are used to represent the values of variables.

base n, (i) a number or algebraic expression that is used as a repeated factor, where an exponent indicates how many times the base is used as a factor. For example, when we write 3^5, the base is 3. (ii) The bottom side of a polygon. (iii) The bottom face of a solid.

base angles n, the angles opposite the equal sides in an isosceles triangle.

binomial n, a polynomial with exactly two terms.

binomial expression n, a sum of two unlike terms, such as $\sqrt{3} + \sqrt{2}$.

build (a fraction) v, to find an equivalent fraction by multiplying numerator and denominator by the same nonzero expression.

building factor n, an expression by which both numerator and denominator of a given fraction are multiplied (in order to build the fraction).

cartesian coordinate systterm n, the grid that associates points in the coordinate plane to ordered pairs of numbers.

C.

cartesian plane n, a plane with a pair of coordinate axes. Also called a **coordinate plane**.

change in (a variable) n, the final value (of the variable) minus the starting value.

change of variables n, (i) a **transformation** of data, (ii) substitution of a new variable for a variable expression, for example, replacing t^2 with x so that the equation $y = at^2 + b$ becomes $y = ax + b$.

circle n, the set of all points in a plane at a fixed distance (the **radius**) from the center.

circumference n, the distance around a circle.

closed interval n, a set of numbers, denoted by $[a, b]$, which includes all the numbers between a and b as well as the numbers a and b thtermselves, where a and b are real numbers and $a < b$. Or the set of numbers denoted by $(-\infty, b]$, which includes the real number b and all numbers less than b, or the set of numbers denoted by $[a, \infty)$, which includes the real number a and all numbers greater than a.

coefficient n, the numerical factor in a term. For example, in the expression $32a + 7b$, the coefficient of a is 32 and the coefficient of b is 7.

coefficient matrix (for a linear systterm with n variables in standard form) n, the matrix of n columns obtained by making each row of the matrix correspond to an equation of the systterm, with the coefficients of the variables filling the n columns (and the constants are not represented in the matrix).

common factor (of two or more expressions) n, a quantity that divides evenly into each of the given expressions.

common log or common logarithm (of a given positive number x) n, the exponent, denoted by $\log(x)$ (or by $\log(x)$) for the number 10 to obtain the value x, that is, $10^{\log(x)} = x$.

commutative law of addition n, the property that when adding terms, the order of the terms does not affect the sum. We express this formally by saying that if a and b are any numbers, then $a + b = b + a$.

commutative law of multiplication n, the property that when multiplying factors, the order of the factors does not affect the product. We express this formally by saying that if a and b are any numbers, then $a \cdot b = b \cdot a$.

compltermentary angles n, two angles whose measures add up to 90°.

complete the square v, to determine the appropriate constant to add to a binomial of the form $ax^2 + bx$ so that the result can be written in the form $a(x + k)^2$.

complex conjugate (of a complex number) n, the complex number with the same real part and opposite imaginary part; for example, the complex conjugate of $1+i$ is $1-i$.

complex fraction n, a fraction that contains one or more fractions in its numerator and/or in its denominator.

complex plane n, a coordinate plane representing complex numbers, with the real parts corresponding to the values on the horizontal axis and imaginary parts corresponding to values on the vertical axis.

complex number n, a number that can be written in the form $a+bi$, where a and b are real numbers and $i^2=-1$.

component n, one of the values of an ordered pair or ordered triple.

compound inequality n, a mathtermatical statterment involving two order symbols. For example, the compound inequality $1<x<2$ says that "1 is less than x, and x is less than 2."

compound interest (or compounded interest) n, an interest earning agreterment in which the interest payment at a given time is computed based on the sum of the original principal and any interest money already accrued.

compounding period n, the time interval between consecutive interest payments to an account that earns interest.

concave down (of a graph) adj, curving so that the ends of a flexible rod would need to be bent downward (compared with a straight rod) to lie along the graph. Or equivalently, curving so that a line segment tangent to the curve will lie above the curve.

concave up (of a graph) adj, curving so that the ends of a flexible rod would need to be bent upward (compared with a straight rod) to lie along the graph. Or equivalently, curving so that a line segment tangent to the curve will lie below the curve.

concavity n, a description of a curve as either concave up or concave down.

concentric (of circles or spheres) adj, having the same center.

conditional equation n, an equation that is true for some (but not all) values of the variable(s).

cone n, a three-dimensional object whose base is a circle and whose vertex is a point above the circle. The points on the segments joining the circle to the vertex make up the cone.

congruent adj, having all measure(s) matching exactly. For example, two line segments are congruent when they have the same length; two triangles are congruent if all three sides and all three angles of one match exactly with the corresponding parts of the other triangle.

conjugate n, (i) (of a complex number) the complex number with the same real part and opposite imaginary part; (ii) (of a binomial expression) the binomial expression with the same first term and opposite second term.

conjugate pair n, (i) (of a complex number) a complex number and its conjugate; (ii) (of a binomial expression) the binomial expression and its conjugate.

consistent (of a systterm of equations) adj, having at least one solution.

consistent and independent (of a systterm of linear equations) adj, having exactly one solution.

constant adj, unchanging, not variable. For example, we say that the product of two variables is constant if the product is always the same number, for any values of the variables.

constant n, a number (as opposed to a variable).

constant of proportionality n, the quotient of two directly proportional variables, or the product of two inversely proportional variables. Also called the **constant of variation**.

constant of variation *see* constant of proportionality.

constraint n, an equation or inequality involving one or more variables, typically specifying a condition that must be true in the given context.

continuous adj, without holes or gaps. For example, a curve is continuous if it can be drawn without lifting the pencil from the page, and a function is continuous if its graph can be drawn without lifting the pencil from the page.

continuous compounding n, an interest earning agreterment in which the amount in the account is Pe^{rt}, where P is the initial principal, r is the annual interest rate, and $e \approx 2.71828$ is the base of the natural logarithm.

conversion factor n, a ratio used to convert from one unit of measure to another.

coordinate n, a number used with a number line or an axis to designate position.

coordinate axis n, one of the two perpendicular number lines used to define the coordinates of points in the plane.

coordinate plane n, a plane with a pair of coordinate axes. Also called the **Cartesian plane** or ***xy*-plane**.

corollary n, a mathtermatical fact that is a consequence of a previously known fact.

costs n, money that an individual or group must pay out. For example, the costs of a company might include payments for wages, supplies, and rent.

counting number n, one of the numbers $1, 2, 3, 4, \ldots$.

cube n, (i) a three-dimensional box whose six faces all consist of squares; (ii) an expression raised to the power 3.

cube v, to raise an expression to the power 3. For example, to cube 2 means to form the product of three 2s: $2^3 = 2 \times 2 \times 2 = 8$.

cube root n, a number that when raised to the power 3 gives a desired value. For example, 2 is the cube root of 8 because $2^3 = 8$.

cubic adj, having to do with the third degree of a variable or with a polynomial of degree 3.

cylinder n, a three-dimensional figure in the shape of a soft drink can. The top and base are circles of identical size, and the line segments joining the two circles are perpendicular to the planes containing the two circles.

D.

decay factor n, the factor by which an initial value of a diminishing quantity is multiplied to obtain the final value.

decimal adj, having to do with a base-10 numeration systterm.

decimal place n, the position of a digit relative to the decimal point. For example, in the number 3.14159, the digit 4 is in the second decimal place, or hundredths place.

decimal point n, the mark "." that is written between the whole number part and the fractional part of a decimal number. For example, the decimal form of $1\frac{3}{10}$ is 1.3.

decreasing adj, (i) (of numbers) moving to the left on a number line: Positive numbers are decreasing when getting closer to zero, and negative numbers are decreasing when they move farther from 0; (ii) (of a graph) having decreasing values of y when moving along the graph from left to right; (iii) (of a function) having a decreasing graph.

degree n, a measure of angle equal to $\frac{1}{360}$ of a complete revolution.

degree n, (i) (of a monomial) the exponent on the variable, or if there are more than one variable, the sum of the exponents of all the variables; (ii) (of a polynomial) the largest degree of the monomials in the polynomial.

demand equation n, an equation that gives the quantity of some product that consumers are willing to purchase in terms of the price of that product.

denominator n, the expression below the fraction bar in a fraction.

dependent adj, (of a systterm of equations) having infinitely many solutions.

dependent variable n, a variable whose value is determined by specifying the value of the independent variable.

descending powers n, expressed with the term with the highest degree written first, then the term with the second highest degree, etc.

diagonal n, (i) a line segment joining one vertex of a quadrilateral to the opposite vertex; (ii) a line segment joining opposite corners of a box; (iii) the entries of a matrix whose row number match the column number, that is, the $(1,1),(2,2),\ldots,(n,n)$ entries

diameter n, (i) a line segment passing through the center of a circle (or sphere) with endpoints on the circle (sphere); (ii) the length of that line segment.

difference n, the result of a subtraction. For example, the expression $a-b$ represents the difference between a and b.

difference of squares n, an expression of the form a^2-b^2.

dimension n, (i) (of a matrix) the numbers of rows and columns respectively of the matrix, also called the **order** of the matrix. For example, a matrix with dimension 2 by 3 (or 2×3) has two rows and three columns; (ii) a measurterment defining a geometric figure, for example, the length and width are dimensions of a rectangle.

direct variation n, a relation between two variables in which one is a constant multiple of the other (so that the ratio between the two variables is the constant), or in which one is a constant multiple of a positive exponent power of the other variable.

directed distance n, the difference between the ending coordinate and the starting coordinate of points on a number line; the directed distance is negative if the ending value is smaller than the starting value. For example, the directed distance from 5 to 2 is $2 - 5 = -3$.

directly proportional adj, describing variables related by direct variation.

discriminant n, (for the quadratic polynomial $ax^2 + bx + c$) the quantity $b^2 - 4ac$.

distributive law n, the property that for any numbers a, b, and c, $a(b+c) = ab + ac$.

divisor n, a quantity that is divided into another quantity. For example, in the expression $a \div b$, the divisor is b.

domain n, the set of all acceptable inputs for a function or equation.

doubling time n, (of exponential growth) the time required for a quantity to double in size.

E.

elementary row operation n, one of the three following operations: (1) an exchange of two rows, (2) multiplying all entries of a row by a nonzero constant, (3) adding a multiple of any row to another row.

elimination n, a method for solving a system of equations that involves adding together the equations of the systterm or multiples of the equations of the systterm.

empirical model n, an equation whose graph (approximately) fits a given set of data (but gives no information about the physical processes involved).

entry n, a value in a matrix, often identified by specifying location by row and column.

equation n, a mathtermatical statterment that two expressions are equal, for example, $1 + 1 = 2$.

equation in two variables n, an equation that involves two variables.

equilateral adj, (of a polygon) having all sides of equal length.

equilibrium point n, the point where the graphs of the supply and dtermand equations intersect

equivalent adj, representing the same value.

equivalent equations n, equations that have the same solutions.

equivalent expressions n, expressions that have the same value for all permissible values of their variables.

error tolerance n, the allowable difference between an estimate and the actual value.

evaluate v, to determine the value of an expression when the variable in the expression is replaced by a number.

exact adj, not simply close, but with absolutely no deviation from an intended value.

exact solution n, the exact value of a solution, i.e., not an approximation.

exponent n, the expression that indicates how many times the base is used as a factor. For example, when we write 3^5, the exponent is 5, and $3^5 = 3 \times 3 \times 3 \times 3 \times 3$.

exponential decay n, a manner of decreasing characterized by a constant decay factor for any fixed specified interval of time, or equivalently, modeled by a function f with the form $f(t) = ab^t$, where a and b are positive constants and $0 < b < 1$.

exponential equation n, an equation containing a variable expression as an exponent.

exponential function n, a function f which can be put in the form $f(x) = ab^x$, where a is a nonzero constant and $b \neq 1$ is a positive constant.

exponential growth n, growth characterized by a constant growth factor for any fixed specified interval of time, or equivalently, modeled by a function f with the form $f(t) = ab^t$, where a and b are positive constants and $b > 1$.

exponential notation n, a way of writing an expression that involves radicals and/or reciprocals in terms of powers that have fractional and/or negative exponents. For example, the exponential notation for $\sqrt{3}$ is $3^{1/2}$.

expression *see* algebraic expression.

extraction of roots n, a method used to solve (quadratic) equations.

extraneous solution n, a value that is not a solution to a given equation but is a solution to an equation derived from the original.

extrapolate v, to estimate the value of a dependent variable for a value of the independent variable that is outside the range of the data.

F.

factor n, an expression that divides evenly into another expression. For example, 2 is a factor of 6.

factor v, to write as a product. For example, to factor 6 we write $6 = 2 \times 3$.

factored form n, (i) (of a polynomial or algebraic expression) an expression written as a product of two or more factors, where the algebraic factors cannot be further factored; (ii) (of an equation of a parabola) the form $y = a(x - r_1)(x - r_2)$.

feasible solution n, an ordered pair which satisfies the constraints of a linear programming problterm.

FOIL n, an acronym for a method for computing the product of two binomials: **F** stands for First terms, **O** for Outer terms, **I** for Inner terms, and **L** stands for Last terms.

formula n, an equation involving two or more variables.

fraction bar n, the line segment separating the numerator and denominator of a fraction. In the fraction $\frac{1}{2}$, the fraction bar is the short segment between the 1 and the 2.

function n, a relationship between two variables in which each value of the input variable determines a unique value of the output variable.

function of two variables n, a relationship between an output variable and an ordered pair of input variables in which each ordered pair of the input variables determines a unique value of the output variable.

function value n, an output value of a function.

fundamental principle of fractions n, the property that the value of a fraction is unchanged when both its numerator and denominator are multiplied by the same nonzero value. We express this formally by saying if a is any number, and b and c are nonzero numbers, then $\frac{a \cdot c}{b \cdot c} = \frac{a}{b}$.

G.

Gaussian reduction n, the process of performing eltermentary row operations on a matrix to obtain a matrix in echelon form.

geometrically similar adj, having the same shape (but possibly different size).

graph n, a visual representation of the values of a variable or variables, typically drawn on a number line or on the Cartesian plane.

graph v, to draw a graph.

graph of an equation (or inequality) n, a picture of the solutions of an equation (or inequality) using a number line or coordinate plane.

graphical solution n, a method for solving equations (or inequalities) by reading values off an appropriate graph. Compare with **algebraic solution** and **numerical solution**.

greatest common factor (GCF) of two or more expressions n, the largest factor that divides evenly into each expression.

growth factor n, the factor by which an initial value of a growing quantity is multiplied to obtain the final value.

guidepoints n, individual points that are plotted to help draw a graph (by hand).

H.

half-life n, (of exponential decay) the time required for a quantity to diminish to half its original size.

half-plane n, either of the two regions of a plane that has been divided into two regions by a straight line

height *see* altitude.

htermisphere n, half a sphere (on one side or the other of a plane passing through the center).

horizontal asymptote n, a line parallel to the x-axis toward which the graph of an equation tends as the value of x grows or diminishes without bound.

horizontal axis n, the horizontal coordinate axis. Often called the x-**axis**.

horizontal intercept n, where the graph meets the horizontal axis. Also called x-**intercept**.

horizontal line test n, a test to determine if a function has an inverse function: If no horizontal line intersects the graph of a function more than once, then the inverse is also a function.

horizontal translation (of a graph) n, the result of moving all points of the graph straight left (or all straight right) by the same distance.

hypotenuse n, the longest side of a right triangle. (It is always the side opposite the right angle.)

I.

identity n, an equation that is true for all permissible values of the variable(s).

imaginary axis n, the vertical axis in the complex plane.

imaginary number n, a complex number of the form bi, where b is a real number and $i^2 = -1$.

imaginary part n, (of a complex number) the coefficient of i when the complex number is written in the form $a + bi$, where a and b are real numbers. For example, the imaginary part of $4 - 7i$ is -7.

imaginary unit n, a nonreal number denoted by i and which satisfies $i^2 = -1$, that is, i is defined to be a square root of -1.

inconsistent adj, (of a system of equations) having no solution.

increasing adj, (i) (of numbers) moving to the right on a number line: Positive numbers are increasing when moving farther from zero, and negative numbers are increasing when they move closer to 0; (ii) (of a graph) having increasing values of y when moving along the graph from left to right; (iii) (of a function) having an increasing graph.

independent adj, (i) (of a system of 2 linear equations in 2 variables) having different graphs for the two equations; (ii) (of a system of n linear equations in n variables) having no one equation equal to a linear combination of the others.

independent variable n, a variable whose value determines the value of the dependent variable.

index n, (of a radical) the number at the left of the radical symbol that indicates the type of root involved; for example, the index of 3 in the expression $\sqrt[3]{x}$ indicates a cube root.

inequality n, a mathematical statement of the form $a < b$, $a \leq b$, $a > b$, $a \geq b$, or $a \neq b$.

inflation n, a persistent increase over time of consumer prices.

inflection point n, a point where a graph changes concavity.

initial value n, the starting value of a variable, often when $t = 0$.

input n, value of the independent variable.

integer n, a whole number or the negative of a whole number.

intercept n, a point where a graph meets a coordinate axis.

intercept method n, a method for graphing a line by finding its horizontal and vertical intercepts.

interest n, money paid for the use of money. For example, after borrowing money, the borrower must pay the lender not only the original amount of money borrowed (known as the **principal**) but also the interest on the principal.

interest rate n, the fraction of the principal that is paid as interest for one year. For example, an interest rate of 10% means that the interest for one year will be 10% of the principal.

interpolate v, to estimate the value of a dependent variable based on data that include both larger and smaller values of the independent variable.

intersection point n, a point in common to two graphs.

interval n, a set of numbers that includes all the numbers between a and b (possibly but not necessarily including a and/or b), where a and b are real numbers. Or the set of all numbers less than b (and possibly including b), or the set all numbers greater than a (and possibly including a).

interval notation n, notation used to designate an interval. For example, $[2, 3]$ is the interval notation to designate all the real numbers from 2 to 3, including both 2 and 3.

inverse function n, a function whose inputs are outputs of a given function f, and whose outputs are the corresponding inputs of f.

inverse square law n, a physical law that states that the magnitude of some quantity is inversely proportional to the square of the distance to the source of that quantity.

inverse variation n, a relation between two variables in which one is a constant divided by the other (so that the product of the two variables is the constant), or in which one is a constant divided by a positive exponent power of the other.

inversely proportional adj, describing variables related by inverse variation.

irrational number n, a number that is not rational but does correspond to a point on the number line.

isolate v, (a variable or expression) to create an equivalent equation (or inequality) in which the variable or expression is by itself on one side of the equation (or inequality).

isosceles triangle n, a triangle with two sides of equal length.

J.

joint variation n, a relationship among three or more variables in which whenever all but two variables are held constant, those remaining two variables vary directly or inversely with each other.

L.

law of exponents n, a basic property about powers and exponents.

lead coefficient n, (of a polynomial) the coefficient of term with highest degree.

leading entry n, (of a row in a matrix) the first nonzero entry of the row, when read from left to right.

leg n, one of the two shorter sides of a right triangle, or the length of that side.

like fractions n, fractions with equivalent denominators.

like terms n, terms with equivalent variable parts.

line segment n, the points on a single line that join two specified points (the **endpoints**) on that line.

linear combination n, (i) the sum of a nonzero constant multiple of one equation and a nonzero constant multiple of a second equation; (ii) the sum of constant multiples of quantities.

linear combinations n, a procedure for solving a linear system of equations which requires taking one or more linear combination of equations.

linear equation n, an equation such as $2x + 3y = 4$ or $x - 3y = 7$ in which each term has degree 0 or 1.

linear programming n, the study of optimizing functions with constraint equations and/or constraint inequalities.

linear regression n, the process of using a line to predict values of a (dependent) variable.

linear system n, a set of linear equations.

linear term n, a term that consists of a constant times a variable.

log *see* **logarithm**.

log scale n, a scale of measurement that uses the logarithm of a physical quantity rather than the quantity itself.

log-log paper n, a type of graph paper in which both horizontal and vertical axes use log scales.

logarithm n, (i) an exponent; (ii) a function whose outputs are exponents associated with a given base.

logarithmic equation n, an equation involving the logarithm of a variable expression.

logarithmic function n, a function of the form $f(x) = \log_b(x)$, where b is a positive constant different from 1.

lowest common denominator (LCD) n, (of two or more fractions) the smallest denominator that is a multiple of the denominators in the given fractions.

lowest common multiple (LCM) n, (of two or more counting numbers) the smallest counting number that the given numbers divide into evenly.

M.

mathematical model n, a representation of relationships among quantities using equations, tables, and/or graphs.

matrix n, a rectangular array of numbers.

maximum adj, largest or greatest.

maximum n, largest value.

maximum value n, (of a variable expression) the largest value that the expression can equal when the variable is allowed to assume all possible values.

mean n, the average of a set of numbers, computed by adding the numbers and dividing by how many are in the set. For example, the mean of 5, 2, and 11 is $\frac{5+2+11}{3} = 6$.

mechanistic model n, an equation whose graph (approximately) fits a given set of data and whose parameters are estimates about the physical properties involved.

median n, the middle number in a set of numbers when written in increasing order. For example, the median of 5, 2, and 11 is 5. If the set has two numbers in the middle when written in order, then the median of the set is the mean of those middle numbers. For example, the median of 6, 1, 9, and 27 is $\frac{6+9}{2} = 7.5$.

minimum adj, least or smallest.

minimum n, smallest value.

minimum value n, (of a variable expression) the smallest value that the expression can equal when the variable is allowed to assume all possible values.

mode n, the number that occurs most frequently in a set of numbers. For example, the mode of 1, 1, 2, and 3 is 1.

model n, a mathematical equation or graph or table used to represent a situation in the world or a situation described in words. For example, the equation $P = R - C$ is a model for the relationship among the variables of profit, revenue, and cost.

model v, to create a model.

monomial n, an algebraic expression with only one term.

monotonic adj, (of a function or graph) either never increasing or never decreasing.

multiplicative property (of absolute values) n, the property that $|a \cdot b| = |a| \cdot |b|$ for any real numbers a and b.

multiplicity n, (i) (of a zero of a polynomial) the number of times the corresponding linear factor appears as a factor of the polynomial. For example, -9 is a zero of multiplicity one and 7 is a zero of multiplicity two for the polynomial $p(x) = x^3 - 5x^2 - 77x + 441$ because $p(x)$ factors as $p(x) = (x+9)(x-7)^2$; (ii) (of a solution to a polynomial equation) the multiplicity of the zero of the corresponding polynomial. For example, -9 is a solution of multiplicity one and 7 is a solution of multiplicity two for the polynomial equation $x^3 = 5x^2 + 77x - 441$ because the equation can be written in the standard form $p(x) = 0$, where $p(x)$ factors as $p(x) = (x+9)(x-7)^2$.

N.

natural base n, the irrational number $e \approx 2.71828182846$, which is useful in calculus, statistics, and other mathematical topics.

natural exponential function n, the function $f(x) = e^x$, where e is the natural base.

natural log or natural logarithm n, the logarithm with base e, where e is the natural base.

natural number n, a counting number.

negative number n, a number that is less than zero.

negative of n, the opposite of.

net change n, the final value of a variable minus the initial value. For example, if an object's weight decreases from 15 pounds to 13 pounds, the net change in weight is -2 pounds.

nonstrict inequality n, a mathematical statement of the form $a \leq b$ or $a \geq b$.

normal adj, perpendicular.

***n*th root n**, a number which when raised to the power n gives a desired value. When $b^n = a$, then b is an nth root of a.

number line n, a line with coordinates marked on it representing the real numbers.

numerator n, the expression in a fraction that is above the fraction bar.

numerical solution n, a method for solving equations by reading values from an appropriate table of values. Compare with **algebraic solution** and **graphical solution**.

O.

objective function n, (in linear programming) the function that is to be optimized.

one-to-one adj, (pertaining to a function) having the property that every output comes from one and only one input.

open interval n, a set of numbers denoted by (a, b), which includes all the numbers between a and b but not the numbers a and b themselves, where a and b are real numbers and $a \neq b$. Or the set of numbers denoted by $(-\infty b)$, which includes all numbers less than b, or the set of numbers denoted by (a, ∞), which includes all numbers greater than a.

operation n, addition, subtraction, multiplication, or division (or raising to a power or taking a root).

opposite n, the number on the number line that is on the other side of 0 and at the same distance. For example, 5 and -5 are opposites.

order n, (of a matrix) the numbers of rows and columns respectively of the matrix, also called the **dimension** of the matrix. For example, a matrix with order 2 by 3 (or 2×3) has two rows and three columns.

order of operations n, rules that prescribe the order in which to carry out the operations in an expression.

order symbol n, one of the four symbols $<$, or $\leq$, or $>$, or $\geq$.

ordered pair n, a pair of numbers enclosed in parentheses, like this: (x, y). Often used to specify a point or a location on the coordinate plane.

ordered triple n, three numbers enclosed in parentheses, like this: (x, y, z). Often used to specify a solution to a system of equations in three variables or a point in three-dimensional space.

origin n, the point where the coordinate axes meet. It has coordinates (0, 0).

output n, value of the dependent variable.

P.

parabola n, a curve with the shape of the graph of $y = ax^2$, where $a \neq 0$.

parallel lines n, lines that lie in the same plane but do not intersect, even if extended indefinitely.

parameter n, a constant in an equation that varies in other equations of the same form. For example, in the slope-intercept formula $y = b + mx$, the constants b and m are parameters.

percent n, a fraction with (an understood) denominator of 100. For example, to express the fraction $\frac{51}{100}$ as a percent, we write 51% or say "51 percent."

percent increase n, the change in some quantity, expressed as a percentage of the starting amount.

perfect square n, the square of an integer. For example, 9 is a perfect square because $9 = 3^2$.

perimeter n, the distance around the edge or boundary of a two-dimensional figure.

perpendicular lines n, lines that meet and form right angles with each other.

piecewise defined function n, a function defined by multiple expressions, one expression for each specified interval of the independent variable.

point-slope form n, one way of writing the equation for a line: $y - y_1 = m(x - x_1)$ or $\frac{y-y_1}{x-x_1} = m$.

polygon n, a simple closed geometric figure in the plane consisting of line segments (called sides) that meet only at their endpoints. For example, triangles are polygons with three sides.

polynomial n, a sum of terms, where each term is either a constant or a constant times a power of a variable, and the exponent is a positive integer.

polynomial function n, a function that can be written in the form $f(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_2 x_2 + a_1 x + a_0$ where $a_0, a_1, a_2, \ldots a_n$ are constants.

positive number n, a number greater than zero.

power n, an expression that consists of a base and an exponent.

power function n, a function of the form $f(x) = ax^p$, where a and p are constants.

prime (or prime number) n, an integer greater than 1 whose only whole number factors are itself and 1.

principal n, the original amount of money deposited in an account or borrowed from a lender. (Compare with **interest**.)

principal root *see* **principal square root**.

principal square root n, the nonnegative square root.

product n, the result of a multiplication. For example, the expression $a \cdot b$ represents the product of a and b.

profit n, the money left after counting all the revenue that came in and subtracting the costs that had to be paid out.

proportion n, an equation in which each side is a ratio.

proportional *see* **directly proportional**, **inversely proportional**.

pyramid n, a three-dimensional object like a cone except that the base is a polygon instead of a circle.

Pythagorean theorem: If the legs of a right triangle are a and b and the hypotenuse is c, then $a^2 + b^2 = c^2$.

Q.

quadrant n, any of the four regions into which the coordinate axes divide the plane. The **first quadrant** consists of the points where both coordinates are positive; the **second quadrant** where the first coordinate is negative and the second coordinate positive; the **third quadrant** consists of points where both coordinates are negative; and the **fourth quadrant** contains the points where the first coordinate is positive and the second coordinate is negative.

quadratic adj, relating to the square of a variable (or of an expression).

quadratic equation n, an equation that equates zero to a polynomial of degree 2 (or an equivalent equation).

quadratic formula n, the formula that gives the solutions of the quadratic equation $ax^2 + bx + c = 0$, namely $x = \frac{-bś\sqrt{b^2-4ac}}{2a}$.

quadratic function n, a function of the form $f(x) = ax^2 + bx + c$.

quadratic polynomial n, a polynomial whose degree is 2.

quadratic regression n, the process of using a quadratic function to predict values of a (dependent) variable.

quadratic term n, a term whose degree is 2.

quadratic trinomial n, a polynomial of degree 2 and having exactly 3 terms.

quadrilateral n, a polygon with exactly 4 sides.

quartic adj, (pertaining to a polynomial) having degree 4.

quotient n, the result of a division. For example, the expression $a \div b$ represents the quotient of a and b.

R.

radical n, a root of a number, such as a square root or a cube root.

radical expression n, a square root, a cube root, or an nth root.

radical equation n, an equation in which the variable appears under a radical sign.

radical notation n, notation using the radical sign to indicate a root.

radical sign n, the symbol $\sqrt{}$, which is used to indicate the principal square root, or the symbol $\sqrt[3]{}$, which is used to indicate cube root, or the symbol $\sqrt[n]{}$, which is used to indicate nth root, where n is a counting number greater than 2.

radicand n, the expression under a radical sign.

radius n, (i) a line segment from the center of a circle (or sphere) to a point on the circle (sphere), (ii) the length of that line segment.

raise to a power v, use as a repeated factor, for example, to raise x to the power 2 is the same as multiplying $x \cdot x$.

range n, (i) the set of all output values for a function; (ii) the difference between the largest and smallest values in a set of data.

rate n, a ratio that compares two quantities (typically) with different units.

rate of change n, the ratio of change in the dependent variable to the corresponding change in the independent variable, measuring the change in the dependent variable per unit change in the independent variable.

ratio n, (i) a way to compare two quantities by division, (ii) a fraction. For example, "the ratio of 1 to 2" can be written as $\frac{1}{2}$.

rational adj, having to do with ratios.

rational exponent n, an exponent that is a rational number. For example, the expression $x^{1/3}$ has a rational exponent of 1/3, and $x^{1/3} = \sqrt[3]{x}$.

rational expression n, a ratio of two polynomials. Also called an **algebraic fraction**.

rational function n, a function of the form $f(x) = \frac{p(x)}{q(x)}$, where p and q are polynomial functions.

rational number n, a number that can be expressed as the ratio of two integers.

rationalize the denominator v, to find an equivalent fraction that contains no radical in the denominator. For example, when we rationalize $\frac{1}{\sqrt{2}}$, we obtain $\frac{\sqrt{2}}{2}$.

real axis n, the horizontal axis in the complex plane.

real line *see* **number line**.

real part n, (of a complex number) the term which does not include i when the complex number is written in the form $a + bi$, where a and b are real numbers. For example, the real part of $4 - 7i$ is 4.

real number n, a number that corresponds to a point on a number line.

reciprocal (of a number) n, the result of dividing 1 by the given number. For example, the reciprocal of 2 is $\frac{1}{2}$. Two numbers are reciprocals of each other when their product is 1.

rectangle n, a four-sided figure (in the plane) with four right angles. The opposite sides are equal in length and parallel.

reduce a fraction v, to find an equivalent fraction whose numerator and denominator share no common factors (other than 1 and -1).

reduced row echelon form n, (of a matrix) a row echelon form matrix that also satisfies (1) the leading entry in each nonzero row is a 1, (2) each leading 1 is the only nonzero entry in its column.

reflection (of a point or graph across a line) n, the transformation that replaces each point of a graph with its mirror image on the other side of the line.

regression line n, the line used for linear regression.

regular polygon n, a polygon all of whose sides have equal length and all of whose angles are congruent.

restricted domain n, a domain of a function that does not include all real numbers.

revenue n, money that an individual or group receives. For example, a person might have revenues from both a salary and from earnings on investments.

right angle n, an angle of $90°$.

right triangle n, a triangle that includes one right angle.

root n, the solution to an equation. See also **cube root**; ***n*th root**; **principal square root**; **square root**.

round v, to give an approximate value of a number by choosing the nearest number of a specified form. For example, to round 3.14159 to two decimal places, we use 3.14.

row echelon form n, (of a matrix) a matrix in which (1) only zeros occur below each nonzero leading entry, (2) the leading entry in any row is to the right of any leading entry above it, and (3) any row consisting entirely of zeros is below all rows with any nonzero entry.

S.

satisfy v, to make an equation true (when substituted for the variable or variables). For example, the number 5 satisfies the equation $x - 2 = 3$.

scale n, marked values on a number line or axes to establish how wide an interval of numbers is represented by a physical distance on the number line.

scale v, (i) to determine the scale on an axis or axes; (ii) to multiply (measurements) by a fixed number (the **scale factor**).

scale factor n, a fixed number by which measurements or values are multiplied.

scaling exponent n, the exponent defining direct variation or a power function. For example, if $y = 3x^4$, then the scaling exponent is 4.

scatterplot n, a type of graph used to represent pairs of data values. Each pair of data values provides the coordinates for one point on the scatterplot. Also called a **scatter diagram**.

scientific notation n, a standard method for writing very large or very small numbers that uses powers of 10. For example, the scientific notation for $12,000$ is 1.2×10^4.

semicircle n, half a circle (on one side or the other of a diameter).

signed number n, a positive or negative number.

significant digit n, (in the decimal form of a number) a digit warranted by the accuracy of the measuring device. When the decimal point is present, the significant digits are all those from the leftmost nonzero digit to the rightmost digit after the decimal point. When there is no decimal point, the significant digits are all those from the leftmost nonzero digit to the rightmost nonzero digit. For example, 123.40 has five significant digits, but $12,340$ has only four significant digits. Also called **significant figure**.

significant figure *see* **significant digit**.

similar *see* **geometrically similar**.

simplify v, to write in an equivalent but simpler or more convenient form. For example, we can simplify the expression $\sqrt{16}$ to 4.

sinusoidal adj, having the shape of a sine or cosine graph.

slope n, a measure of the steepness of a line or of the rate of change of one variable with respect to another.

slope-intercept form n, a standard form for the equation of a nonvertical line: $y = b + mx$.

slope-intercept method n, a method for graphing a line that uses the slope and the y-intercept.

solution n, a value for the variable that makes an equation or an inequality true. A solution to an equation in two variables is an ordered pair that satisfies the equation. A solution to a system is an ordered pair that satisfies each equation of the system.

solve v, (i) (an equation) to find any and all solutions to an equation, inequality, or system; (ii) (a formula) to write an equation for one variable in terms of any other variables, for example, when we solve $5x + y = 3$ for y to get $y = -5x + 3$; (iii) (a triangle) to find the measures of all three sides and of all three angles.

sphere n, a three-dimensional object in the shape of a ball. A sphere consists of all the points in space at a fixed distance (the radius) from the center of the sphere.

square n, (i) any expression times itself; (ii) a rectangle whose sides are all the same length.

square v, to multiply by itself, that is, to raise to the power 2.

square matrix n, a matrix with the same number of rows as columns.

square root n, a number that when squared gives a desired value. For example, 7 is a square root of 49 because $7^2 = 49$.

standard form n, (i) (of a linear, quadratic, or other polynomial equation) an equation in which the right side is 0, so the equation has the form $p(x) = 0$; (ii) (of a system of linear equations) a system in which the variables occur only on the left side of each equation and in alphabetic order.

strict inequality n, a mathematical statement of the form $a < b$ or $a > b$.

subscript n, a small number written below and to the right of a variable. For example, in the equation $x_1 = 3$, the variable x has the subscript 1.

substitution method n, a method for solving a system of equations that begins by expressing one variable in terms of the other.

sum n, the result of an addition. For example, the expression $a + b$ represents the sum of a and b.

surface area n, the total area of the faces or surfaces of a three-dimensional object.

supplementary angles n, two angles whose measures add up to 180°.

supply equation n, an equation that gives the quantity of some product that producers are willing to produce in terms of the price of that product.

symmetry n, a geometric property of having sameness on opposite sides of a line (or plane) or about a point.

system of equations n, two or more equations involving the same variables.

T.

term n, (i) (in a sum) a quantity that is added to another. For example, in the expression $x + y - 4$, x, y, and -4 are the terms; (ii) an algebraic expression that is not a sum or difference, for example, 4x is one term.

test point n, (for an inequality) a point in the plane (or on a number line) used to determine which side of the plane (or number line) is included in the solution.

transform v, to apply a **transformation**.

transformation n, (i) (of data) applying a function to one or both components in a set of data, typically so that the resulting data becomes approximately linear; (ii) (of a graph) a change that occurs in the graph of an equation when one or more of the parameters defining that equation are altered.

translation n, (of a graph or geometric figure) sliding horizontally and/or vertically without rotating or changing any shapes.

trapezoid n, a four-sided figure in the plane with one pair of parallel sides.

triangle n, a three-sided figure in the plane.

triangle inequality n, the inequality $|a + b| \leq |a| + |b|$, which is true for any two real numbers a and b.

triangular form n, a system of linear equations in which the first variable does not occur in the second equation, the first two variables do not occur in the third equation (and the first three variables do not occur in the fourth equation if there are more than three variables, and so on).

trinomial n, a polynomial with exactly three terms.

turning point n, (of a graph) where the graph either changes from increasing to decreasing or vice versa.

U.

union n, the set obtained by collecting all the elements of one set along with all the elements of another set.

unit circle n, a circle of radius 1 unit (usually centered at the origin).

unlike fractions n, fractions whose denominators are not equivalent.

unlike terms n, terms with variable parts that are not equivalent.

upper triangular form n, (of a matrix) a matrix with all zeros in the lower left corner. More precisely, the entry in the ith row and jth column is 0 whenever $i > j$.

V.

variable adj, not constant, subject to change.

variable n, a numerical quantity that changes over time or in different situations.

variation *see* **direct variation**; **inverse variation**.

verify v, to prove the truth or validity of an assertion.

vertex n, (*plural* **vertices**), (i) a point where two sides of a polygon meet; (ii) a corner or extreme point of a geometric object; (iii) the highest or lowest point on a parabola.

vertex angle n, the angle between the equal sides in an isosceles triangle.

vertex form n, one way of writing a quadratic equation, $y = a(x - x_v)2 + y_v$, which displays the vertex, (x_v, y_v).

vertical asymptote n, a line $x = a$ parallel to the y-axis toward which the graph of an equation tends as the value of x approaches a.

vertical axis n, the vertical coordinate axis. Often called the ***y*-axis**.

vertical compression n, (of a graph) the result of replacing each point of the graph with the point obtained by scaling the y-coordinate by a fixed factor (when that factor is between 0 and 1).

vertical intercept n, where the graph meets the vertical axis. Also called the ***y*-intercept**.

vertical line test n, a test to decide whether a graph defines a function: A graph represents a function if and only if every vertical line intersects the graph in at most one point.

vertical stretch n, (of a graph) the result of replacing each point of the graph with the point obtained by scaling the y-coordinate by a fixed factor (when that factor is greater than 1).

vertical translation n, (of a graph) the result of moving all points of the graph straight up (or all straight down) by the same distance.

vertices n, the plural of **vertex**.

volume n, a measure of the three-dimensional space enclosed by a three-dimensional object, typically expressed in terms of cubic units, such as cubic meters or cubic feet.

W.

whole number n, one of the numbers $0, 1, 2, 3, \ldots$.

X.

***x*-axis** *see* **horizontal axis**.

***x*-intercept** *see* **horizontal intercept**.

***xy*-plane** *see* **coordinate plane**.

Y.

***y*-axis** *see* **vertical axis**.

***y*-intercept** *see* **vertical intercept**.

Z.

zero n, (i) the number 0, with the property that when it is added to any other number, the resulting sum is equal to that second number; (ii) an input to a function which yields an output of 0.

zero-factor principle *see* Zero-Factor Principle , p. 1014 below.

Properties of Numbers. Associative Laws.

Addition: If a, b, and c are any numbers, then $(a+b)+c=a+(b+c)$.

Multiplication If a, b, and c are any numbers, then $(a \cdot b) \cdot c = a \cdot (b \cdot c)$.

Associative Laws.

Addition: If a, b, and c are any numbers, then $(a+b)+c=a+(b+c)$.

Multiplication If a, b, and c are any numbers, then $(a \cdot b) \cdot c = a \cdot (b \cdot c)$.

Commutative Laws.

Addition: If a and b are any numbers, then $a+b=b+a$.

Multiplication If a and b are any numbers, then $a \cdot b = b \cdot a$.

Distributive Law.

$a(b+c)=ab+ac$ for any numbers a, b, and c.

Properties of Equality.

Addition: If $a=b$ and c is any number, then $a+c=b+c$.

Subtraction: If $a=b$ and c is any number, then $a-c=b-c$.

Multiplication If $a=b$ and c is any number, then $a \cdot c = b \cdot c$.

Division If $a=b$ and c is any nonzero number, then $\frac{a}{c}=\frac{b}{c}$.

Fundamental Principle of Fractions.

If a is any number, and b and c are nonzero numbers, then $\dfrac{a \cdot c}{b \cdot c}=\dfrac{a}{b}$.

Laws of Exponents.

1 $a^m \cdot a^n = a^{m+n}$

2
- $\dfrac{a^m}{a^n}=a^{m-n} \qquad (n<m)$
- $\dfrac{a^m}{a^n}=\dfrac{1}{a^{n-m}} \qquad (n>m)$

3 $(a^m)^n = a^{m+n}$

4 $(ab)^n = a^n b^n$

5 $\left(\dfrac{a}{b}\right)^n = \dfrac{a^n}{b^n}$

Product Rule for Radicals.

If a and b are both nonnegative, then $\sqrt{ab} = \sqrt{a}\sqrt{b}$.

Quotient Rule for Radicals.

If $a \geq 0$ and $b > 0$, then $\sqrt{\dfrac{a}{b}} = \dfrac{\sqrt{a}}{\sqrt{b}}$.

Zero-Factor Principle.

If $ab = 0$ then either $a = 0$ or $b = 0$.

Properties of Absolute Value.

$$|a+b| \leq |a| + |b| \quad \text{Triangle inequality}$$
$$|ab| = |a|\,|b| \quad \text{Multiplicative property}$$

Appendix D

Technology (Graphing calculators)

Technology D.0.1 Graphing an Equation. We can use a graphing calculator to graph an equation. On most calculators, we follow three steps.

To Graph an Equation:

1. Press `Y=` and enter the equation you wish to graph.
2. Press `WINDOW` and select a suitable graphing window.
3. Press `GRAPH`

Technology D.0.2 Graphing an Equation. We can use a graphing calculator to graph an equation. On most calculators, we follow three steps.

To Graph an Equation:

1. Press `Y=` and enter the equation you wish to graph.
2. Press `WINDOW` and select a suitable graphing window.
3. Press `GRAPH`

Technology D.0.3 Choosing a Graphing Window. Knowing the intercepts can also help us choose a suitable window on a graphing calculator. We would like the window to be large enough to show the intercepts. For the graph in the example above, we can enter the equation

$$Y = (9000 - 150X)/{-180}$$

in the window

Xmin$= -20$	Xmax$= 70$
Ymin$= -70$	Ymax$= 30$

Technology D.0.4 Making a Table of Values with a Calculator. We can use a graphing calculator to make a table of values for a function defined by an equation. For the function in Example 1.2.8, p. 32,

$$h = 1776 - 16t^2$$

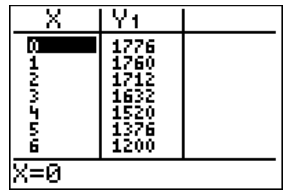
we follow the steps:

- Enter the equation: Press the `Y=` key, clear out any other equations, and define $Y_1 = 1776 - 16X^2$.

- Choose the x-values for the table. Press `2ndWINDOW` to access the *TblSet* (Table Setup) menu and set it to look like the figure at left below.

This setting will give us an initial x-value of 0 ($TblStart = 0$) and an increment of one unit in the x-values, ($\Delta Tbl = 1$). It also fills in values of both variables automatically.

- Press `2nd GRAPH` to see the table of values, as shown in the figure at right below. From this table, we can check the heights we found in Example 1.2.8, p. 32.



Now try making a table of values with $TblStart = 0$ and $\Delta Tbl = 0.5$. Use the ↑ and ↓ arrow keys to scroll up and down the table.

Technology D.0.5 Evaluating a Function. We can use the table feature on a graphing calculator to evaluate functions. Consider the function of Checkpoint 1.2.17, p. 34, $f(x) = 5 - x^3$.

- Press `Y=`, clear any old functions, and enter

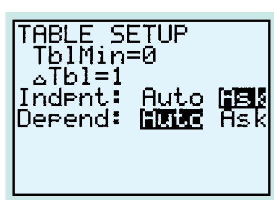
  $Y_1 = 5 - X^3$

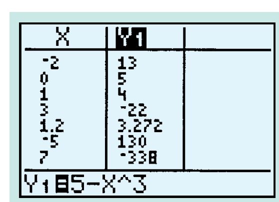
- Press `TblSet` (`2nd WINDOW`) and choose *Ask* after *Indpnt*, as shown in the figure at left below, and press `ENTER`. This setting allows you to enter any x-values you like.

- Press `TABLE` (using `2nd GRAPH`).

- To follow Checkpoint 1.2.17, p. 34, key in `(-)` 2 `ENTER` for the x-value, and the calculator will fill in the y-value. Continue by entering 0, 1, 3, or any other x-values you choose.

One such table is shown in the figure at right below.





If you would like to evaluate a new function, you do not have to return to the `Y=` screen. Use the → and ↑ arrow keys to highlight Y_1 at the top of the second column. The definition of Y_1 will appear at the bottom of the display, as shown above. You can key in a new definition here, and the second column will be updated automatically to show the y-values of the new function.

Technology D.0.6 Making a Table of Values with a Calculator. We can use a graphing calculator to make a table of values for a function defined by an equation. For the function in Example 1.2.8, p. 32,

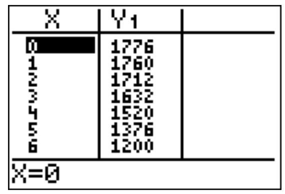
$$h = 1776 - 16t^2$$

we follow the steps:

- Enter the equation: Press the Y= key, clear out any other equations, and define $Y_1 = 1776 - 16X^2$.
- Choose the x-values for the table. Press 2ndWINDOW to access the *TblSet* (Table Setup) menu and set it to look like the figure at left below.

This setting will give us an initial x-value of 0 ($TblStart = 0$) and an increment of one unit in the x-values, ($\Delta Tbl = 1$). It also fills in values of both variables automatically.

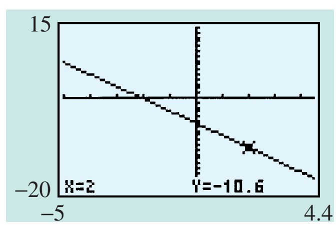
- Press 2nd GRAPH to see the table of values, as shown in the figure at right below. From this table, we can check the heights we found in Example 1.2.8, p. 32.



Now try making a table of values with $TblStart = 0$ and $\Delta Tbl = 0.5$. Use the ↑ and ↓ arrow keys to scroll up and down the table.

Technology D.0.7 Finding Coordinates with a Graphing Calculator. We can use the TRACE feature of the calculator to find the coordinates of points on a graph. For example, graph the equation $y = -2.6x - 5.4$ in the window

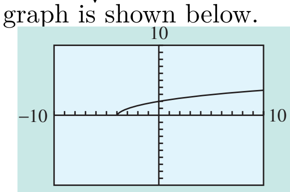
$$\text{Xmin}= -5 \qquad \text{Xmax}= 4.4$$
$$\text{Ymin}= -20 \qquad \text{Ymax}= 15$$



Press TRACE, and a ``bug'' begins flashing on the display. The coordinates of the bug appear at the bottom of the display, as shown in the figure. Use the left and right arrows to move the bug along the graph. You can check that the coordinates of the point $(2, -10.6)$ do satisfy the equation $y = -2.6x - 5.4$.

The points identified by the Trace bug depend on the window settings and on the type of calculator. If we want to find the y-coordinate for a particular x-value, we enter the x-coordinate of the desired point and press ENTER.

Technology D.0.8 Using a Calculator to Graph a Function. We can also use a graphing calculator to obtain a table and graph for the function in Example 1.3.5, p. 60. We graph a function just as we graphed an equation. For this function, we enter $Y_1 = \sqrt{}(X + 4)$ and press ZOOM 6 for the standard window. The calculator's graph is shown below.



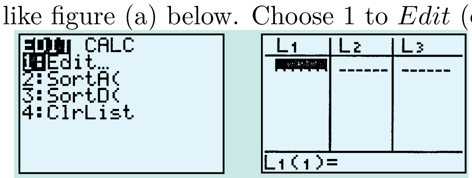
Technology D.0.9 Using the Trace Feature. You can use the Trace feature on a graphing calculator to approximate solutions to equations. Graph the function $f(x)$ in Example 1.3.14, p. 64 in the window

$$\text{Xmin}= -4 \qquad \text{Xmax}= 4$$
$$\text{Ymin}= -20 \qquad \text{Ymax}= 40$$

and trace along the curve to the point $(2.4680851, 15.512401)$. We are close to a solution, because the y-value is close to 15. Try entering x-values close to 2.4680851, for instance, $x = 2.4$ and $x = 2.5$, to find a better approximation for the solution.

We can use the intersect feature on a graphing calculator to obtain more accurate estimates for the solutions of equations.

Technology D.0.10 Using a Calculator for Linear Regression. You can use a graphing calculator to make a scatterplot, find a regression line, and graph the regression line with the data points. On the TI-83 calculator, we use the statistics mode, which you can access by pressing `STAT`. You will see a display that looks like figure (a) below. Choose 1 to *Edit* (enter or alter) data.



(a) (b)

Now follow the instructions in Example 8.1.12, p. 815 for using your calculator's statistics features.

Technology D.0.11 Using the Intersect Feature. We can use the *intersect* feature on a graphing calculator to solve equations.

Example D.0.12 Use a graphing calculator to solve $\dfrac{3}{x-2} = 4$

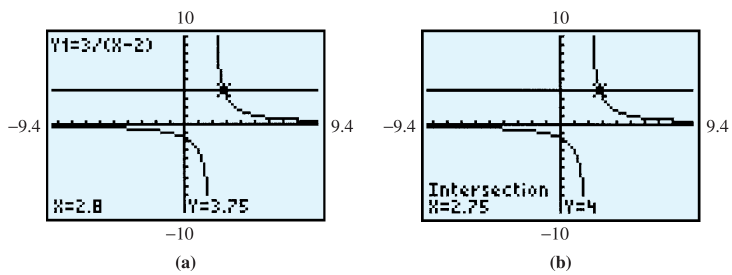
Solution. We would like to find the points on the graph of $y = \dfrac{3}{x-2}$ that have y-coordinate equal to 4. We graph the two functions

$$Y_1 = 3/(X-2)$$
$$Y_2 = 4$$

in the window

$$\text{Xmin} = -9.4 \quad \text{Xmax} = 9.4$$
$$\text{Ymin} = -10 \quad \text{Ymax} = 10$$

The point where the two graphs intersect locates the solution of the equation. If we trace along the graph of Y_1, the closest we can get to the intersection point is $(2.8, 3.75)$, as shown in figure (a). We get a better approximation using the *intersect* feature.



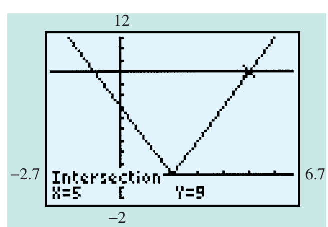
Use the arrow keys to position the Trace bug as close to the intersection point as you can. Then press `2nd` `TRACE` to see the Calculate menu. Press 5 for intersect; then respond to each of the calculator's questions, *First curve?*, *Second curve?*, and *Guess?* by pressing `ENTER`. The calculator will then display the intersection point, $x = 2.75$, $y = 4$, as shown in figure (b). The solution of the original equation is $x = 2.75$. □

Technology D.0.13 Solving Absolute Value Equations. We can use a graphing calculator to solve the equations in Example 2.5.3, p. 236.

The graph shows the graphs of $Y_1 = \text{abs}(3X-6)$ and $Y_2 = 9$ in the window

$$\text{Xmin} = -2.7 \quad \text{Xmax} = 6.7$$
$$\text{Ymin} = -2 \quad \text{Ymax} = 12$$

We use the `Trace` or the *intersect* feature to locate the intersection points at $(-1, 9)$ and $(5, 9)$.

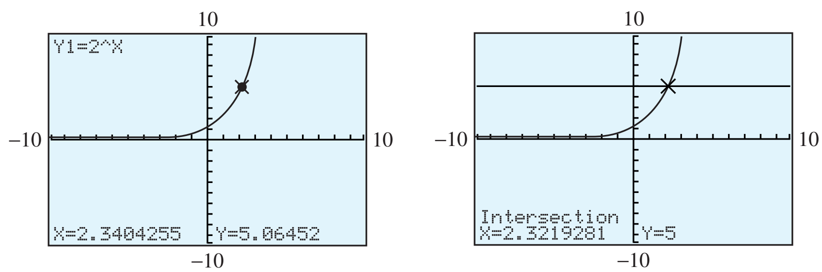


Technology D.0.14 Graphical Solution of Exponential Equations. It is not always so easy to express both sides of the equation as powers of the same base. In the following sections, we will develop more general methods for finding exact solutions to exponential equations. But we can use a graphing calculator to obtain approximate solutions.

Example D.0.15 Use the graph of $y = 2^x$ to find an approximate solution to the equation $2^x = 5$ accurate to the nearest hundredth.

Solution. Enter $Y_1 = 2$ ^ X and use the standard graphing window (`ZOOM` 6) to obtain the graph shown in figure (a). We are looking for a point on this graph with y-coordinate 5.

Using the TRACE feature, we see that the y-coordinates are too small when $x < 2.1$ and too large when $x > 2.4$. The solution we want lies somewhere between $x = 2.1$ and $x = 2.4$, but this approximation is not accurate enough.



(a) (b)

To improve our approximation, we will use the **intersect** feature. Set $Y_2 = 5$ and press `GRAPH`. The x-coordinate of the intersection point of the two graphs is the solution of the equation $2^x = 5$ Activating the **intersect** command results in figure (b), and we see that, to the nearest hundredth, the solution is 2.32.

We can verify that our estimate is reasonable by substituting into the equation:

$$2^{2.32} \stackrel{?}{=} 5$$

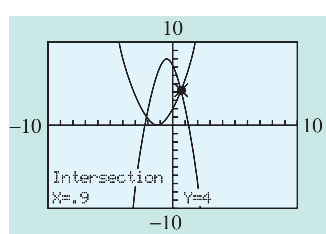
We enter 2 ^ 2.32 `ENTER` to get 4.993322196. This number is not equal to 5, but it is close, so we believe that $x = 2.32$ is a reasonable approximation to the solution of the equation $2^x = 5$. □

Technology D.0.16 Solving Systems with the Graphing Calculator. We can use the intersect feature of the graphing calculator to solve systems of quadratic equations. Consider the system

$$y = (x + 1.1)^2$$
$$y = 7.825 - 2x - 2.5x^2$$

We will graph these two equations in the standard window. The two intersection points are visible in the window, but we do not find their exact coordinates when we trace the graphs.

We can use the intersect command to locate one of the solutions, as shown below. You can check that the point $(0.9, 4)$ is an exact solution to the system by substituting $x = 0.9$ and $y = 4$ into each equation of the system. (The calculator is not always able to find the exact coordinates, but it usually gives a very good approximation.)



You can find the other solution of the system by following the same steps and moving the bug close to the other intersection point. You should verify that the other solution is the point $(-2.1, 1)$.

Technology D.0.17 Using a Calculator for Quadratic Regression. We can use a graphing calculator to find an approximate quadratic fit for a set of data. The procedure is similar to the steps for linear regression outlined in Section 1.2, p. 27.

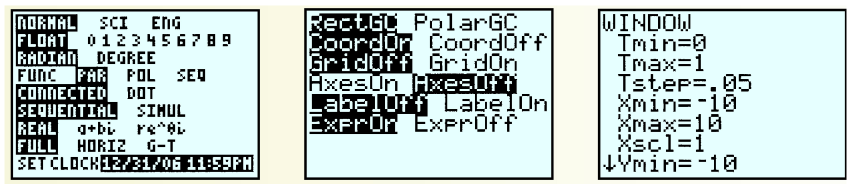
Technology D.0.18 Bézier Curves on the Graphing Calculator.

Investigation 59

A We can draw Bézier curves on the graphing calculator using the parametric mode. First, press the `MODE` key and highlight *PAR*, as shown in figure (a). To remove the x- and y-axes from the display, press `2nd` `ZOOM` to get the *Format* menu, then choose *AxesOff* as shown in figure (b). Finally, we set the viewing window: Press `WINDOW` and set

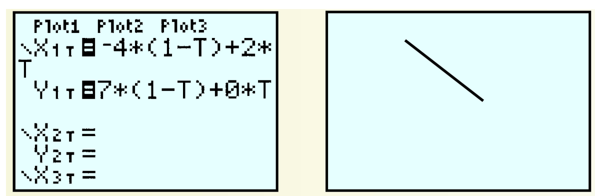
$$\text{Tmin} = 0 \quad \text{Tmax} = 1 \quad \text{TStep} = 0.05$$
$$\text{Xmin} = -10 \quad \text{Xmax} = 10 \quad \text{Ymin} = -10 \quad \text{Ymax} = 10$$

as shown in figure (c).



(a) (b) (c)

As an example, we will graph the linear curve from part (A). Press `Y=` and enter the definitions for $x(t)$ and $y(t)$, as shown below. Press `GRAPH` and the calculator displays the line segment joining $(-4, 7)$ and $(2, 0)$. Experiment by modifying the endpoints to see how the graph changes.



(a) (b)

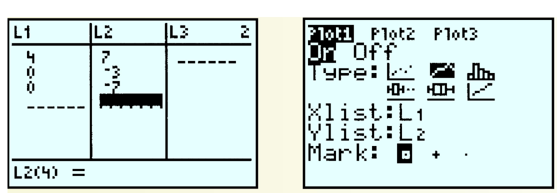
B Designing a Numeral 7

1 Press `2nd` `Y=` and enter the formulas for the quadratic Bézier curve defined by the endpoints $(4, 7)$ and $(0, -7)$, and the control point $(0, 5)$ under X_{1T} and Y_{1T}. (These are the same formulas you found in step B.1, p. 696 of Investigation 48, p. 696.)

2 Find the functions f and g for the linear Bézier curve joining the points $(-4, 7)$ and $(4, 7)$. Simplify the formulas for those functions and enter them into your calculator under X_{2T} and Y_{2T}. Press `GRAPH` to see the graph.

3 Now we will alter the image slightly: Go back to X_{1T} and Y_{1T} and change the control point to $(0, -3)$. (These are the formulas you found in step B.4, p. 697 of Investigation 48, p. 696.) How does the graph change?

4 We can see exactly how the control point affects the graph by connecting the three data points with line segments. Press `STAT` `ENTER` and enter the coordinates of $(4, 7)$, $(0, -3)$, and $(0, -7)$ in L_1 and L_2, as shown in figure (a). Press `2nd` `Y=` `ENTER`, turn on *Plot1*, and select the second plot type, as shown in figure (b). You should see the line segments superimposed on your numeral 7. How are those segments related to the curve?



(a) (b)

5 Now edit L_2 so that the control point is $(0, 5)$, and again define

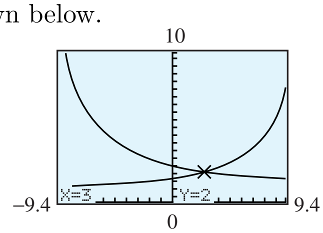
Technology D.0.19 Solving Equations with Fractions Graphically. We can solve the equation in Example 7.5.3, p. 777 graphically by considering two functions, one for each side of the equation. Graph the two functions

$$Y_1 = \frac{30}{12+x} \qquad \text{and} \qquad Y_2 = \frac{30}{12-x}$$

in the window

$$\text{Xmin} = -9.4 \qquad \text{Xmax} = 9.4$$
$$\text{Ymin} = 0 \qquad \text{Ymax} = 10$$

to obtain the graph shown below.



The function Y_1 gives the time it takes Rani to paddle 30 miles downstream, and Y_2 gives the time it takes her to paddle 18 miles upstream. Both of these times depend on the speed of the current, x.

We are looking for a value of x that makes Y_1 and Y_2 equal. This occurs at the intersection point of the two graphs, $(3, 2)$. Thus, the speed of the current is 3 miles per hour, as we found in Example 7.5.3, p. 777. The y-coordinate of the intersection point gives the time Rani paddled on each part of her trip: 2 hours each way.

Technology D.0.20 Using the Intersect Feature to Solve a System. Because the `TRACE` feature does not show every point on a graph, we may not find the exact solution to a system by tracing the graphs. In the next Example we demonstrate the *intersect* feature of the calculator.

Example D.0.21 Solve the system

$$3x - 2.8y = 21.06$$
$$2x + 1.2y = 5.3$$

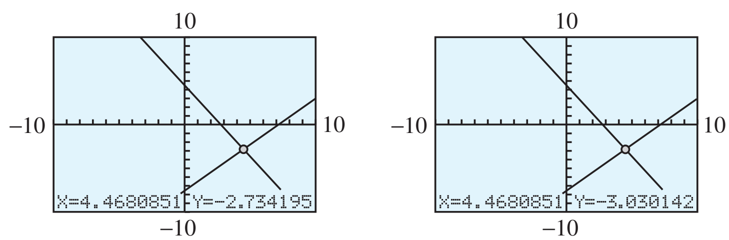
Solution. We can graph this system in the standard window by solving each equation for y. We enter

$$Y_1 = (21.06 - 3X)/-2.8$$
$$Y_2 = (5.3 - 2X)/1.2$$

and then press `ZOOM` 6. (Don't forget the parentheses around the numerator of each expression.)

We Trace along the first line to find the intersection point. It appears to be at $x = 4.468051$, $y = -2.734195$, as shown in figure (a). However, if we press the up or down arrow to read the coordinates off the second line, we see that for the same x-coordinate we obtain a different y-coordinate, as in figure (b).

The different y-coordinates indicate that we have not found an intersection point, although we are close. The *intersect* feature can give us a better estimate, $x = 4.36$, $y = -2.85$.



(a) **(b)**

We can substitute these values into the original system to check that they satisfy both equations.

$$3(\mathbf{4.36}) - 2.8(\mathbf{-2.85}) = 21.06$$
$$2(\mathbf{4.36}) + 1.2(\mathbf{-2.85}) = 5.3$$

□

Technology D.0.22 TI-84 and TI-83 calculators have a command for finding the reduced row echelon form of a matrix.

Example D.0.23 Solve the system

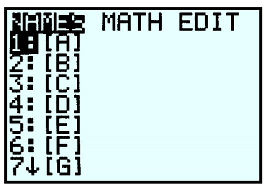
$$\begin{aligned} a + \ \ 2b + 4c + \ \ 8d &= 12 \\ -2a + \ \ 2b - 2c + \ \ 2d &= \ 1 \\ 6a + \ \ 6b + 6c + \ \ 6d &= 19 \\ 4a + 20b - 8c + 14d &= 41 \end{aligned}$$

by finding the reduced row echelon form of the augmented matrix.

Solution. The augmented matrix is

$$\left[\begin{array}{cccc|c} 1 & 2 & 4 & 8 & 12 \\ -2 & 2 & -2 & 2 & 1 \\ 6 & 6 & 6 & 6 & 19 \\ 4 & 20 & -8 & 14 & 41 \end{array}\right]$$

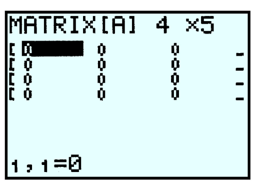
We enter this matrix into the calculator as follows: First access the *MATRIX* menu by pressing 2nd $\boxed{x^{-1}}$ on a TI-84 orMATRX on a TI-83. You will see the menu shown in figure (a). We will use matrix [A], which is already selected, and we presss $\boxed{\rightarrow}$ $\boxed{\rightarrow}$ ENTER to *EDIT* (or enter) the matrix, shown in figure(b).



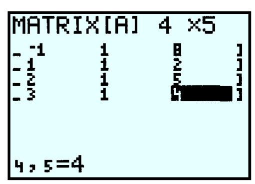
(a)

(b)

We want to enter a 4 × 5 matrix, so we press 4ENTER5ENTER, and we see the display in figure (a) below. Now type in the first row of the matrix, pressing ENTER after each entry. The calculator automatically moves to the second row. Continue filling in the rest of the augmented matrix, as shown in figure (b).

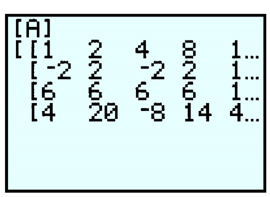


(a)

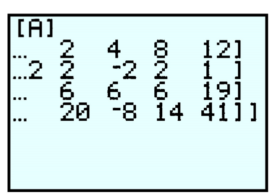


(b)

To make sure you have entered the values correctly, press 2ndMODE to quit to the home screen, then open the matrix menu again. Press 1ENTER to retrieve matrix [A]; the calculator display should look like figure (a) below. To check the rest of the matrix, press the right arrow key until you see the last column, as in figure (b).

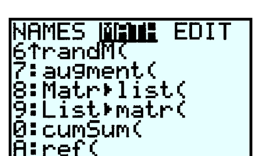


(a)



(b)

Now we are ready to compute the reduced row echelon form of the matrix. Access the matrix menu again, but this time press the right arrow once to highlight *MATH* as shown in figure (a). Scroll down until the rref(command is highlighted, as shown in figure (b), and press ENTER.



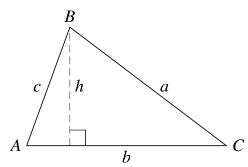
Appendix E

Geometry formulas

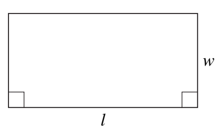
GEOMETRY

Perimeter and Area of Plane Figures

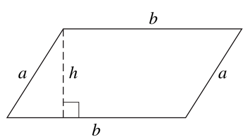
Triangle
$P = a + b + c$
$A = \frac{1}{2}bh$



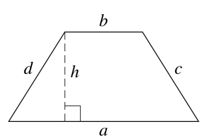
Rectangle
$P = 2l + 2w$
$A = lw$



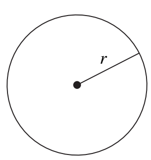
Parallelogram
$P = 2a + 2b$
$A = bh$



Trapezoid
$P = a + b + c + d$
$A = \frac{1}{2}h(a + b)$

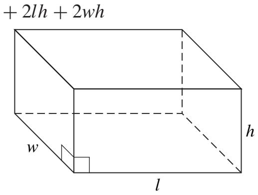


Circle
$C = 2\pi r$
$A = \pi r^2$

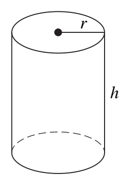


Volume and Surface Area of Solid Figures

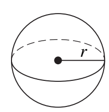
Rectangular Prism
$V = lwh$
$S = 2lw + 2lh + 2wh$



Right Circular Cylinder
$V = \pi r^2 h$
$S = 2\pi r^2 + 2\pi rh$



Sphere
$V = \frac{4}{3}\pi r^3$
$S = 4\pi r^2$



Right Circular Cone
$V = \frac{1}{3}\pi r^2 h$
$S = \pi r^2 + \pi rs$